NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2001-2002

MA1102R Calculus

November 2001 — Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **NINE (9)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions in **Section A**. Each question in Section A carries 10 marks.

3. Answer not more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1. [10 marks]

Evaluate each of the following limits.

(a) $\lim_{x\to 1} \sqrt[3]{\dfrac{x^2-1}{x^3-1}}$.

(b) $\lim_{x\to -\infty} e^x(x^2+3x)$.

(c) $\lim_{x\to 2} \dfrac{|x^2-x-6|-4}{x^2-4}$.

Question 2. [10 marks]

(a) Find the largest subset of $\mathbb{R}$ on which the following expression defines a function with it as the domain.

$$f(x) = \frac{1}{\sqrt{x^2-3x-4}}.$$

(b) Using the $\epsilon - \delta$ definition of the limit, prove that

$$\lim_{x\to 4}(1-\frac{x}{2}) = -1.$$

Question 3. [10 marks]

(a) Let $g : (0,\infty) \longrightarrow \mathbb{R}$ be defined by $g(x) = x^{\ln x} + x\ln x$. Find the derivative of g.

(b) Suppose that h is a differentiable function and $\dfrac{d}{dx}[h(3x)] = e^{x^2}$. Determine the derivative $\dfrac{dh(x)}{dx}$.

Question 4. [10 marks]

(a) State the Mean Value Theorem.

(b) Prove that the equation $3\tan x + x^3 = 2$ has **exactly** one real solution in $(0, \frac{\pi}{4})$.

Question 5. [10 marks]

Evaluate each of the following integrals:

(a) $\displaystyle\int \frac{1}{x^2+x^4}dx.$

(b) $\displaystyle\int x^4(\ln x)^2 dx.$

(c) $\displaystyle\int_4^9 \frac{e^{\sqrt{x}}}{\sqrt{x}(1+e^{\sqrt{x}})}dx.$ [You may leave your answer in terms of e.]

Question 6. [10 marks]

A metal box without the top is to be constructed from a square sheet of metal with sides of length 10cm by first cutting square pieces of the same size from the corners of the sheet and then folding up the sides. Find the dimensions of the box with the largest volume that can be constructed.

SECTION B

Answer not more than **two** *questions from this section. Each question in this section carries 20 marks.*

Question 7. [20 marks]

Let $f(x) = \begin{cases} x\sin(\dfrac{1}{x}) & \text{if} \quad x>0, \\ 1 & \text{if} \quad x=0, \\ \dfrac{2^x-e^x}{3^x-x} & \text{if} \quad x<0. \end{cases}$

(a) Find each of the following limits, if it exists. If the limit does not exist, explain why it does not exist. If it is an infinite limit, determine whether it is ∞ or $-\infty$. Justify your answers.

(i) $\lim_{x\to 0^+} f(x)$.

(ii) $\lim_{x\to 0} f(x)$.

(iii) $\lim_{x\to \infty} f(x)$.

(iv) $\lim_{x\to -\infty} f(x)$.

(b) Determine whether or not f is continuous at $x=0$. Justify your answer.

(c) Is f differentiable at $x=0$? Justify your answer.

Question 8. [20 marks]

Let $f(x) = \dfrac{x^2}{x^2+3}$.

(a) Find the intervals on which (i) f is increasing and (ii) f is decreasing.

(b) Find the horizontal and vertical asymptotes (if any) of the graph of f.

(c) Find the intervals on which (i) the graph of f is concave upward and (ii) the graph of f is concave downward.

(d) Find all the relative maxima and minima of f.

(e) Find the points of inflection of the graph of f.

(f) Sketch the graph of f.

(g) From the graph of f, or otherwise, deduce the range of f.

Question 9. [20 marks]

Let $f : (1, \infty) \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \int_e^x \frac{1}{\ln t} dt.$$

(a) Find $f'(x)$.

(b) Prove that f is an increasing function.

(c) Find $(f^{-1})'(0)$.

(d) Prove that $\dfrac{1}{\ln t} > \dfrac{1}{t}$ for all $t > e$.

(e) Using (d), or otherwise, prove that $\lim\limits_{x\to\infty} f(x) = \infty$.

