EXAMPLE SESSION 4

Continuity and Differentiability

1. Show that the equation $x = \sqrt{1 - x^2}$ has a solution between $\frac{1}{2}$ and $\frac{3}{4}$.

The question suggests use of the *Intermediate Value Theorem*.

Consider the function

$$g(x) = x - \sqrt{1 - x^2} \quad \text{for} \quad x \in [-1,].$$

Then $g(x)$ is continuous on $[\frac{1}{2}, \frac{3}{4}]$ because $\sqrt{1 - x^2}$ and x

are continuous on $[-1, 1]$.

$$g(\tfrac{1}{2}) = \frac{1}{2} - \sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{1}{2} - \sqrt{1 - \frac{1}{4}}$$

$$= \frac{1}{2} - \sqrt{\frac{3}{4}} = \frac{1 - \sqrt{3}}{2} < 0$$

$$g(\tfrac{3}{4}) = \frac{3}{4} - \sqrt{1 - \left(\frac{3}{4}\right)^2} = \frac{3}{4} - \sqrt{1 - \frac{9}{16}}$$

$$= \frac{3}{4} - \sqrt{\frac{7}{16}} = \frac{3 - \sqrt{7}}{4} > 0$$

Therefore, by the *Intermediate Value Theorem*

there exists a $c \in \left(\frac{1}{2}, \frac{3}{4}\right)$ such that $g(c) = 0$.

That is, $c = \sqrt{1 - c^2}$.

2. Find the derivative of $f(x) = x|x|$.

First rewrite the function as a piecewise polynomial.

$$f(x) = \begin{cases} x \cdot x, & x \geq 0 \\ x \cdot (-x), & x < 0 \end{cases} = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}.$$

Since x^2 is differentiable on $(0, \infty)$ with derivative equaling $2x$ and $-x^2$ is likewise differentiable on $(-\infty, 0)$ with derivative $-2x$,

$$f'(x) = \begin{cases} 2x, & x > 0 \\ -2x, & x < 0 \end{cases}.$$

Now $\lim_{x \to 0^+} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \dfrac{x^2 - 0}{x - 0} = \lim_{x \to 0^+} x = 0$

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{-x^2 - 0}{x - 0} = \lim_{x \to 0^+} -x = 0$$

Therefore, $f'(0) = 0$.

3. Find the derivative of the following function wherever it exists.

$$f(x) = \begin{cases} x^3 + x, & x \leq 2 \\ x^2 + 7x, & x > 2 \end{cases}.$$

Observe first that since f is a piecewise polynomial,

$$f'(x) = \begin{cases} 3x^2 + 1, & x < 2 \\ 2x + 7, & x > 2 \end{cases}.$$

Also $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + 7x) = 4 + 14 = 18$ and

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^3 + x) = 8 + 2 = 10$.

Thus $\lim_{x \to 2} f(x)$ does not exist and so f is not continuous at $x = 2$.

Therefore, f is not differentiable at $x = 2$.

4. Find the derivative of

$$f(x) = \begin{cases} x^4 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases} .$$

Note that away from 0, $x^4 \sin\left(\frac{1}{x}\right)$ is defined and differentiable and whose derivative is evaluated as follows by the *product* and *chain rule* :---

$$4x^3 \sin\left(\frac{1}{x}\right) + x^4 \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right)$$

$$= 4x^3 \sin\left(\frac{1}{x}\right) - x^2 \cos\left(\frac{1}{x}\right)$$

i .e.

$$f'(x) = \begin{cases} 4x^3 \sin\left(\frac{1}{x}\right) - x^2 \cos\left(\frac{1}{x}\right), x > 0 \\ 4x^3 \sin\left(\frac{1}{x}\right) - x^2 \cos\left(\frac{1}{x}\right), x < 0 \end{cases}$$

Now for $x = 0$, we shall have to use the definition of the derivative.

Recall $f(0) = 0$.

