Problems on convergence and uniform convergence of series of functions
by Ng Tze Beng

In this note I describe a method of summing a series by grouping terms of the same sign. If the series formed by this grouping is convergent, then the original series is convergent and converges to the same value. The estimation of the sum of the terms of the same sign is crucial to this method. In a method, the sum of each grouping is estimated by the integral of a non-negative or non-positive differentiable function with precisely one local stationary point in a closed and bounded interval. The argument of convergence is then given by observing that each term of the series formed by grouping the terms of the same sign is equal to the terms of a convergent series up to $o(n^{-\beta})$, $\beta > 1$. In another method, the sum of the integral forms an improper integral up to $o(n^{-\beta})$, $\beta > 1$. Thus, the knowledge of the convergence of the improper integral is crucial in this method.

Problem 1 The series $S(x) = \sum_{n=1}^{\infty} \frac{\sin(\sqrt{n}\, x\pi)}{n}$ converges pointwise on **R**. $S(x)$ converges uniformly on $[k, K]$ for any $k > 0$ and any $K > k$. It does not converge uniformly on $[0, K]$, any $K > 0$.

The technique to solve this problem is to group all the terms with the same sign and show that the series formed by the grouping of terms of the same sign is convergent.

Firstly, we note that $S(0)$ is obviously convergent and $S(0) = 0$. Now fix any $x > 0$.

For each integer $m \geq 0$, let $N_m = \{n : n$ is an integer and $\left(\frac{m}{x}\right)^2 < n \leq \left(\frac{m+1}{x}\right)^2\}$.
Then N_m may be empty. If $N_m = \varnothing$, then $\left(\frac{m+1}{x}\right)^2 - \left(\frac{m}{x}\right)^2 = \frac{2m+1}{x^2} < 1$.
If $N_m \neq \varnothing$, then for each $n \in N_m$, $m < \sqrt{n}\, x \leq m+1$ and so $m\pi < \sqrt{n}\, x\pi \leq (m+1)\pi$. This means that $\sin(\sqrt{n}\, x\pi)$ is of the same sign or 0 for all n in N_m. More precisely, for m even, $\sin(\sqrt{n}\, x\pi) \geq 0$ for all n in N_m and for m odd, $\sin(\sqrt{n}\, x\pi) \leq 0$ for all n in N_m. We partition $[0, \infty)$ into non-overlapping intervals $\left\{\left[\left(\frac{m}{x}\right)^2, \left(\frac{m+1}{x}\right)^2\right], m = 0, 1, 2, \cdots\right\}$, where for t in each interval $\left[\left(\frac{m}{x}\right)^2, \left(\frac{m+1}{x}\right)^2\right]$, $h(t) = \sin(\sqrt{t}\, x\pi)$ is of the same sign in the interior and 0 at the end points. Note that the maximum of $|h(t)|$ on $\left[\left(\frac{m}{x}\right)^2, \left(\frac{m+1}{x}\right)^2\right]$ occurs at $\left(\frac{m+1/2}{x}\right)^2$ and is equal to 1. For integer $m \geq 0$, define

$$S_m(x) = \begin{cases} \sum_{n \in N_m} \frac{\sin(\sqrt{n}\, x\pi)}{n} & \text{if } N_m \neq \varnothing, \\ 0 \text{ if } N_m = \varnothing. \end{cases}$$

Thus, all terms in $S_m(x)$ are of the same sign or 0. If we can show that the alternating series $\sum_{m=0}^{\infty} S_m(x)$ is convergent for $x > 0$, then $S(x)$ is convergent for $x > 0$.

Suppose now $N_m \neq \varnothing$. For each $n \in N_m$, $\left(\frac{m}{x}\right)^2 < n \leq \left(\frac{m+1}{x}\right)^2$ so that for $m > 0$,

$$\left(\frac{x}{m+1}\right)^2 \leq \frac{1}{n} < \left(\frac{x}{m}\right)^2 . \quad \text{----------------------------------} \quad (1)$$

Now $N_0 \neq \varnothing$, if and only if, $0 < x^2 \leq 1$. Thus, $n \in N_0$ implies that $x^2 \leq \frac{1}{n} \leq 1$.
Hence, for $N_m \neq \varnothing$ and $m > 0$, it follows from (1) that

$$\left(\frac{x}{m+1}\right)^2\left|\sum_{n\in N_m}\sin(\sqrt{n}\,x\pi)\right| \le |S_m(x)| \le \left(\frac{x}{m}\right)^2\left|\sum_{n\in N_m}\sin(\sqrt{n}\,x\pi)\right|. \text{ --------} \quad (2)$$

If $N_0 \ne \varnothing$,

$$x^2\sum_{n\in N_0}\sin(\sqrt{n}\,x\pi) \le S_0(x) \le \sum_{n\in N_0}\sin(\sqrt{n}\,x\pi). \quad \text{--------------------} \, (3)$$

Now we are going to estimate the sum $\sum_{n\in N_m}\sin(\sqrt{n}\,x\pi)$ when $N_m \ne \varnothing$.

Let
$$D_m(x) = \begin{cases} \sum_{n\in N_m}\sin(\sqrt{n}\,x\pi) \text{ if } N_m \ne \varnothing, \\ 0 \text{ if } N_m = \varnothing. \end{cases}.$$

Then we can write (2) as

$$\left(\frac{x}{m+1}\right)^2|D_m(x)| \le |S_m(x)| \le \left(\frac{x}{m}\right)^2|D_m(x)|, m>0 \text{ ------------------} \, (4)$$

and (3) as

$$x^2D_0(x) \le S_0(x) \le D_0(x). \text{ -------------------------------------} \, (5).$$

Now, if $N_m \ne \varnothing$, write $N_m = \{a_1, a_2, \ldots, a_L\}$, where a_i are integers and $a_{i+1} - a_i = 1$, $i = 1, \ldots, L-1$. If $|Nm| = L \ge 1$, then, in general, we have the following three cases.

(A) There exists k such that $a_k \le \left(\frac{m+1/2}{x}\right)^2$ and $a_{k+1} > \left(\frac{m+1/2}{x}\right)^2$ or

(B) $a_1 \ge \left(\frac{m+1/2}{x}\right)^2$ or

(C) $a_L \le \left(\frac{m+1/2}{x}\right)^2$.

If $|Nm| = L \ge 2$, then we can have only cases (A) and (B). This is because if $a_L \le \left(\frac{m+1/2}{x}\right)^2$ and $L \ge 2$, then $\left(\frac{m+1/2}{x}\right)^2 - \left(\frac{m}{x}\right)^2 > 1$ and so $\left(\frac{m+1}{x}\right)^2 - \left(\frac{m+1/2}{x}\right)^2 > 1$ and it follows that $a_L + 1 < \left(\frac{m+1}{x}\right)^2$ contradicting that $a_L + 1 > \left(\frac{m+1}{x}\right)^2$.

The function $h(t) = \sin(\sqrt{t}\,x\pi)$ for t in $\left[\left(\frac{m}{x}\right)^2, \left(\frac{m+1}{x}\right)^2\right]$ is increasing on $\left[\left(\frac{m}{x}\right)^2, \left(\frac{m+1/2}{x}\right)^2\right]$ and decreasing on $\left[\left(\frac{m+1/2}{x}\right)^2, \left(\frac{m+1}{x}\right)^2\right]$ for m even and is decreasing on $\left[\left(\frac{m}{x}\right)^2, \left(\frac{m+1/2}{x}\right)^2\right]$ and increasing on $\left[\left(\frac{m+1/2}{x}\right)^2, \left(\frac{m+1}{x}\right)^2\right]$ for m odd. Hence $|h(t)|$ for t in $\left[\left(\frac{m}{x}\right)^2, \left(\frac{m+1}{x}\right)^2\right]$ is increasing on $\left[\left(\frac{m}{x}\right)^2, \left(\frac{m+1/2}{x}\right)^2\right]$ and decreasing on $\left[\left(\frac{m+1/2}{x}\right)^2, \left(\frac{m+1}{x}\right)^2\right]$. Then we have if $L \ge 2$, by using the monotonicity of $|h(t)|$ described above, for case (A),

$$\sum_{i=1}^{L}|h(a_i)| \le \int_{a_1}^{a_L}|h(t)|dt + \max(|h(a_k)|, |h(a_{k+1})|) \le \int_{a_1}^{a_L}|h(t)|dt + 1$$

and for case (B)

$$\sum_{i=1}^{L}|h(a_i)| \le \int_{a_1}^{a_L}|h(t)|dt + 1$$

and so
$$\sum_{i=1}^{L}|h(a_i)| \le \int_{m^2/x^2}^{(m+1)^2/x^2}|h(t)|dt + 1.$$

If $L = 1$, then plainly $|h(a_1)| \le 1 \le \int_{m^2/x^2}^{(m+1)^2/x^2}|h(t)|dt + 1.$

Thus, we have if $N_m \ne \varnothing$, $\sum_{i=1}^{L}|h(a_i)| \le \int_{m^2/x^2}^{(m+1)^2/x^2}|h(t)|dt + 1.$

Consequently, for $m \ge 0$,

$$|D_m(x)| \le \int_{m^2/x^2}^{(m+1)^2/x^2} |h(t)|dt + 1, \qquad \text{---------------------------- (6)}$$

since $D_m(x) = 0$ when $N_m = \varnothing$.