END OF PAPER

Solution

Section A

1. (a) $\displaystyle\lim_{x\to1}\sqrt[3]{\frac{x^2-1}{x^3-1}}=\sqrt[3]{\lim_{x\to1}\frac{(x-1)(x+1)}{(x-1)(x^2+x+1)}}=\sqrt[3]{\lim_{x\to1}\frac{x+1}{x^2+x+1}}=\sqrt[3]{\frac{2}{3}}.$

(b)
$$\begin{aligned}
&\lim_{x\to-\infty} e^x(x^2+3x)\\
=&\lim_{x\to-\infty}\frac{x^2+3x}{e^{-x}} &&\text{by L'Hôpital's rule,}\\
=&\lim_{x\to-\infty}\frac{2x+3}{-e^{-x}} &&\text{by L'Hôpital's rule,}\\
=&\lim_{x\to-\infty}\frac{2}{e^{-x}}=0.
\end{aligned}$$

(c)
$$\begin{aligned}
&\lim_{x\to2}\frac{|x^2-x-6|-4}{x^2-4}\\
=&\lim_{x\to2}\frac{-(x^2-x-6)-4}{x^2-4} &&\text{since } x^2-x-6=(x-3)(x+2)<0 \text{ for } -2<x<3,\\
=&\lim_{x\to2}\frac{-x^2+x+2}{x^2-4}\\
=&\lim_{x\to2}-\frac{(x-2)(x+1)}{(x-2)(x+2)}\\
=&\lim_{x\to2}-\frac{(x+1)}{x+2}=-\frac{3}{4}.
\end{aligned}$$

2. (a) Note that $x^2-3x-4=(x-4)(x+1)>0$ if and only if $x<-1$ or $x>4$. Hence, the largest subset of $\mathbb{R}$ on which f is defined is $(-\infty,-1)\cup(4,\infty)$.

(b) Given $\epsilon>0$. Choose $\delta=2\epsilon$. If $0<|x-4|<\delta$, then $|(1-\frac{x}{2})-(-1)|=|\frac{4-x}{2}|=\frac{1}{2}|x-4|<\frac{1}{2}\delta=\frac{1}{2}\cdot2\epsilon=\epsilon$. Hence, $\lim_{x\to4}(1-\frac{x}{2})=-1$.

3. (a)
$$\begin{aligned}
\frac{d}{dx}g(x)&=\frac{d}{dx}e^{(\ln x)^2}+\frac{d}{dx}(x\ln x)\\
&=e^{(\ln x)^2}\frac{d}{dx}(\ln x)^2+(\ln x+x\cdot\frac{1}{x})\\
&=e^{(\ln x)^2}\cdot2\ln x\cdot\frac{1}{x}+(\ln x+1)\\
&=\frac{2x^{\ln x}\ln x}{x}+\ln x+1.
\end{aligned}$$

(b) By chain rule of differentiation, $3h'(3x)=\frac{d}{dx}[h(3x)]=e^{x^2}$. By letting $t=3x$, we obtain $h'(t)=\frac{1}{3}e^{t^2/9}$.

4. (a) The Mean Value Theorem states that if f is continuous on $[a, b]$ and differentiable in (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) Let $f(x) = 3\tan(x) + x^3 - 2$. We wish to show that there is exactly one c in $(0, \frac{\pi}{4})$ such that $f(c) = 0$. First, observe that f is continuous on $[0, \frac{\pi}{4}]$ and differentiable in $(0, \frac{\pi}{4})$. Next, we have $f(0) = -2$ and $f(\frac{\pi}{4}) = 3 - (\frac{\pi}{4})^3 - 2 = 1 - (\frac{\pi}{4})^3 > 0$. By Intermediate Value theorem, there exists a real number c in $(0, \frac{\pi}{4})$ such that $f(c) = 0$.
As $f'(x) = 3\sec^2 x + 3x^2 > 0$ for all in $(0, \frac{\pi}{4})$, f is an increasing function on $[0, \frac{\pi}{4}]$, hence injective on $[0, \frac{\pi}{4}]$. This shows that there cannot be two distinct numbers c and c' in $(0, \frac{\pi}{4})$ such that $f(c) = f(c') = 0$.
Alternatively, suppose that there are two distinct numbers c and c' in $(0, \frac{\pi}{4})$ with $c < c'$ such that $f(c) = f(c') = 0$. By applying the Mean Value Theorem or Rolle's Theorem to the function f defined on $[c, c']$, there exists a number ζ in (c, c') such that $f'(\zeta) = (f(c') - f(c))/(c' - c) = 0$. But this contradicts the fact the $f'(\zeta) = 3\sec^2\zeta + 3\zeta^2 > 0$ as ζ in $(0, \frac{\pi}{4})$.