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^4 \sin\left(\frac{1}{x}\right) - 0}{x - 0} = \lim_{x \to 0^+} x^3 \sin\left(\frac{1}{x}\right)$$

$$= 0$$

by the *Squeeze Theorem* since $0 \leq \left| x^3 \sin\left(\frac{1}{x}\right) \right| \leq |x^3|$ and $\lim_{x \to 0^+} |x^3| = 0.$

Similarly by the *Squeeze Theorem*,

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} x^3 \sin\left(\frac{1}{x}\right) = 0.$$

Therefore, $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=0$, i.e. $f'(0)=0$.Hence

$$f'(x)=\begin{cases} 4x^3\sin\left(\frac{1}{x}\right)-x^2\cos\left(\frac{1}{x}\right),\ x\neq 0 \\ 0,\ x=0 \end{cases}.$$

5. Let $f(x)=\begin{cases} \alpha x+1,\ x<1 \\ \beta x^2+3x+1,\ x\geq 1 \end{cases}$.

Find a and b so that the derivative $f'(1)$ exist.

Since we know by [Ng, Theorem 5.1.5] that differentiability at $x = 1$ implies continuity at $x = 1$. So we first formulate condition for f to be continuous at $x = 1$; then formulate condition for the derivative to exist at $x = 1$.

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(\alpha x+1)=\alpha+1$ and

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(\beta x^2+3x+1) = \beta+3+1=\beta+4\ (=f(1))$

Therefore for continuity, $\alpha+1=\beta+4\ (=f(1))$,

i.e. $\alpha=\beta+3$ ----------- (1)

Recall $f(1)=\beta+4$.

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-}\frac{\alpha\,(1+h)+1-(\beta+4)}{h}$$

$$=\lim_{h\to 0^-}\frac{\alpha\,h+(\alpha-\beta-3)}{h}$$

$$=\lim_{h\to 0^-}\frac{\alpha\,h}{h} \quad \text{since } \alpha-\beta-3=0 \text{ by (1)}$$

$$=\alpha.$$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{\beta(1+h)^2+3(1+h)+1-(\beta+4)}{h}$$

$$=\lim_{h\to 0^+}\frac{\beta(1+2h+h^2)+3(1+h)+1-(\beta+4)}{h}$$

$$=\lim_{h\to 0^+}\frac{\beta(2h+h^2)+3h}{h}=\lim_{h\to 0^+}2\beta+\beta h+3$$

$$=2\beta+3.$$

Therefore, for differentiability at $x=1$, we must have

$$\alpha=2\beta+3 \qquad \text{-------------} \qquad (2)$$

Equation (2) - Equation (1) gives $0 = b$.

Then substituting this value of b in (2) gives a = 3.

Thus a = 3 and b = 0.

6. Differentiate $f(x)=\sin(\cos^2(2x))$.

Remember the *chain rule*

$$\frac{d}{dx}(g(h(x)))|_{x_0}=\frac{d}{dx}g|_{h(x_0)}\cdot\frac{d}{dx}h|_{x_0}$$

Applying the *chain rule* 3 times

$$\frac{d}{dx}f(x)=\frac{d}{dx}\sin(\cos^2(2x))$$

$$=\cos(\cos^2(2x))\cdot\frac{d}{dx}\cos^2(2x)$$

$$= \cos\left(\cos^2(2x)\right) \cdot 2\cos(2x) \cdot \frac{d}{dx}\cos(2x)$$

$$= \cos\left(\cos^2(2x)\right) \cdot 2\cos(2x) \cdot (-\sin(2x)) \cdot \frac{d}{dx}(2x)$$

$$= \cos\left(\cos^2(2x)\right) \cdot 2\cos(2x) \cdot (-\sin(2x)) \cdot 2$$

$$= -4\sin(2x)\cos(2x)\cos\left(\cos^2(2x)\right)$$

$$= -2\sin(4x)\cos\left(\cos^2(2x)\right) \quad \text{since} \quad 2\sin(2x)\cos(2x) = \sin(4x).$$