Now if $L \ge 2$, then we have for case (A),

$$\sum_{i=1}^{L} |h(a_i)| + 1 \ge \int_{m^2/x^2}^{(m+1)^2/x^2} |h(t)|dt$$

and for case (B), $\displaystyle\sum_{i=1}^{L} |h(a_i)| + \int_{m^2/x^2}^{a_1} |h(t)|dt \ge \int_{m^2/x^2}^{(m+1)^2/x^2} |h(t)|dt$

and since for this case $a_1 - \left(\frac{m}{x}\right)^2 \le 1$, $\int_{m^2/x^2}^{a_1} |h(t)|d \le 1$, we have again,

$$\sum_{i=1}^{L} |h(a_i)| + 1 \ge \int_{m^2/x^2}^{(m+1)^2/x^2} |h(t)|dt.$$

If $L = 1$, then $a_1 - \left(\frac{m}{x}\right)^2 \le 1$ and $\left(\frac{m+1}{x}\right)^2 - a_1 \le 1$ and so we have,

$$|h(a_1)| + 1 \ge \int_{m^2/x^2}^{(m+1)^2/x^2} |h(t)|dt.$$

Thus, for any $m \ge 0$ and, $N_m \ne \varnothing$,

$$|D_m(x)| \ge \int_{m^2/x^2}^{(m+1)^2/x^2} |h(t)|dt - 1. \qquad \text{--------------------------- (7)}$$

Note that $N_m = \varnothing$ implies that $\left(\frac{m+1}{x}\right)^2 - \left(\frac{m}{x}\right)^2 = \frac{2m+1}{x^2} < 1$ and so $\int_{m^2/x^2}^{(m+1)^2/x^2} |h(t)|dt \le 1$. It follows that $\int_{m^2/x^2}^{(m+1)^2/x^2} |h(t)|dt - 1 \le 0 = |D_m(x)|$.

Hence, (7) holds for any $m \ge 0$ without any condition. Therefore, combining (6) and (7) we have,

$$\int_{m^2/x^2}^{(m+1)^2/x^2} |h(t)|dt - 1 \le |D_m(x)| \le \int_{m^2/x^2}^{(m+1)^2/x^2} |h(t)|dt + 1. \qquad \text{------------ (8)}$$

Next, we shall evaluate the integral $\int_{m^2/x^2}^{(m+1)^2/x^2} |h(t)|dt$.

Now, $\displaystyle\int h(t)dt = \int \sin(\sqrt{t}\, x\pi)dt = \int 2\sin(u\pi)\frac{u}{x^2}du$, by change of variable $u = \sqrt{t}\, x$,

$$= -\frac{2}{x^2\pi}u\cos(u\pi) + \frac{2}{x^2\pi}\int \cos(u\pi)du \text{ , by integration by parts,}$$

$$= -\frac{2}{x^2\pi}u\cos(u\pi) + \frac{2}{x^2\pi^2}\sin(u\pi) + C.$$

Therefore,

$$\int_{m^2/x^2}^{(m+1)^2/x^2} h(t)dt = \left[-\frac{2}{x^2\pi}u\cos(u\pi) + \frac{2}{x^2\pi^2}\sin(u\pi)\right]_m^{m+1}$$

$$= \frac{2}{x^2\pi}\{-(m+1)(-1)^{m+1} + m(-1)^m\}$$

$$= \frac{2}{x^2\pi}(2m+1)(-1)^m. \qquad \text{----------------------------- (9)}$$

Thus, (8) and (9) give us

$$\frac{2}{x^2\pi}(2m+1) - 1 \le |D_m(x)| \le \frac{2}{x^2\pi}(2m+1) + 1, \;\; m \ge 0. \qquad \text{------- (10)}$$

Thus, it follows from (4) and (10) that for $m > 0$,

$$|S_m(x)| \le \left(\frac{x}{m}\right)^2 |D_m(x)| \le \frac{2}{m^2\pi}(2m+1) + \frac{x^2}{m^2} = \frac{4}{m\pi} + \frac{2}{m^2\pi} + \frac{x^2}{m^2} \,.$$

And for $m \ge 0$,

$$|S_m(x)| \ge \left(\frac{x}{m+1}\right)^2 |D_m(x)| \ge \frac{2}{(m+1)^2\pi}(2m+1) - \frac{x^2}{(m+1)^2}$$

$$\ge \frac{4}{(m+1)\pi} - \frac{2}{(m+1)^2\pi} - \frac{x^2}{(m+1)^2} \,.$$

Hence, for $m > 0$,

$$\frac{4}{(m+1)\pi}-\frac{2}{m^2\pi}-\frac{x^2}{m^2}\le\frac{4}{(m+1)\pi}-\frac{2}{(m+1)^2\pi}-\frac{x^2}{(m+1)^2}\le|S_m(x)|\le\frac{4}{m\pi}+\frac{2}{m^2\pi}+\frac{x^2}{m^2}, \qquad \text{----------------------- (11)}$$

and from (5) and (10),

$$\frac{2}{\pi}-x^2\le S_0(x)\le\frac{2}{x^2\pi}+1. \qquad \text{---------------------- (12)}$$

Thus, for m even and $m > 0$,

$$\frac{4}{(m+1)\pi}-\frac{2}{m^2\pi}-\frac{x^2}{m^2}\le S_m(x)\le\frac{4}{m\pi}+\frac{2}{m^2\pi}+\frac{x^2}{m^2}\ , \text{-------------- (13)}$$

and for m odd > 0,

$$-\frac{4}{m\pi}-\frac{2}{m^2\pi}-\frac{x^2}{m^2}\le S_m(x)\le-\frac{4}{(m+1)\pi}+\frac{2}{m^2\pi}+\frac{x^2}{m^2}\ .\text{--------- (14)}$$

Combining (13) and (14) we get, for $m > 0$,

$$c_m-\frac{2}{m^2\pi}-\frac{x^2}{m^2}\le S_m(x)\le d_m+\frac{2}{m^2\pi}+\frac{x^2}{m^2}\ , \quad \text{------------------------ (15)}$$

where $c_m=\begin{cases}\frac{4}{(m+1)\pi}, & m \text{ even}\\ -\frac{4}{m\pi}, & m \text{ odd}\end{cases}$ and $d_m=\begin{cases}\frac{4}{m\pi}, & m \text{ even}\\ -\frac{4}{(m+1)\pi}, & m \text{ odd}\end{cases}$.

Observe that both the series $\sum_{m=1}^{\infty} c_m$ and $\sum_{m=1}^{\infty} d_m$ are convergent series and $\sum_{m=1}^{\infty} c_m=-\frac{4}{\pi}$ while $\sum_{m=1}^{\infty} d_m=0$. Therefore, it follows from (15) and (12) that the series $\sum_{m=0}^{\infty} S_m(x)$ is a convergent series since $\frac{2}{\pi}-x^2+\sum_{m=1}^{\infty}\left\{c_m-\frac{2}{m^2\pi}-\frac{x^2}{m^2}\right\}$ and $\frac{2}{x^2\pi}+1+\sum_{m=1}^{\infty}\left\{d_m+\frac{2}{m^2\pi}+\frac{x^2}{m^2}\right\}$ are convergent series for each $x > 0$. We can now conclude that $S(x)$ converges for $x > 0$. This is a consequence of the general result: if the series formed by regrouping a given series into terms of the same sign is convergent, then the given series is convergent. We shall prove this special case as the general case can be proven in the same manner.

For a fixed $x > 0$, $\sum_{m=0}^{\infty} S_m(x)$ is convergent implies that given any $\varepsilon > 0$, there exists an integer $N > 0$ such that for all $m \ge n \ge N$,

$$\left|\sum_{k=n}^{k=m} S_k(x)\right|<\varepsilon/3.$$

The number of positive integers in $\left(0,\left(\frac{N}{x}\right)^2\right]$ is bounded above by $\left[\left(\frac{N}{x}\right)^2\right]+1$. if $a>\left[\left(\frac{N}{x}\right)^2\right]+1$, then $a\in N_n$ for some $n \ge N$. Thus if $b > a$, then $b\in N_m$ for some m such that $m \ge n$. Note that for fixed $x > 0$, each N_m depends on *this* fixed x. We shall now denote N_m by $N_m(x)$ to emphasize its dependence on x. Then

$$\begin{aligned}\left|\sum_{j=a}^{j=b}\frac{\sin(\sqrt{j}\,x\pi)}{j}\right| &= \left|\sum_{j\in N_n(x)\ ,\ j\ge a}\frac{\sin(\sqrt{j}\,x\pi)}{j}+S_{n+1}(x)+\cdots+S_m(x)-\sum_{j\in N_m(x)\ ,\ j>b}\frac{\sin(\sqrt{j}\,x\pi)}{j}\right|\\ &\le\left|\sum_{j\in N_m(x)\ ,\ j\ge a}\frac{\sin(\sqrt{j}\,x\pi)}{j}\right|+|S_{n+1}(x)+\cdots+S_m(x)|+\left|\sum_{j\in N_m(x)\ ,\ j>b}\frac{\sin(\sqrt{j}\,x\pi)}{j}\right|\\ &\le|S_n(x)|+|S_{n+1}(x)+\cdots+S_m(x)|+|S_m(x)|\\ &<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

Hence $S(x)$ is convergent.

Observe that the number of terms in each $S_m(x)$ depends on x. Indeed the number of terms tends to infinity as x tends to 0 on the right. For this reason it is reasonable to suggest that the original series $S(x)$ cannot converge uniformly on the interval $[0, K]$ for any $K > 0$. To prove this we shall need a better estimate of $S_m(x)$. We shall do this later. For now we shall show that $S(x)$ converges uniformly on the interval $[k, K]$ for any $k > 0$ and any $K > k$.