5. (a) $\displaystyle\int \frac{1}{x^2 + x^4} dx = \int \frac{1}{x^2} - \frac{1}{1 + x^2} dx = -\frac{1}{x} - \tan^{-1} x + C.$

(b)
$$\begin{aligned}\int x^4 (\ln x)^2 dx &= \frac{1}{5}x^5(\ln x)^2 - \int \frac{1}{5}x^5 \cdot 2\ln x \cdot \frac{1}{x} dx \\ &= \frac{1}{5}x^5(\ln x)^2 - \frac{2}{5}\int x^4 \ln x dx \\ &= \frac{1}{5}x^5(\ln x)^2 - \frac{2}{25}x^5 \ln x + \frac{2}{25}\int x^5 \cdot \frac{1}{x} dx \\ &= \frac{1}{5}x^5(\ln x)^2 - \frac{2}{25}x^5 \ln x + \frac{2}{125}x^5 + C.\end{aligned}$$

(c) Let $t = 1 + e^{\sqrt{x}}$. Then $dt = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}} dx$. When $x = 4$, $t = 1 + e^2$. When $t = 9$, $x = 1 + e^3$. Hence,

$$\int_4^9 \frac{e^{\sqrt{x}}}{\sqrt{x}(1 + e^{\sqrt{x}})} dx = \int_{1+e^2}^{1+e^3} \frac{2}{t} dt = \Big[2\ln t\Big]_{1+e^2}^{1+e^3} = 2\ln\left(\frac{1 + e^3}{1 + e^2}\right).$$

6. Let x be the length in cm of the sides of the squares that are cut out. Since the original sheet of metal is 10cm on a side, we have $0 \le x \le 5$. The resulting box has a height of x cm and a base which is a square of side $(10 - 2x)$ cm. This means

that the volume V of the box is given by $V(x) = x(10-2x)^2 = 4x^3 - 40x^2 + 100x$ for x in $[0,5]$. Next we shall find the critical points of V.

$V'(x) = 12x^2 - 80x + 100 = 4(3x-5)(x-5)$. Therefore, $V'(x) = 0$ for $x = \frac{5}{3}$ and $x = 5$. Thus the only critical point of V in $(0,5)$ is $\frac{5}{3}$. Since $V(0) = 0, V(\frac{5}{3}) = \frac{2000}{27}$ and $V(5) = 0$, it follows that the maximum value of V occurs for $x = \frac{5}{3}$. The corresponding value of $(10-2x)$ is $\frac{20}{3}$. Consequently, the box with the maximum volume has a height of $\frac{5}{3}$ cm and a square base of side $\frac{20}{3}$ cm.

Section B

7. (a) (i) For $x > 0$, $-x \leq x\sin(\frac{1}{x}) \leq x$. Also $\lim\limits_{x\to 0^+} x = \lim\limits_{x\to 0^+} -x = 0$. Therefore, by squeeze theorem, $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x\sin(\dfrac{1}{x}) = 0$.

(ii) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{2^x - e^x}{3^x - x} = \dfrac{1-1}{1-0} = 0$.

Combining the result in (i), we have $\lim\limits_{x\to 0} f(x) = 0$.

(iii) Let $x = 1/t$. Hence t approaches 0 from the right as x approaches to infinity. Therefore, $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} x\sin(\dfrac{1}{x}) = \lim\limits_{t\to 0^+} \dfrac{\sin t}{t} = 1$.

(iv) Note that $\lim\limits_{x\to-\infty} 2^x = 0$, $\lim\limits_{x\to-\infty} e^x = 0$, $\lim\limits_{x\to-\infty} 3^x = 0$ and $\lim\limits_{x\to-\infty} x = -\infty$. Hence, $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{2^x - e^x}{3^x - x} = 0$.

(b) f is not continuous at $x = 0$ since $\lim\limits_{x\to 0} f(x) = 0 \neq 1 = f(1)$.

