Example Session 1

Functions

1. The expression $f(x) = \frac{4}{2x^2-5}$.

   Determine its range and domain.

- Examine the expression. When does it make sense?

  Remember

  $\frac{k}{0}$, $\frac{0}{0}$ do not make any sense. $\frac{4}{2x^2-5}$ make sense when $2x^2 - 5 \neq 0$.

I.e. $2x^2 \neq 5$ or $x^2 \neq \frac{5}{2}$. I.e. $x \neq \pm\sqrt{\frac{5}{2}}$.

$\therefore$ The largest domain is $\boldsymbol{R} - \left\{\pm\sqrt{\frac{5}{2}}\right\}$.

Now for the range of f .

Remember: y is in the range of f $\Leftrightarrow \exists\, x$ in the domain of f such that $f(x) = y$.

So we look at $\frac{4}{2x^2-5} = y$ ----------- (*)

For what values of y can we solve (*) for x in the domain of f ?

- Note that $\frac{4}{2x^2-5} \neq 0$. Thus (*) has no solution for $y = 0$.

Now we assume $y \neq 0$. Then from (*) we get $2x^2 - 5 = \frac{4}{y}$ so that

$2x^2 = 5 + \frac{4}{y} = \frac{5y+4}{y}$ and so

$$x^2 = \frac{1}{2} \cdot \frac{5y+4}{y} \text{----------- } (**)$$

- Now $x^2 \geq 0$. For (**) to be solvable we must have

$$\frac{5y+4}{y} \geq 0 \quad \text{------ } (***)$$

I.e. $5y+4\geq 0$ and $y>0$

or $5y+4\leq 0$ and $y<0$

I.e. $y\geq -\frac{4}{5}$ and $y>0$

or $y\leq -\frac{4}{5}$ and $y<0$•

I.e. $y>0$ or $y\leq -\frac{4}{5}$.

For these values of y we can take $x=\sqrt{\frac{4+5y}{2y}}$.

$\therefore$ The range of f is $\{\, y : y>0$ or $y\leq -\frac{4}{5}\}$.

2. For the expression $g(x)=|x-1|+|x+2|$

find the largest subset of **R** for which g determines a function with it as the domain and determine its range.

- The modulus function has at the point 0 a 'turning point' , where on the left of 0 it is a standard polynomial function and on the right of 0 it is also a polynomial function.

So we should try to rewrite the expression in terms of the more familiar functions.

Note that the modulus function is defined for the whole of the real numbers.

$\therefore$ The expression g is defined for the whole of the real numbers. So the largest domain is **R**.

$$g(x)=|x-1|+|x+2|$$

$$=\begin{cases} -(x-1)-(x+2), & x<-2 \\ -(x-1)+(x+2), & -2\leq x<1 \\ (x-1)+(x+2), & 1\leq x \end{cases}$$

$$=\begin{cases} -2x-1, & x<-2 \\ 3, & -2\leq x<1 \\ 2x+1, & 1\leq x \end{cases} \bullet$$

Thus we need to know the values g can take in the three intervals,

$x < -2$, $-2 \le x < 1$ and $1 \le x$.

We note that $-2x-1$ and $2x+1$ are injective functions. Note that

$x < -2 \Leftrightarrow -2x > 4 \Leftrightarrow -2x - 1 > 3$.

$\therefore$ g takes the set $\{x : x < -2\}$ onto the set $\{y : y > 3\}$

because for any $y > 3$, $-2x - 1 = y$ can be solved for $x < -2$.

Note also that $x \ge 1 \Leftrightarrow 2x \ge 2 \Leftrightarrow 2x + 1 \ge 3$.

$\therefore g$ takes the set $\{x : x \ge 1\}$ onto the set $\{y : y \ge 3\}$.

We have seen that for $-2 \le x < 1$, $g(x) = 3$.

Thus g takes the set $\{x : -2 \le x < 1\}$ onto the set with one element $\{3\}$.

Thus the range of g is image of $\{x : x < -2\}$ under g

$\cup$ image of $\{x : -2 \le x < 1\}$ under g

$\cup$ image of $\{x : 1 \le x\}$ under g

$= \{y : y > 3\} \cup \{3\} \cup \{y : y \ge 3\}$

$= \{y : y \ge 3\}$

$= [3, \infty)$.

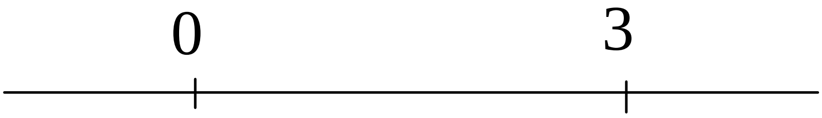
3. Solve the inequality $|x| + |x - 3| \ge 5$ ------ (*)

- As in the last example try rewriting (*) by removing the modulus sign.

Note that

$|x|$ is a polynomial on the left of 0 and also on the right of 0

$|x - 3|$ is a polynomial on the left of 3 and also on the right of 3.



So we shall solve the inequality (*) in each of the three intervals.

$\{x : x < 0\}$, $\{x : 0 \le x < 3\}$, $\{x : x \ge 3\}$

<u>Case $x < 0$.</u>

Then the inequality become $-x - (x-3) \ge 5$

i.e. $-2x \ge 2$.

Dividing by -2 give $x \le -1$.

$\therefore$ The solution set for this case is $\{x : x \le -1\}$.

<u>Case</u> $0 \le x < 3$. The inequality (*) becomes $x - (x-3) \ge 5$

i.e. $3 \ge 5$ which is absurd. The solution set for this case is empty.

<u>Case</u> $x \ge 3$. The inequality (*) becomes $x + (x-3) \ge 5$

I.e. $2x \ge 8$ or $x \ge 4$. $\therefore$ The solution set in this case is $\{x : x \ge 4\}$.

Thus the solution set to (*) is the union of the solution set for all three cases and is

$\{x : x \le -1 \text{ or } x \ge 4\}$ $= (-\infty, -1] \cup [4, \infty)$.