NATIONAL UNIVERSITY OF SINGAPORE
DEPARTMENT OF MATHEMATICS

1998 - 99 1st Semester MA1102 CALCULUS MOCK TEST

1. (1989) Let the function f be defined on the real numbers by

$$f(x) = \begin{cases} (x+3)^2 , & x \le -3 \\ 2x , & -3 < x < 1 \\ x^2 + 1 , & x \ge 1 \end{cases} .$$

(a) Find the *range* of the function f .

(b) Find the values of x (if any) where (i) $f(x) = -5$, (ii) $f(x) = -7$ and (iii) $f(x) = 0$.

(c) Determine all x in **R** at which the function f is continuous.

(d) Is the function f differentiable at $x = 1$? Justify your answer.

(e) Compute $\int_0^2 f(x)dx$.

(f) Sketch the graph of the function f.

2. (1996) Evaluate, if it exists, each of the following limits.

(a) $\lim_{x\to+\infty} \dfrac{10+9x^3-x^2}{3x^3-7+5x}$. (b) $\lim_{x\to 0} \dfrac{\sqrt{5x^2+4}-2}{x^2}$. (c) $\lim_{x\to 0} \dfrac{\sin(x)}{7x-x^2}$.

(d) $\lim_{x\to\infty} \left(\sqrt{16x^2+3}-4x\right)$. (e) $\lim_{x\to 1} \dfrac{\sin(x-1)}{\sqrt{x}-1}$. (f) $\lim_{x\to 0} (1+7x^2)^{\frac{1}{x^2}}$.

3. (1995) (a) Let g be the function defined on **R** by $g(x) = x^5 + x + 5$.

(i) Use the *Intermediate Value Theorem* to show that there is a number c in **R** such that $g(c) = 0$.

(ii) Using *Rolle's Theorem* or otherwise, show that there is exactly one such c with $g(c) = 0$.

(b) Determine the absolute extrema of the function h on [–2, 4] defined by $h(x) = x^4 - 4x^2 + 16$.

(c) Find $\dfrac{d^2y}{dx^2}$ by implicit differentiation if $y^2 - \sin(y) = 2x$.

4. (1987) Let $f(x) = \dfrac{2+x-x^2}{(x-1)^2}$.

(a) Find the intervals on which f is (i) *increasing* and (ii) *decreasing*.

(b) Find the critical points of f.

(c) Determine the relative extrema of f.

(d) Find the intervals on which f is concave upward or concave downward.

(e) Determine the point(s) of inflection of the graph of f.

(f) Find the vertical and horizontal asymptotes of f if any.

(g) Sketch the graph of f.

5. Find the following integrals.

(a) $\int_0^1 \frac{x^2}{(x^2+1)^2}\,dx$. (1986)

(b) $\int (\ln(3x))^2\,dx$. (1986)

(c) $\int \cos(\sin(y))\cos(y)\,dy$. (1996)

(d) $\int \frac{e^x}{e^{2x}+3e^x+2}\,dx$. (1996)

(e) $\int \frac{1}{\sqrt{2\sqrt{x}+5}}\,dx$. (1996)

6. (1994) (a) Evaluate $\int_2^5 (|x-3|+|x-4|)\,dx$.

(b) Prove that $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{2n}\sin\left(\frac{\pi}{2}\,\frac{i}{n}\right) = 1.$

(c) Let f be a function defined on **R** such that $f(x) > 0$ for all x in **R**. Prove that

$g(x) = x\int_0^x f(t)\,dt$

is increasing on $(0,\infty)$.

7. (1996) (a) Differentiate each of the following functions.

(i) $g(x) = \int_{x^3}^{x^4} \frac{1}{1+t^2}\,dt$.

(ii) $h(x) = 3^{(x^2)}$.

(iii) $k(x) = (1+x^2)^{\ln(x)}$.

(b) Let f be the function defined on the set **R** of real numbers by

$$f(x) = \int_1^x \sqrt{8+t^2}\,dt .$$

(i) Without integrating, show that the function f is injective.

(ii) Determine $(f^{-1})'(0)$.

(Extra practice)

8. (1994) (a) Determine a value for the constant k for which the function f defined by

$$f(x) = \begin{cases} \frac{\sin(3x)}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

will be continuous at $x = 0$.

(b) Suppose that the function f satisfies

(1) $f(x+y) = f(x)f(y)$ for all values of x and y,

(2) $f(0) = 1$ and

(3) f is differentiable at $x = 0$ and $f\ \ (0) = 1$.

Show that f is differentiable at x for all x and $f\ \ (x) = f(x)$.