EXAMPLE SESSION 5

Curves, Absolute extrema

1. Given curve defined by the equation

$$y = x^4 - 6x. \quad \text{-------------} \quad (1)$$

Find the equation of the tangent line to the curve parallel to the line

$$10x + y + 3 = 0. \text{ ----------} \quad (2)$$

Find too the equation of the tangent line perpendicular to the line

$$x - 2y + 1 = 0. \quad \text{-- -----------} \ (3)$$

Differentiate (1) giving $\frac{dy}{dx} = 4x^3 - 6$ -------------- (4)

From (2), the gradient of the line is -10.

Thus setting $\frac{dy}{dx} = 4x^3 - 6 = -10$ yield $4x^3 - 6 = -10$ so that $4x^3 = -4$. Since we are seeking real solution $x = -1$. Substitute this in (1) gives $y = (-1)^4 - 6(-1) = 1 + 6 = 7$.

Therefore, equation of the tangent line is $\frac{y-7}{x+1} = -10$.

Gradient of line (3) is $\frac{1}{2}$. Thus the gradient of any line perpendicular to (3) is -2.

Thus for a tangent line to (1) to be perpendicular to line (3), we must have

$$\frac{dy}{dx} = 4x^3 - 6 = -2$$

giving rise to $4x^3 = 4$ which implies that $x = 1$. Substituting this value in (1) gives

$$y = 1^4 - 6 \cdot 1 = -5.$$

Therefore, equation of the tangent line to (1) perpendicular to line (3) is given by

$$\frac{y-(-5)}{x-1}=-2 \qquad \text{i.e.,} \quad y+5=-2x+2.$$

2. Find the absolute extrema of the function defined on the interval [0, 4] by

$$f(x)=\begin{cases} x^2-2x+5, & 0\le x\le 2 \\ x^3-3x^2-9x+27, & 2<x\le 4 \end{cases}.$$

First check that f is continuous on [0,4].

Then try to differentiate f wherever you can on the open interval (0, 4) with the intention to locate the stationary points. Note that f is continuous on [0,4]-{2}.

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(x^2-2x+5)=4-4+5=5 \quad \text{and}$$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(x^3-3x^2-9x+27) \quad =8-12-18+27=5=f(2)$$

Thus $\lim_{x\to 2} f(x)=f(2)$.

Therefore f is continuous at x = 2. So f is continuous on [0,4].

$$f'(x)=\begin{cases} 2x-2, & 0<x<2 \\ 3x^2-6x-9, & 2<x<4 \end{cases}.$$

Thus solving $f'(x)=0$ for $0<x<2$ gives $2x-2=0$. Thus x =1. Also solving $f'(x)=0$ for $2<x<4$ gives $3x^2-6x-9=0$.

I.e. $\quad 3(x^2-2x-3)=3(x-3)(x+1)=3$ -- (1)

Solution in 2 < x < 4 is x = 3. Thus we have 2 stationary points in (0,4)-{2}.

Since 2 is a probable critical point we shall take into consideration the value of f at x = 2.

Thus the points of interest are 1,2,3,0,4.

$f(0)=5$, $f(1)=4$, $f(2)=5$, $\quad f(3)=3^3-3^3-9\times 3+27=0$ and $f(4)=4^3-3\times 4^2-9\times 4+27=7$

Therefore, the absolute maximum is 7 and the absolute minimum is 0.