Note that $x \in [k, K]$ implies that $\frac{1}{x} \le \frac{1}{k}$. So from (12) we have for all x in $[k, K]$,

$$\frac{2}{\pi} - K^2 \le S_0(x) \le \frac{2}{k^2\pi} + 1, \quad \text{------------------------} \quad (*)$$

and for all $m > 0$, for all $x \in [k, K]$,

$$c_m - \frac{2}{m^2\pi} - \frac{K^2}{m^2} \le S_m(x) \le d_m + \frac{2}{m^2\pi} + \frac{K^2}{m^2}\,. \quad \text{--------------} \ (16)$$

Since the two series defined by the terms on the left and right of (16) and (*) are uniformly convergent, the series

$$\sum_{m=0}^{\infty} S_m(x)$$

is uniformly convergent on $[k, K]$. It remains to show that the original series $S(x)$ is uniformly convergent on $[k, K]$. The proof is similar to the proof for convergence. Since $\sum_{m=0}^{\infty} S_m(x)$ is uniformly convergent, given any $\varepsilon > 0$, there exists an integer $N > 0$ such that for all $m \ge n \ge N$ and for all x in $[k, K]$,

$$\left|\sum_{k=n}^{k=m} S_k(x)\right| < \varepsilon/3.$$

Now the number of positive integers in $\left(0, \left(\frac{m}{x}\right)^2\right]$ is bounded above by $\left(\frac{m}{k}\right)^2$. Therefore, if $a > \left[\left(\frac{N}{k}\right)^2\right] + 1$, then $a \in N_n(x)$ for some $n \ge N$ and for each x in $[k, K]$. Thus if $b > a$, then $b \in N_m(x)$ for some m such that $m \ge n$. Hence, for all x in $[k, K]$,

$$\begin{aligned}
\left|\sum_{j=a}^{j=b} \frac{\sin(\sqrt{j}\,x\pi)}{j}\right| &= \left|\sum_{j\in N_n(x)\,,\,j\ge a} \frac{\sin(\sqrt{j}\,x\pi)}{j} + S_{n+1}(x) + \cdots + S_m(x) - \sum_{j\in N_m(x)\,,\,j>b} \frac{\sin(\sqrt{j}\,x\pi)}{j}\right| \\
&\le \left|\sum_{j\in N_m(x)\,,\,j\ge a} \frac{\sin(\sqrt{j}\,x\pi)}{j}\right| + |S_{n+1}(x) + \cdots + S_m(x)| + \left|\sum_{j\in N_m(x)\,,\,j>b} \frac{\sin(\sqrt{j}\,x\pi)}{j}\right| \\
&\le |S_n(x)| + |S_{n+1}(x) + \cdots + S_m(x)| + |S_m(x)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This shows that $S(x)$ is uniformly convergent on $[k, K]$.

It follows that $S(x)$ is pointwise convergent on $(0, \infty)$. Since sine is an odd function, consequently, $S(x)$ is pointwise convergent on $(-\infty, 0)$. $S(x)$ is plainly convergent at 0 and so $S(x)$ is pointwise convergent on **R**.

Now we proceed to show that $S(x)$ is not uniformly convergent on $[0, K]$ any $K > 0$. For this purpose we shall use the function, $g(t) = \dfrac{\sin(\sqrt{t}\,x\pi)}{t}$ on each interval $\left[\left(\frac{m}{x}\right)^2, \left(\frac{m+1}{x}\right)^2\right], m = 0, 1, 2, \cdots$. Fix any $x > 0$. Note that $|g(t)|$ has precisely one local maximum (hence absolute maximum) in each interval except on $\left[0, \left(\frac{1}{x}\right)^2\right]$, where $g(t)$ is decreasing. $g(t)$ alternates in sign as $h(t)$.

We can deduce this as follows. The derivative of g, $g'(t) = \dfrac{\cos(\sqrt{t}\,x\pi)\sqrt{t}\,x\pi/2 - \sin(\sqrt{t}\,x\pi)}{t^2}$.

Thus $g'(t) = 0$ if and only if $\tan(\sqrt{t}\,x\pi) = \sqrt{t}\,x\pi/2$. But $\tan(\theta) = \theta/2$ has only one solution in each interval $[m\pi, (m+1)\pi]$ $m > 0$ and none in $[0, \pi]$. This is because if there were two solutions in $[m\pi, (m+1)\pi]$, then by the Mean Value Theorem there would be a point η in the interval such that $\tan'(\eta) = \sec^2(\eta) = 1/2$. This contradicts that $\sec^2(\eta) \geq 1$. This means $g(t)$ has only one stationary point which is the maximum on the interval $\left[\left(\frac{m}{x}\right)^2, \left(\frac{m+1}{x}\right)^2\right]$ when m is even and bigger than 0 and a minimum on the interval $\left[\left(\frac{m}{x}\right)^2, \left(\frac{m+1}{x}\right)^2\right]$ when m is odd. Thus, for $m > 0$, $|g(t)|$ increases from 0 to its maximum and decreases to zero after that on each interval $\left[\left(\frac{m}{x}\right)^2, \left(\frac{m+1}{x}\right)^2\right]$. Now the maximum for $|g(t)|$ in $\left[\left(\frac{m}{x}\right)^2, \left(\frac{m+1}{x}\right)^2\right]$ is less than $\left(\frac{x}{m}\right)^2$ for $m > 0$. Note also that $\lim_{t \to 0^+} g(t) = +\infty$.

With $\left(\frac{x}{m}\right)^2$ in place of 1 as in the case for $h(t)$, following the estimate for $D_m(x)$, we can deduce similarly that for $m \geq 1$,

$$\int_{m^2/x^2}^{(m+1)^2/x^2} |g(t)|dt - \left(\frac{x}{m}\right)^2 \leq |S_m(x)| \leq \int_{m^2/x^2}^{(m+1)^2/x^2} |g(t)|dt + \left(\frac{x}{m}\right)^2.$$

It follows that for $m \geq 1$,

$$\int_{m^2/x^2}^{(m+1)^2/x^2} g(t)dt - \left(\frac{x}{m}\right)^2 \leq S_m(x) \leq \int_{m^2/x^2}^{(m+1)^2/x^2} g(t)dt + \left(\frac{x}{m}\right)^2 . \text{ ---------- } (17)$$

For $m = 0$, $N_0 \neq \varnothing$, if and only if, $0 < x^2 \leq 1$ and so since g is decreasing on $(0, \frac{1}{x^2}]$,

$$S_0(x) \leq \int_0^{1/x^2} g(t)dt,$$

where the integral is a convergent improper integral. Also we have

$$\int_0^1 g(t)dt + S_0(x) \geq \int_0^{1/x^2} g(t)dt.$$

This means

$$\int_0^{1/x^2} g(t)dt - \int_0^1 g(t)dt \leq S_0(x) \leq \int_0^{1/x^2} g(t)dt. \text{ ---------------------- } (18)$$

Observe that (18) holds when $N_0 = \varnothing$, i.e., $S_0(x) = 0$ and $\frac{1}{x^2} < 1$. We shall prove the pointwise convergence of $S(x)$ using this estimate. Central to this approach is the convergence of the improper integral $\int_0^\infty g(t)dt$.

$$\int g(t)dt = \int \frac{\sin(\sqrt{t}\, x\pi)}{t}dt = 2\int \frac{\sin(u)}{u}du, \text{ by change of variable } u = \sqrt{t}\, x\pi.$$

Thus,

$$\int_0^\infty g(t)dt = \int_0^\infty \frac{\sin(\sqrt{t}\, x\pi)}{t}dt = 2\int_0^\infty \frac{\sin(u)}{u}du = 2 \cdot \frac{\pi}{2} = \pi, \text{ ----------- } (19)$$

assuming the convergence of the improper integral $\int_0^\infty \frac{\sin(u)}{u}du = \frac{\pi}{2}$. This can be established by complex variable technique. Note also that

$$\int_{m^2/x^2}^{(m+1)^2/x^2} g(t)dt = 2\int_{m\pi}^{(m+1)\pi} \frac{\sin(u)}{u}du \text{ ------------------------------- } (20)$$

and

$$\int_0^1 g(t)dt = \int_0^1 \frac{\sin(\sqrt{t}\, x\pi)}{t}dt = 2\int_0^{x\pi} \frac{\sin(u)}{u}du . \text{ -------------------- } (21)$$

Rewriting (17) using (20) we get for $m \geq 1$,

$$2\int_{m\pi}^{(m+1)\pi} \frac{\sin(u)}{u}du - \left(\frac{x}{m}\right)^2 \leq S_m(x) \leq 2\int_{m\pi}^{(m+1)\pi} \frac{\sin(u)}{u}du + \left(\frac{x}{m}\right)^2 \text{ ------- } (22)$$

and using (21),

$$2\int_0^\pi \frac{\sin(u)}{u}du - 2\int_0^{x\pi} \frac{\sin(u)}{u}du \leq S_0(x) \leq 2\int_0^\pi \frac{\sin(u)}{u}du. \text{ -------------- } (23)$$

Hence, because the series $\sum_{m=0}^{\infty} \int_{m\pi}^{(m+1)\pi} \frac{\sin(u)}{u}$ is convergent as the improper integral $\int_0^\infty \frac{\sin(u)}{u}du$ is convergent and $\sum_{m=1}^{\infty} \left(\frac{x}{m}\right)^2$ is also convergent, the two series defined by the terms on the left and right of (23) and (22) are convergent. Therefore, $\sum_{m=0}^{\infty} S_m(x)$ is convergent for each $x > 0$. And we can now conclude as before that $S(x)$ is convergent for each $x > 0$ and $S(x) = \sum_{m=0}^{\infty} S_m(x)$. Now we shall show that the convergence of $S(x)$ on $[0, K]$ for any $K > 0$ is not uniform. If $S(x)$ converges uniformly on $[0, K]$, then $S(x)$ is continuous on $[0, K]$ and so $\lim_{x\to 0^+} S(x) = S(0)$.

But from (22) and (23) for $x > 0$,

$$2\int_0^\infty \frac{\sin(u)}{u}du - x^2\sum_{m=1}^{\infty}\frac{1}{m^2} - 2\int_0^{x\pi}\frac{\sin(u)}{u}du \le S(x) \le 2\int_0^\infty \frac{\sin(u)}{u}du + x^2\sum_{m=1}^{\infty}\frac{1}{m^2},$$

so that $\pi - x^2\frac{\pi^2}{6} - 2\int_0^{x\pi}\frac{\sin(u)}{u}du \le S(x) \le \pi + x^2\frac{\pi^2}{6}$ because $\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^2}{6}$.

Since $\lim_{x\to 0^+}\int_0^{x\pi}\frac{\sin(u)}{u}du = 0$, by the Squeeze Theorem,

$$\lim_{x\to 0^+} S(x) = \pi \neq S(0) = 0.$$

This contradiction shows that the convergence is not uniform on $[0, K]$.

Problem 2. For $\frac{1}{2} < \beta < 1$, the series $T(x) = \sum_{n=1}^{\infty}\frac{\sin(\sqrt{n}\,x\pi)}{n^\beta}$ converges pointwise on **R,** converges uniformly on $[k, K]$ for any $k > 0$ and any $K > k$ but not uniformly on $[0, K]$, any $K > 0$.

Plainly, $T(0)$ is convergent and $T(0) = 0$. Fix any $x > 0$. For integer $m \ge 0$, let

$$T_m(x) = \begin{cases} \sum_{n\in N_m}\frac{\sin(\sqrt{n}\,x\pi)}{n^\beta} \text{ if } N_m \neq \varnothing, \\ 0 \text{ if } N_m = \varnothing. \end{cases} \quad \text{-------------------- (24)}$$

Following problem 1, we can show that for $m \ge 1$,

$$\left(\frac{x}{m+1}\right)^{2\beta}|D_m(x)| \le |T_m(x)| \le \left(\frac{x}{m}\right)^{2\beta}|D_m(x)|, \quad \text{-------------------- (25)}$$

and

$$x^{2\beta}D_0(x) \le T_0(x) \le D_0(x).$$

Then from (10) and (25) we get,

$$\left(\frac{x}{m+1}\right)^{2\beta}\left\{\frac{2}{\pi x^2}(2m+1) - 1\right\} \le |T_m(x)| \le \left(\frac{x}{m}\right)^{2\beta}\left\{\frac{2}{\pi x^2}(2m+1)+1\right\},\ m \ge 1 \text{ --- (26)}$$

and

$$\frac{2}{\pi x^{2-2\beta}} - x^{2\beta} \le T_0(x) \le \frac{2}{\pi x^2} + 1. \quad \text{---------------------- (27)}$$

Subsequently, for $m \ge 1$,

$$|T_m(x)| \le \frac{4}{\pi x^{2-2\beta}}\frac{1}{m^{2\beta-1}} + \frac{2}{\pi x^{2-2\beta}m^{2\beta}} + \left(\frac{x}{m}\right)^{2\beta},$$

$$|T_m(x)| \ge \frac{4}{\pi x^{2-2\beta}}\frac{1}{(m+1)^{2\beta-1}} - \frac{2}{\pi x^{2-2\beta}(m+1)^{2\beta}} - \left(\frac{x}{m+1}\right)^{2\beta}$$

$$\ge \frac{4}{\pi x^{2-2\beta}}\frac{1}{(m+1)^{2\beta-1}} - \frac{2}{\pi x^{2-2\beta}m^{2\beta}} - \left(\frac{x}{m}\right)^{2\beta}.$$

Thus, we have for m even and $m > 0$,

$$T_m(x) \le \frac{4}{\pi x^{2-2\beta}}\frac{1}{m^{2\beta-1}} + \frac{2}{\pi x^{2-2\beta}m^{2\beta}} + \left(\frac{x}{m}\right)^{2\beta}$$

and

$$T_m(x) \ge \frac{4}{\pi x^{2-2\beta}}\frac{1}{(m+1)^{2\beta-1}} - \frac{2}{\pi x^{2-2\beta}m^{2\beta}} - \left(\frac{x}{m}\right)^{2\beta}. \quad \text{--------- (28).}$$

For odd $m \ge 1$,

$$T_m(x) \le -\frac{4}{\pi x^{2-2\beta}}\frac{1}{(m+1)^{2\beta-1}} + \frac{2}{\pi x^{2-2\beta}m^{2\beta}} + \left(\frac{x}{m}\right)^{2\beta}$$

and

$$T_m(x) \ge -\frac{4}{\pi x^{2-2\beta}}\frac{1}{m^{2\beta-1}} - \frac{2}{\pi x^{2-2\beta}m^{2\beta}} - \left(\frac{x}{m}\right)^{2\beta}. \quad \text{------------- (29)}$$

Let

$$c_m(x) = \begin{cases} \frac{4}{\pi x^{2-2\beta}}\frac{1}{(m+1)^{2\beta-1}},\ m \text{ even and } m > 0 \\ -\frac{4}{\pi x^{2-2\beta}}\frac{1}{m^{2\beta-1}},\ m \text{ odd} \end{cases},$$

$$d_m(x) = \begin{cases} \dfrac{4}{\pi x^{2-2\beta}} \dfrac{1}{m^{2\beta-1}}, \ m \text{ even and } m > 0 \\ -\dfrac{4}{\pi x^{2-2\beta}} \dfrac{1}{(m+1)^{2\beta-1}}, \ m \text{ odd} \end{cases},$$

$$c_0(x) = \frac{2}{\pi x^{2-2\beta}} - x^{2\beta},$$

and

$$d_0(x) = \frac{2}{\pi x^2} + 1.$$

Then from (28) and (29) we have,

$$c_m(x) - \frac{2}{\pi x^{2-2\beta} m^{2\beta}} - \left(\frac{x}{m}\right)^{2\beta} \le T_m(x) \le d_m(x) + \frac{2}{\pi x^{2-2\beta} m^{2\beta}} + \left(\frac{x}{m}\right)^{2\beta}, \ m \ge 1, \ \text{------ (30)}$$

$$c_0(x) \le T_0(x) \le d_0(x).$$

Now $\sum_{m=0}^{\infty} c_m(x)$ and $\sum_{m=0}^{\infty} d_m(x)$ are convergent with $\sum_{m=0}^{\infty} c_m(x) = -\frac{2}{\pi x^{2-2\beta}} - x^{2\beta}$ and $\sum_{m=0}^{\infty} d_m(x) = \frac{2}{\pi x^2} + 1$. Since 2β > 1, $\sum_{m=1}^{\infty} \left\{ \frac{2}{\pi x^{2-2\beta} m^{2\beta}} + \left(\frac{x}{m}\right)^{2\beta} \right\}$ is convergent being a constant times a convergent *p*-series. Therefore, we conclude from (30) that $\sum_{m=0}^{\infty} T_m(x)$ is convergent.

We can now conclude that for $x > 0$, $T(x)$ is convergent and $T(x) = \sum_{m=0}^{\infty} T_m(x)$.

Now, for x in $[k, K]$, $K > k > 0$, we deduce from (30) that, for $m \ge 1$,

$$c_m(x) - \frac{2}{\pi k^{2-2\beta} m^{2\beta}} - \left(\frac{K}{m}\right)^{2\beta} \le T_m(x) \le d_m(x) + \frac{2}{\pi k^{2-2\beta} m^{2\beta}} + \left(\frac{K}{m}\right)^{2\beta} \ \text{------- (31)}$$

and

$$\frac{2}{\pi K^{2-2\beta}} - K^{2\beta} \le T_0(x) \le \frac{2}{\pi k^2} + 1.$$

We now claim that $\sum_{m=0}^{\infty} c_m(x)$ converges uniformly on $[k, K]$. This is seen as follows.

The partial sum $\sum_{k=0}^{n} c_k(x) = \begin{cases} c_0(x) + c_1(x) \text{ if } n \text{ is odd} \\ c_0(x) + c_1(x) + c_n(x) \text{ if n is even} \end{cases}$. Hence, for $m > n$, and for all x in $[k, K]$.

$$\left| \sum_{k=n+1}^{m} c_k(x) \right| \le |c_m(x)| + |c_n(x)| \le \frac{4}{\pi x^{2-2\beta}} \frac{1}{m^{2\beta-1}} + \frac{4}{\pi x^{2-2\beta}} \frac{1}{n^{2\beta-1}} \le \frac{8}{\pi k^{2-2\beta}} \frac{1}{n^{2\beta-1}}.$$

Since $\lim_{n\to\infty} \frac{8}{\pi k^{2-2\beta}} \frac{1}{n^{2\beta-1}} = 0$, given any ε > 0, there is an integer N such that for all $n \ge N$, $\frac{8}{\pi k^{2-2\beta}} \frac{1}{n^{2\beta-1}} < \varepsilon$. Therefore, for all $m, n \ge N$ and $m > n$,

$$\left| \sum_{k=n+1}^{m} c_k(x) \right| \le \frac{8}{\pi k^{2-2\beta}} \frac{1}{n^{2\beta-1}} < \varepsilon \quad \text{for all } x \text{ in } [k, K].$$

This means that $\sum_{k=0}^{n} c_k(x)$ is uniformly convergent.

The partial sum $\sum_{k=0}^{n} d_k(x) = \begin{cases} d_o(x) \text{ if } n \text{ iseven} \\ d_0(x) + d_n(x) \text{ if } n \text{ is odd} \end{cases}$. It follows that for $m > n$, and for all x in $[k, K]$,

$\left| \sum_{k=n+1}^{m} d_k(x) \right| \le |d_m(x)| + |d_n(x)| \le \frac{4}{\pi x^{2-2\beta}} \frac{1}{m^{2\beta-1}} + \frac{4}{\pi x^{2-2\beta}} \frac{1}{n^{2\beta-1}} \le \frac{8}{\pi k^{2-2\beta}} \frac{1}{n^{2\beta-1}}$. Therefore, as in the case of the previous series, for all $m, n \ge N$ and $m > n$,

$$\left| \sum_{k=n+1}^{m} d_k(x) \right| \le \frac{8}{\pi k^{2-2\beta}} \frac{1}{n^{2\beta-1}} < \varepsilon \quad \text{for all } x \text{ in } [k, K].$$

This means $\sum_{k=0}^{\infty} d_k(x)$ is uniformly convergent on $[k, K]$. This implies that the two series defined by the terms on the left and right of (31) are uniformly convergent on $[k, K]$. It follows that $\sum_{m=0}^{\infty} T_m(x)$ is uniformly convergent on $[k, K]$. We deduce as in problem 1 that $T(x)$ is uniformly convergent on $[k, K]$. This implies that $T(x)$ is pointwise convergent on $(0, \infty)$ and so on $(-\infty, 0)$. Since it is convergent at 0, $T(x)$ is pointwise convergent on $\mathbf{R}$.

We next show that $T(x)$ is not uniformly convergent on $[0, K]$.
From (26) we obtain for m even for any $x > 0$ in $[0, K]$,

$$\begin{aligned} T_m(x) &\geq \frac{1}{(m+1)^{2\beta}}\frac{2}{\pi x^{2-2\beta}}(2m+1) - \left(\frac{x}{m+1}\right)^{2\beta} \\ &\geq \frac{1}{(m+1)^{2\beta}}\frac{2}{\pi x^{2-2\beta}}\left\{(2m+1) - \frac{\pi x^2}{2}\right\} \\ &\geq \frac{1}{(m+1)^{2\beta}}\frac{2}{\pi x^{2-2\beta}}\left\{(2m+1) - \frac{\pi K^2}{2}\right\}. \quad \text{---------------------- (32)} \end{aligned}$$

For any integer $N > 0$, choose an even integer $2n > N$ so that $4n > \frac{\pi K^2}{2}$. Then choose $y > 0$ so small such that $0 < y^{2-2\beta} < \frac{1}{(2n+1)^{2\beta}}\frac{1}{\pi}$. Then by using (32), we have

$$\begin{aligned} T_{2n}(y) &\geq \frac{1}{(2n+1)^{2\beta}}\frac{2}{\pi y^{2-2\beta}}\left\{(4n+1) - \frac{\pi K^2}{2}\right\} \\ &\geq \frac{1}{(2n+1)^{2\beta}}\frac{2}{\pi y^{2-2\beta}} \\ &\geq 2. \end{aligned}$$

Hence $\sum_{m=0}^{\infty} T_m(x)$ does not converge uniformly on $[0, K]$. Consequently, $T(x)$ cannot converge uniformly on $[0, K]$.

Problem 3. The series $T(x) = \sum_{n=1}^{\infty} \frac{\sin(\sqrt{n}\, x\pi)}{\sqrt{n}}$ diverges for all $x \neq 0$.

Suppose $x > 0$. With the notation as in Problem 1, note that for $N_m \neq \varnothing$, $n \in N_m$ implies that $\left(\frac{m}{x}\right)^2 < n \leq \left(\frac{m+1}{x}\right)^2$ so that for $m > 0$,

$$\frac{x}{m+1} \leq \frac{1}{\sqrt{n}} < \frac{x}{m}\,. \quad \text{------------------------------------ (33)}$$

Let $T_m(x) = \begin{cases} \sum_{n \in N_m} \frac{\sin(\sqrt{n}\, x\pi)}{\sqrt{n}} \text{ if } N_m \neq \varnothing, \\ 0 \text{ if } N_m = \varnothing. \end{cases}$. Then the partial sums of the series $\sum_{m=0}^{\infty} T_m(x)$ is a subsequence of the partial sums of $T(x)$. If $T(x)$ is convergent, then $\sum_{m=0}^{\infty} T_m(x)$ is also convergent.
It follows from (33) that

$$\begin{aligned} |T_m(x)| &\geq \frac{x}{m+1}|D_m(x)| \\ &\geq \frac{x}{(m+1)}\left\{\frac{2}{x^2\pi}(2m+1) - 1\right\} \text{ by inequality (8).} \end{aligned}$$

But $\lim_{m\to\infty} \frac{x}{(m+1)}\left\{\frac{2}{x^2\pi}(2m+1) - 1\right\} = \frac{4}{\pi x} \neq 0$. Thus, $|T_m(x)|$ does not converge to 0. It follows that $\sum_{m=0}^{\infty} T_m(x)$ is divergent for $x > 0$. This implies that $T(x)$ is divergent for $x > 0$. Since sine is an odd function, for $x < 0$, $T(x) = -T(-x)$ and so $T(x)$ is divergent for $x < 0$. In conclusion, $T(x)$ is divergent for all $x \neq 0$.

Problem 4. The series $T(x)=\sum_{n=1}^{\infty}\dfrac{\cos(\sqrt{n}\,x\frac{\pi}{2})}{\sqrt{n}}$ diverges for all x.

We can use the same technique as in Problem 3 to show this. Obviously, $T(0)=\sum_{n=1}^{\infty}\dfrac{1}{\sqrt{n}}$ is divergent.

Suppose $x>0$.

For each integer $m\ge 1$, let $N_m=\{n : n$ is an integer and $\left(\frac{2m-1}{x}\right)^2<n\le\left(\frac{2m+1}{x}\right)^2\}$.

For $m=0$, let $N_0=\{n : n$ is an integer and $0<n\le\left(\frac{1}{x}\right)^2\}$.

Then N_m may be empty. If $N_m=\varnothing$ and $m\ge 1$, then $\left(\frac{2m+1}{x}\right)^2-\left(\frac{2m-1}{x}\right)^2=\frac{8m}{x^2}<1$.

If $N_m\ne\varnothing$ and $m\ge 1$, then for each $n\in N_m$, $2m-1<\sqrt{n}\,x\le 2m+1$ and so

$$\frac{x}{2m+1}<\frac{1}{\sqrt{n}}\le\frac{x}{2m-1} \text{ ------------------------ (34)}$$

and $(2m-1)\frac{\pi}{2}<\sqrt{n}\,x\frac{\pi}{2}\le(2m+1)\frac{\pi}{2}$. Note that $N_0\ne\varnothing$, if and only if, $\left(\frac{1}{x}\right)^2\ge 1$. When $N_0\ne\varnothing$, $n\in N_0$ implies that

$$x<\frac{1}{\sqrt{n}}\le 1. \text{ ---------------------------------- (35)}$$

This means that $\cos(\sqrt{n}\,x\frac{\pi}{2})$ is of the same sign or 0 for all n in N_m. More precisely, for m even, $\cos(\sqrt{n}\,x\frac{\pi}{2})\ge 0$ for all n in N_m and for m odd, $\cos(\sqrt{n}\,x\frac{\pi}{2})\le 0$ for all n in N_m. We partition $[0,\infty)$ into non-overlapping intervals

$$\left\{[0,\tfrac{1}{x^2}]\right\}\cup\left\{\left[\left(\tfrac{2m-1}{x}\right)^2,\left(\tfrac{2m+1}{x}\right)^2\right], m=1,2,\cdots\right\},$$

where for t in each interval $\left[\left(\frac{2m-1}{x}\right)^2,\left(\frac{2m+1}{x}\right)^2\right]$, $h(t)=\cos(\sqrt{t}\,x\frac{\pi}{2})$ is of the same sign in the interior and 0 at the end points. Note that the maximum of $|h(t)|$ on $\left[\left(\frac{2m-1}{x}\right)^2,\left(\frac{2m+1}{x}\right)^2\right]$ occurs at $\left(\frac{2m}{x}\right)^2$ and is equal to 1. For integer $m\ge 0$, define

$$T_m(x)=\begin{cases}\displaystyle\sum_{n\in N_m}\frac{\cos(\sqrt{n}\,x\frac{\pi}{2})}{\sqrt{n}} \text{ if } N_m\ne\varnothing,\\ 0 \text{ if } N_m=\varnothing.\end{cases},$$

and

$$E_m(x)=\begin{cases}\displaystyle\sum_{n\in N_m}\cos(\sqrt{n}\,x\tfrac{\pi}{2}) \text{ if } N_m\ne\varnothing,\\ 0 \text{ if } N_m=\varnothing.\end{cases}.$$

We deduce from (34) and (35), that for $m\ge 0$,

$$|T_m(x)|\ge\frac{x}{2m+1}|E_m(x)|. \text{ ---------------------- (36)}$$

So now we shall estimate $|E_m(x)|$ via the use of the function $h(t)$. We can estimate $h(t)$ in exactly the same manner as $h(t)$ in Problem 1. We obtain similar to (7) in Problem 1, for $m>0$,

$$|E_m(x)|\ge\int_{(2m-1)^2/x^2}^{(2m+1)^2/x^2}|h(t)|dt-1, \text{ -------------------------- (37)}$$

and

$$|E_0(x)|\ge\int_0^{1/x^2}|h(t)|dt-1.$$

Now

$$\int h(t)dt=\int\cos(\sqrt{t}\,x\tfrac{\pi}{2})dt=\int 2\cos(u\tfrac{\pi}{2})\frac{u}{x^2}du, \text{ by change of variable } u=\sqrt{t}\,x,$$

$$=\frac{4}{x^2\pi}u\sin(u\tfrac{\pi}{2})-\frac{4}{x^2\pi}\int\sin(u\tfrac{\pi}{2})du, \text{ by integration by parts,}$$

$$= \frac{4}{x^2\pi} u \sin(u\frac{\pi}{2}) + \frac{8}{x^2\pi^2} \cos(u\frac{\pi}{2}) + C.$$

Therefore, for $m \geq 1$,

$$\int_{(2m-1)^2/x^2}^{(2m+1)^2/x^2} h(t)dt = \left[\frac{4}{x^2\pi} u \sin(u\frac{\pi}{2}) + \frac{8}{x^2\pi^2} \cos(u\frac{\pi}{2}) \right]_{2m-1}^{2m+1}$$
$$= \frac{4}{x^2\pi} \{(2m+1)(-1)^m + (2m-1)(-1)^m\}$$
$$= \frac{16m}{x^2\pi}(-1)^m.$$

Thus, for $m \geq 1$,

$$\int_{(2m-1)^2/x^2}^{(2m+1)^2/x^2} |h(t)|dt = \frac{16m}{x^2\pi}. \qquad \text{----------------------} \quad (38)$$

Combining (37) and (38) gives us that for $m \geq 1$,

$$|E_m(x)| \geq \frac{16m}{x^2\pi} - 1.$$

It follows then from (36) that for $m > 0$,

$$|T_m(x)| \geq \frac{x}{2m+1} |E_m(x)| \geq \frac{x}{2m+1} \left\{ \frac{16m}{x^2\pi} - 1 \right\}.$$

But $\lim_{m \to \infty} \frac{x}{2m+1} \left\{ \frac{16m}{x^2\pi} - 1 \right\} = \frac{8}{\pi x} \neq 0$. Thus $|T_m(x)|$ does not converge to 0. It follows that $\sum_{m=0}^{\infty} T_m(x)$ is divergent for $x > 0$. This implies that $T(x)$ is divergent for $x > 0$. Since cosine is an even function, for $x < 0$, $T(x) = T(-x)$ and so $T(x)$ is divergent for $x < 0$. In conclusion, $T(x)$ is divergent for all x in **R**. Consequently, $\sum_{n=1}^{\infty} \frac{\cos(\sqrt{n}\,x)}{\sqrt{n}}$ diverges for all x.

Problem 5. For $p > 1$, the series $\sum_{n=1}^{\infty} \sin\left(\left(\frac{x}{n}\right)^p\right)$ converges on $[0, \infty)$ but not uniformly.

For $x \geq 0$, $\left|\sin\left(\left(\frac{x}{n}\right)^p\right)\right| \leq \left(\frac{x}{n}\right)^p = x^p \cdot \frac{1}{n^p}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$, by the Weierstrass M Test, the series $\sum_{n=1}^{\infty} \sin\left(\left(\frac{x}{n}\right)^p\right)$ converges absolutely and uniformly on $[0, K]$ for any $K > 0$. Hence it is pointwise convergence on $[0, \infty)$.

But for $x \geq 0$, $\sin\left(\left(\frac{x}{n}\right)^p\right) \geq \left(\frac{x}{n}\right)^p - \frac{1}{6}\left(\frac{x}{n}\right)^{3p}$. For each integer $N > 1$, let $a = N$. Then

$$\sin\left(\left(\frac{a}{N}\right)^p\right) \geq 1 - \frac{1}{6} = \frac{5}{6}.$$

Consequently,

$$\sum_{n=N}^{\infty} \sin\left(\left(\frac{a}{n}\right)^p\right) \geq \frac{5}{6}.$$

This means that there does not exist an integer N such that for any $n \geq N$,

$$\sum_{k=N}^{n} \sin\left(\left(\frac{x}{k}\right)^p\right) < \frac{5}{6} \text{ for all } x \text{ in } [0, \infty).$$

This means $\sum_{n=1}^{\infty} \sin\left(\left(\frac{x}{n}\right)^p\right)$ does not converge uniformly on $[0, \infty)$.

Problem 6. The series $T(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin(\frac{x}{\sqrt{n}})$ converges for all x in **R** but not uniformly on **R**.

It is easy to deduce the convergence of this series on a closed and bounded interval by using Weierstrass M Test.

Take any $K > 0$. Observe that for any x in $[-K, K]$,

$$\left|\frac{1}{n}\sin(\frac{x}{\sqrt{n}})\right| \le \frac{|x|}{n\sqrt{n}} \le \frac{K}{n\sqrt{n}}.$$

Since $\sum_{n=1}^{\infty}\frac{K}{n\sqrt{n}}$ is convergent, by the Weierstrass M Test, $\sum_{n=1}^{\infty}\frac{1}{n}\sin(\frac{x}{\sqrt{n}})$ is uniformly convergent on $[-K, K]$. For any x in **R**, there exists a $K > 0$ such that $x \in [-K, K]$ and so by what we have just proved, $T(x)$ is convergent. This means $T(x)$ is pointwise convergent on **R**. Now we show that the convergent is not uniform on **R**.

We shall use the inequality $\sin(x) \ge x - \frac{x^3}{6}$ for $x \ge 0$.
Take any integer $N > 1$. Let $x_N = \sqrt{N}$. Then

$$\sin\left(\frac{x_N}{\sqrt{n}}\right) \ge \frac{x_N}{\sqrt{n}} - \frac{1}{6}\left(\frac{x_N}{\sqrt{n}}\right)^3 = \frac{x_N}{\sqrt{n}} - \frac{1}{6}\frac{x_N^3}{n\sqrt{n}} = \frac{\sqrt{N}}{\sqrt{n}} - \frac{1}{6}\frac{N\sqrt{N}}{n\sqrt{n}} \text{ . ------ (39)}$$

Hence,

$$\frac{1}{n}\sin\left(\frac{x_N}{\sqrt{n}}\right) \ge \frac{\sqrt{N}}{n\sqrt{n}} - \frac{1}{6}\cdot\frac{N\sqrt{N}}{n^2\sqrt{n}} \text{ . -------------------- (40)}$$

Therefore,

$$\sum_{n=N}^{2N}\frac{1}{n}\sin\left(\frac{x_N}{\sqrt{n}}\right) \ge \sum_{n=N}^{2N}\frac{\sqrt{N}}{n\sqrt{n}} - \frac{1}{6}\sum_{n=N}^{2N}\frac{N\sqrt{N}}{n^2\sqrt{n}}. \text{ ------------------- (41)}$$

Observe that

$$\sum_{n=N}^{2N}\frac{\sqrt{N}}{n\sqrt{n}} \ge \sum_{n=N}^{2N}\frac{\sqrt{N}}{2N\sqrt{2N}} = \frac{1}{2\sqrt{2}}\sum_{n=N}^{2N}\frac{1}{N} \ge \frac{1}{2\sqrt{2}}\cdot\frac{N+1}{N} = \frac{1}{2\sqrt{2}}\left(1+\frac{1}{N}\right) \text{ ---- (42)}$$

and

$$\frac{1}{6}\sum_{n=N}^{2N}\frac{N\sqrt{N}}{n^2\sqrt{n}} \le \frac{1}{6}\sum_{n=N}^{2N}\frac{N\sqrt{N}}{N^2\sqrt{N}} = \frac{1}{6}\sum_{n=N}^{2N}\frac{1}{N} = \frac{1}{6}\left(1+\frac{1}{N}\right). \text{ -------- (43)}$$

It follows from (42) and (43) that (41) becomes

$$\begin{aligned}\sum_{n=N}^{2N}\frac{1}{n}\sin\left(\frac{x_N}{\sqrt{n}}\right) &\ge \frac{1}{2\sqrt{2}}\left(1+\frac{1}{N}\right) - \frac{1}{6}\left(1+\frac{1}{N}\right) = \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{3}\right)\left(1+\frac{1}{N}\right)\\ &\ge \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{3}\right) > 0 .\end{aligned}$$

This means that for any $N > 1$, $\sum_{n=N}^{\infty}\frac{1}{n}\sin\left(\frac{x_N}{\sqrt{n}}\right) \ge \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{3}\right)$. Hence

$$M_N = \sup\left\{\sum_{n=N}^{\infty}\frac{1}{n}\sin\left(\frac{x}{\sqrt{n}}\right) : x \in \mathbf{R}\right\} \ge \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{3}\right)$$

Therefore, M_N does not tend to 0 as N tends to infinity. Consequently $T(x)$ cannot converge uniformly on **R**.

Problem 7. The series $S(x) = \sum_{n=1}^{\infty}\frac{\sin(n^{1/3}x\pi)}{n}$ converges pointwise on **R**. $S(x)$ converges uniformly on $[k, K]$ for any $k > 0$ and any $K > k$. It does not converge uniformly on $[0, K]$, any $K > 0$.

This can be proven in the same manner as Problem 1.

Problem 8 The series $T(x) = \sum_{n=1}^{\infty} \frac{\cos(\sqrt{n}\, x\frac{\pi}{2})}{n}$ converges pointwise on $\mathbf{R} - \{\mathbf{0}\}$. $T(x)$ converges uniformly on $[k, K]$ for any $k > 0$ and any $K > k$.

The technique to solve this problem is the same as Problem 1 to group all the terms with the same sign and show that the series formed by the grouping of terms of the same sign is convergent.

Firstly, we note that $T(0)$ is obviously divergent as $T(0) = \sum_{n=1}^{\infty} \frac{1}{n}$, which is divergent. Now fix any $x > 0$.

For each integer $m \geq 1$, let $N_m = \{n : n$ is an integer and $\left(\frac{2m-1}{x}\right)^2 < n \leq \left(\frac{2m+1}{x}\right)^2\}$.

For $m = 0$, let $N_0 = \{n : n$ is an integer and $0 < n \leq \left(\frac{1}{x}\right)^2\}$.

Then N_m may be empty. If $N_m = \varnothing$ and $m \geq 1$, then $\left(\frac{2m+1}{x}\right)^2 - \left(\frac{2m-1}{x}\right)^2 = \frac{8m}{x^2} < 1$.

If $N_0 = \varnothing$, then $\left(\frac{1}{x}\right)^2 < 1$, i.e., $0 < \frac{1}{x} < 1$. Conversely, $0 < \frac{1}{x} < 1$ implies that $N_0 = \varnothing$.

If $N_m \neq \varnothing$ and $m \geq 1$ then for each $n \in N_m$, $2m - 1 < \sqrt{n}\, x \leq 2m + 1$ and so

$$\frac{x^2}{(2m+1)^2} < \frac{1}{n} \leq \frac{x^2}{(2m-1)^2} \text{ ------------------------- } (1)$$

and $(2m-1)\frac{\pi}{2} < \sqrt{n}\, x\frac{\pi}{2} \leq (2m+1)\frac{\pi}{2}$. Note that $N_0 \neq \varnothing$ if and only if $\left(\frac{1}{x}\right)^2 \geq 1$. When $N_0 \neq \varnothing$, $n \in N_0$ implies that

$$x^2 \leq \frac{1}{n} \leq 1. \text{ ---------------------------------- } (2)$$

This means that $\cos(\sqrt{n}\, x\frac{\pi}{2})$ is of the same sign or 0 for all n in N_m. More precisely, for m even, $\cos(\sqrt{n}\, x\frac{\pi}{2}) \geq 0$ for all n in N_m and for m odd, $\cos(\sqrt{n}\, x\frac{\pi}{2}) \leq 0$ for all n in N_m. We partition $[0, \infty)$ into non-overlapping intervals

$$\left\{[0, \tfrac{1}{x^2}]\right\} \cup \left\{\left[\left(\tfrac{2m-1}{x}\right)^2, \left(\tfrac{2m+1}{x}\right)^2\right], m = 1, 2, \cdots\right\},$$

where for t in each interval $\left[\left(\frac{2m-1}{x}\right)^2, \left(\frac{2m+1}{x}\right)^2\right]$, $h(t) = \cos(\sqrt{t}\, x\frac{\pi}{2})$ is of the same sign in the interior and 0 at the end points. Note that the maximum of $|h(t)|$ on $\left[\left(\frac{2m-1}{x}\right)^2, \left(\frac{2m+1}{x}\right)^2\right]$ occurs at $\left(\frac{2m}{x}\right)^2$ and is equal to 1. For integer $m \geq 0$, define

$$T_m(x) = \begin{cases} \sum_{n \in N_m} \frac{\cos(\sqrt{n}\, x\frac{\pi}{2})}{n} \text{ if } N_m \neq \varnothing, \\ 0 \text{ if } N_m = \varnothing. \end{cases},$$

and

$$E_m(x) = \begin{cases} \sum_{n \in N_m} \cos(\sqrt{n}\, x\frac{\pi}{2}) \text{ if } N_m \neq \varnothing, \\ 0 \text{ if } N_m = \varnothing. \end{cases}.$$

We deduce from (1) that for $m > 0$,

$$\frac{x^2}{(2m-1)^2}|E_m(x)| \geq |T_m(x)| \geq \frac{x^2}{(2m+1)^2}|E_m(x)|, \text{ ----------------- } (3)$$

and from (2) we have

$$E_0(x) \geq |T_0(x)| \geq x^2 E_0(x). \text{ ---------------------- } (4)$$

So now we shall estimate $|E_m(x)|$ via the use of the function $h(t)$. We can estimate $h(t)$ in exactly the same manner as $h(t)$ in Problem 1. Similar to the case in Problem 1, for $m > 0$ we have the following deduction.

Assume $m > 0$.
If $N_m \neq \varnothing$, write $N_m = \{a_1, a_2, \ldots, a_L\}$, where a_i are integers and $a_{i+1} - a_i = 1$, $i = 1, \ldots, L-1$. If $|Nm| = L \geq 1$, then, in general, we have the following three cases.

(A) There exists k such that $a_k \leq \left(\frac{2m}{x}\right)^2$ and $a_{k+1} > \left(\frac{2m}{x}\right)^2$ or

(B) $a_1 \geq \left(\frac{2m}{x}\right)^2$ or

(C) $a_L \leq \left(\frac{2m}{x}\right)^2$.

If $|Nm| = L \geq 2$, then we can have only cases (A) and (B). This is because if $a_L \leq \left(\frac{2m}{x}\right)^2$ and $L \geq 2$, then $\left(\frac{2m}{x}\right)^2 - \left(\frac{2m-1}{x}\right)^2 > 1$ and so $\left(\frac{2m+1}{x}\right)^2 - \left(\frac{2m}{x}\right)^2 > 1$ and it follows that $a_L + 1 < \left(\frac{2m+1}{x}\right)^2$ contradicting that $a_L + 1 > \left(\frac{2m+1}{x}\right)^2$.
The function, $h(t) = \cos(\sqrt{t}\, x\frac{\pi}{2})$ defined for t in $\left[\left(\frac{2m-1}{x}\right)^2, \left(\frac{2m+1}{x}\right)^2\right]$, is increasing on $\left[\left(\frac{2m-1}{x}\right)^2, \left(\frac{2m}{x}\right)^2\right]$ and decreasing on $\left[\left(\frac{2m}{x}\right)^2, \left(\frac{2m+1}{x}\right)^2\right]$ for m even and is decreasing on $\left[\left(\frac{2m-1}{x}\right)^2, \left(\frac{2m}{x}\right)^2\right]$ and increasing on $\left[\left(\frac{2m}{x}\right)^2, \left(\frac{2m+1}{x}\right)^2\right]$ for m odd. Hence, $|h(t)|$ for t in $\left[\left(\frac{m}{x}\right)^2, \left(\frac{m+1}{x}\right)^2\right]$ is increasing on $\left[\left(\frac{2m-1}{x}\right)^2, \left(\frac{2m}{x}\right)^2\right]$ and decreasing on $\left[\left(\frac{2m}{x}\right)^2, \left(\frac{2m+1}{x}\right)^2\right]$. Then we have, if $L \geq 2$, by using the monotonicity of $|h(t)|$ described above, for case (A),

$$\sum_{i=1}^{L} |h(a_i)| \leq \int_{a_1}^{a_L} |h(t)|dt + \max(|h(a_k)|, |h(a_{k+1})|) \leq \int_{a_1}^{a_L} |h(t)|dt + 1$$

and for case (B),

$$\sum_{i=1}^{L} |h(a_i)| \leq \int_{a_1}^{a_L} |h(t)|dt + 1$$

and so
$$\sum_{i=1}^{L} |h(a_i)| \leq \int_{(2m-1)^2/x^2}^{(2m+1)^2/x^2} |h(t)|dt + 1.$$

If $L = 1$, then plainly $|h(a_1)| \leq 1 \leq \int_{(2m-1)^2/x^2}^{(2m+1)^2/x^2} |h(t)|dt + 1$.
Thus, we have if $N_m \neq \varnothing$, $\sum_{i=1}^{L} |h(a_i)| \leq \int_{(2m-1)^2/x^2}^{(2m+1)^2/x^2} |h(t)|dt + 1$.
Consequently, for $m > 0$,

$$|E_m(x)| \leq \int_{(2m-1)^2/x^2}^{(2m+1)^2/x^2} |h(t)|dt + 1, \qquad \text{--------------------------- (5)}$$

since $E_m(x) = 0$ when $N_m = \varnothing$.

Now if $L \geq 2$, then we have, for case (A),

$$\sum_{i=1}^{L} |h(a_i)| + 1 \geq \int_{(2m+1)^2/x^2}^{(2m+1)^2/x^2} |h(t)|dt$$

and for case (B),

$$\sum_{i=1}^{L} |h(a_i)| + \int_{(2m)^2/x^2}^{a_1} |h(t)|dt \geq \int_{(2m)^2/x^2}^{(2m+1)^2/x^2} |h(t)|dt$$

and since for this case $a_1 - \left(\frac{2m}{x}\right)^2 \leq 1$, $\int_{(2m)^2/x^2}^{a_1} |h(t)|d \leq 1$, we obtain again,

$$\sum_{i=1}^{L} |h(a_i)| + 1 \geq \int_{(2m)^2/x^2}^{(2m+1)^2/x^2} |h(t)|dt.$$

If $L = 1$, then $a_1 - \left(\frac{2m}{x}\right)^2 \leq 1$ and $\left(\frac{2m+1}{x}\right)^2 - a_1 \leq 1$ and so we have,

$$|h(a_1)|+1 \ge \int_{(2m-1)^2/x^2}^{(2m+1)^2/x^2} |h(t)|dt.$$

Thus, for any $m > 0$ and $N_m \neq \varnothing$,

$$|E_m(x)| \ge \int_{(2m-1)^2/x^2}^{(2m+1)^2/x^2} |h(t)|dt - 1. \text{ -------------------------- (6)}$$

Note that $N_m = \varnothing$ implies that $\left(\frac{2m+1}{x}\right)^2 - \left(\frac{2m-1}{x}\right)^2 = \frac{8m}{x^2} < 1$ and so $\int_{(2m-1)^2/x^2}^{(2m+1)^2/x^2} |h(t)|dt \le 1$.

It follows that $\int_{(2m-1)^2/x^2}^{(2m+1)^2/x^2} |h(t)|dt - 1 \le 0 = |E_m(x)|$.

Hence, (6) holds for any $m > 0$ without any condition. Therefore combining (5) and (6) we have,

$$\int_{(2m-1)^2/x^2}^{(2m+1)^2/x^2} |h(t)|dt - 1 \le |E_m(x)| \le \int_{(2m-1)^2/x^2}^{(2m+1)^2/x^2} |h(t)|dt + 1. \text{ ------------ (7)}$$

For $m = 0$, $h(t) = \cos(\sqrt{t}\, x\frac{\pi}{2})$ is non-negative and decresing on $[0, \frac{1}{x^2}]$with maximum value 1 at $t = 0$. Thus, if $N_0 \neq \varnothing$,

$$\int_0^{1/x^2} |h(t)|dt - 1 \le |E_0(x)| \le \int_0^{1/x^2} |h(t)|dt. \text{ ------------------ (7)*}$$

Now,

$$\int h(t)dt = \int \cos(\sqrt{t}\, x\frac{\pi}{2})dt = \int 2\cos(u\frac{\pi}{2})\frac{u}{x^2}du, \text{ by change of variable } u = \sqrt{t}\, x,$$
$$= \frac{4}{x^2\pi}u\sin(u\frac{\pi}{2}) - \frac{4}{x^2\pi}\int \sin(u\frac{\pi}{2})du, \text{ by integration by parts,}$$
$$= \frac{4}{x^2\pi}u\sin(u\frac{\pi}{2}) + \frac{8}{x^2\pi^2}\cos(u\frac{\pi}{2}) + C.$$

Therefore, for $m \ge 1$,

$$\int_{(2m-1)^2/x^2}^{(2m+1)^2/x^2} h(t)dt = \left[\frac{4}{x^2\pi}u\sin(u\frac{\pi}{2}) + \frac{8}{x^2\pi^2}\cos(u\frac{\pi}{2})\right]_{2m-1}^{2m+1}$$
$$= \frac{4}{x^2\pi}\{(2m+1)(-1)^m + (2m-1)(-1)^m\}$$
$$= \frac{16m}{x^2\pi}(-1)^m.$$

Thus, for $m \ge 1$,

$$\int_{(2m-1)^2/x^2}^{(2m+1)^2/x^2} |h(t)|dt = \frac{16m}{x^2\pi}. \text{ ----------------------- (8)}$$

Combining (7) and (8) gives us that for $m \ge 1$,

$$|E_m(x)| \ge \frac{16m}{x^2\pi} - 1.$$

It follows then from (3) that for $m > 0$,

$$|T_m(x)| \ge \frac{x^2}{(2m+1)^2}|E_m(x)| \ge \frac{x^2}{(2m+1)^2}\left\{\frac{16m}{x^2\pi} - 1\right\}$$

and $$/|T_m(x)| \le \frac{x^2}{(2m-1)^2}|E_m(x)| \le \frac{x^2}{(2m-1)^2}\left\{\frac{16m}{x^2\pi} + 1\right\}.$$

Now

$$\int_0^{1/x^2} h(t)dt = \left[\frac{4}{x^2\pi}u\sin(u\frac{\pi}{2}) + \frac{8}{x^2\pi^2}\cos(u\frac{\pi}{2})\right]_0^1$$
$$= \frac{4}{x^2\pi} - \frac{8}{x^2\pi^2}.$$

Therefore, it follows from (4), (7)* and the above deduction of the integral that, for $0 < x \le 1$,

$$\frac{4}{x^2\pi} - \frac{8}{x^2\pi^2} \ge E_0(x) \ge |T_0(x)| \ge x^2E_0(x) \ge x^2\left(\frac{4}{x^2\pi} - \frac{8}{x^2\pi^2} - 1\right).$$

For $m > 0$,

$$\frac{16m}{(2m+1)^2\pi} - \frac{x^2}{(2m+1)^2} \le |T_m(x)| \le \frac{16m}{(2m-1)^2\pi} + \frac{x^2}{(2m-1)^2}.$$

Thus, for m even and $m > 0$,

$$\frac{16m}{(2m+1)^2\pi} - \frac{x^2}{(2m+1)^2} \le T_m(x) \le \frac{16m}{(2m-1)^2\pi} + \frac{x^2}{(2m-1)^2}. \quad \text{--------------} \quad (9)$$

And for m odd > 0,

$$-\frac{16m}{(2m-1)^2\pi} - \frac{x^2}{(2m-1)^2} \le T_m(x) \le -\frac{16m}{(2m+1)^2\pi} + \frac{x^2}{(2m+1)^2}. \quad \text{------------} \quad (10)$$

We also have that for for $0 < x \le 1$,

$$\frac{4}{\pi} - \frac{8}{\pi^2} - 1 \le \frac{4}{\pi} - \frac{8}{\pi^2} - x^2 \le T_0(x) \le \frac{4}{x^2\pi} - \frac{8}{x^2\pi^2}.$$

Note that here $T_0(x) = 0$, if $N_0 = \varnothing$, if and only if, $x^2 > 1$.

$$\frac{4}{\pi} - \frac{8}{\pi^2} - 1 \le T_0(x) \le \frac{4}{x^2\pi} - \frac{8}{x^2\pi^2}. \quad \text{---------------------------} \quad (11)$$

Observe that the above inequality also holds when $x > 1$ when $T_0(x) = 0$.

Thus, for $k \ge 1$, $\dfrac{32k}{(4k+1)^2\pi} - \dfrac{x^2}{(4k+1)^2} \le T_{2k}(x) \le \dfrac{32k}{(4k-1)^2\pi} + \dfrac{x^2}{(4k-1)^2}$,

$$-\frac{32k-16}{(4k-3)^2\pi} - \frac{x^2}{(4k-3)^2} \le T_{2k-1}(x) \le -\frac{32k-16}{(4k-1)^2\pi} + \frac{x^2}{(4k-1)^2},$$

and

$$-\frac{32k+16}{(4k+1)^2\pi} - \frac{x^2}{(4k+1)^2} \le T_{2k+1}(x) \le -\frac{32k+16}{(4k+3)^2\pi} + \frac{x^2}{(4k+3)^2}.$$

For $m \ge 1$, let

$$c_m = \begin{cases} \dfrac{16m}{(2m-1)^2\pi}, & m \text{ even} \\ -\dfrac{16m}{(2m+1)^2\pi}, & m \text{ odd} \end{cases}, \quad d_m = \begin{cases} \dfrac{16m}{(2m+1)^2\pi}, & m \text{ even} \\ -\dfrac{16m}{(2m-1)^2\pi}, & m \text{ odd} \end{cases},$$

$$e_m = \begin{cases} \dfrac{x^2}{(2m-1)^2}, & m \text{ even} \\ \dfrac{x^2}{(2m+1)^2}, & m \text{ odd} \end{cases} \quad \text{and } f_m = \begin{cases} \dfrac{x^2}{(2m+1)^2}, & m \text{ even} \\ \dfrac{x^2}{(2m-1)^2}, & m \text{ odd} \end{cases}.$$

Hence, for $m \ge 1$,

$$d_m - f_m \le T_m(x) \le c_m + e_m.$$

Let $e_0 = \dfrac{4}{x^2\pi} - \dfrac{8}{x^2\pi^2}$, $f_0 = 0$, $c_0 = 0$ and $d_0 = \dfrac{4}{\pi} - \dfrac{8}{\pi^2} - 1$.

Note that $\sum\limits_{m=1}^{\infty} c_m$ and $\sum\limits_{m=1}^{\infty} d_m$ are convergent alternating series since $c_m \to 0$ and $d_m \to 0$.
For $m \ge 1$, $0 \le e_m, f_m \le \dfrac{x^2}{m^2}$. Hence, by Comparison Test, $\sum\limits_{m=1}^{\infty} e_m$ and $\sum\limits_{m=1}^{\infty} f_m$ are convergent.
Thus, $\sum\limits_{m=1}^{\infty}(c_m + e_m)$ and $\sum\limits_{m=1}^{\infty}(d_m - f_m)$ are convergent and so are Cauchy series. Therefore, $\sum\limits_{m=1}^{\infty} T_m(x)$ is a Cauchy series and so is convergent. We then deduce as in Problem 1 that
$T(x) = \sum\limits_{n=1}^{\infty} \dfrac{\cos(\sqrt{n}\, x\frac{\pi}{2})}{n}$ is convergent for $x > 0$. Since cosine is an even function
$T(x) = \sum\limits_{n=1}^{\infty} \dfrac{\cos(\sqrt{n}\, x\frac{\pi}{2})}{n}$ is also convergent for $x < 0$. Thus it is convergent for $x \ne 0$.

For now we shall show that $T(x)$ converges uniformly on the interval $[k, K]$ for any $k > 0$ and any $K > k$.

Note that $x \in [k, K]$ implies that $\frac{1}{x} \le \frac{1}{k}$. So from (11) we have for all x in $[k, K]$,

$$\frac{4}{\pi} - \frac{8}{\pi^2} - 1 \le T_0(x) \le \frac{4}{k^2\pi} - \frac{8}{k^2\pi^2} \quad \text{------------------------} \quad (*)$$

and for all $m > 0$, for all $x \in [k, K]$,

$$d_m - \frac{K^2}{m^2} \le T_m(x) \le c_m + \frac{K^2}{m^2}$$

and

$$d_0 - f_0 \le T_0(x) \le c_0 + \frac{4}{k^2\pi}\left(1 - \frac{2}{\pi}\right). \quad \text{-------------------} \ (12)$$

Since the two series defined by the terms on the left and right of (12) and (*) are uniformly convergent, the series

$$\sum_{m=0}^{\infty} T_m(x)$$

is uniformly convergent on $[k, K]$. It remains to show that the original series $T(x)$ is uniformly convergent on $[k, K]$. The proof is similar to the proof for convergence. Since $\sum_{m=0}^{\infty} T_m(x)$ is uniformly convergent, given any $\varepsilon > 0$, there exists an integer $N > 0$ such that for all $m \ge n \ge N$ and for all x in $[k, K]$.

$$\left|\sum_{k=n}^{k=m} T_k(x)\right| < \varepsilon/3.$$

Now the number of positive integers in $\left(0, \left(\frac{2N-1}{x}\right)^2\right]$ is bounded above by $\left[\left(\frac{2N-1}{k}\right)^2\right] + 1$.

Therefore, if $a > \left[\left(\frac{2N-1}{k}\right)^2\right] + 1$, then $a \in N_n(x)$ for some $n \ge N$ and for each x in $[k, K]$.

Thus, if $b > a$, then $b \in N_m(x)$ for some m such that $m \ge n$.

Hence, for all x in $[k, K]$,

$$\left|\sum_{j=a}^{j=b} \frac{\cos(\sqrt{j}\,x\frac{\pi}{2})}{j}\right| = \left|\sum_{j\in N_n(x),\, j\ge a} \frac{\cos(\sqrt{j}\,x\frac{\pi}{2})}{j} + T_{n+1}(x) + \cdots + T_m(x) - \sum_{j\in N_m(x),\, j>b} \frac{\cos(\sqrt{j}\,x\frac{\pi}{2})}{j}\right|$$

$$\le \left|\sum_{j\in N_m(x),\, j\ge a} \frac{\cos(\sqrt{j}\,x\frac{\pi}{2})}{j}\right| + |T_{n+1}(x) + \cdots + T_m(x)| + \left|\sum_{j\in N_m(x),\, j>b} \frac{\cos(\sqrt{j}\,x\frac{\pi}{2})}{j}\right|$$

$$\le |T_n(x)| + |T_{n+1}(x) + \cdots + T_m(x)| + |T_m(x)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This shows that $T(x)$ is uniformly convergent on $[k, K]$.