(c) As f is not continuous at $x = 0$, f is not differentiable at $x = 0$.

8. Let's first compute $f'(x)$ and $f''(x)$. We have $f'(x) = \dfrac{6x}{(x^2+3)^2}$ and

$$f''(x) = \frac{6(x^2+3)^2 - 2(x^2+3)(2x)(6x)}{(x^2+3)^4} = \frac{-18(x+1)(x-1)}{(x^2+3)^3}.$$

(a) From the expression of $f'(x)$, we see that $f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$. Therefore, f is decreasing on $(-\infty, 0]$ and is increasing on $[0, \infty)$.

(b) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x^2}{x^2+3} = \lim\limits_{x\to\infty} \dfrac{1}{1+\frac{3}{x^2}} = 1$. Similarly, $\lim\limits_{x\to-\infty} f(x) = 1$. Therefore, $y = 1$ is a horizontal asymptote of the graph of f.

Since f is continuous on $\mathbb{R}$, there is no vertical asymptote of the graph of f.

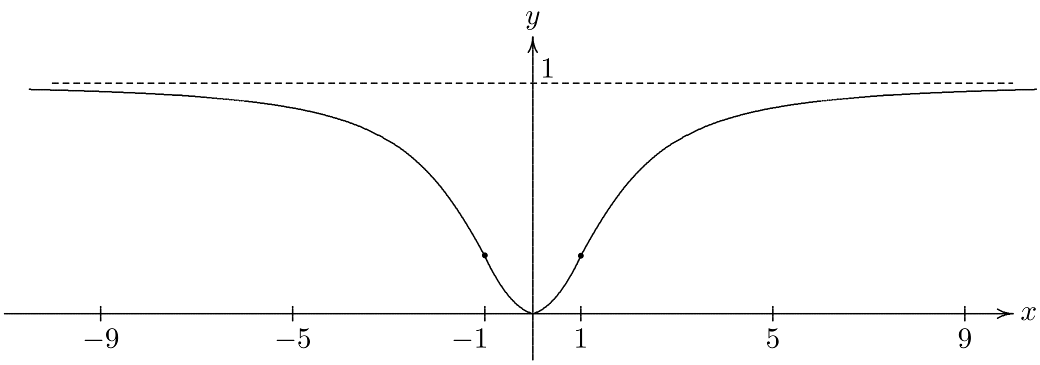
(c) From the expression of $f''(x)$, we see that $f''(x) < 0$ for x in $(-\infty, -1) \cup (1, \infty)$ and $f''(x) > 0$ for x in $(-1, 1)$.

Thus the graph of f is concave downward in $(-\infty, -1) \cup (1, \infty)$ and concave upward in $(-1, 1)$.

(d) $f'(x) = 0$ if and only if $x = 0$. Therefore, there is only one critical point for f. By the first derivative test for relative extremum, f has a relative minimum at $x = 0$.

(e) Setting $f''(x) = 0$, we see that $f''(x) = 0$ if and only if $x = -1$ or 1. From (c), the graph of f is concave downward in $(-\infty, 1)$ and concave upward on $(-1, 1)$. So this is a change of concavity of the graph of f at $x = -1$. Similarly, this is a change of concavity of the graph of f at $x = 1$. Consequently, there are two points of inflections of the graph of f at $x = -1$ and $x = 1$.

(f) The graph of f is shown below.



(g) The range of f is $[0, 1)$. Clearly $0 \le f(x) < 1$. Conversely, let y be in $[0, 1)$. Then by solving x in terms of y in the equation $y = \dfrac{x^2}{3 + x^2}$, we obtain $x = \pm(\frac{3y}{1-y})^{\frac{1}{2}}$ which is well-defined as y is in $[0, 1)$. Thus $f(-(\frac{3y}{1-y})^{\frac{1}{2}}) = f((\frac{3y}{1-y})^{\frac{1}{2}}) = y$. Therefore all points in $[0, 1)$ are in the range of f. This shows that the range of $f = [0, 1)$.

9. (a) By the fundamental theorem of Calculus, $f'(x) = \frac{1}{\ln x} > 0$ for all $x > 1$.

(b) By (a), $f'(x) = \frac{1}{\ln x} > 0$ for all $x > 1$. Hence, f is increasing on $(1, \infty)$. Alternatively, one may check directly using the definition of increasing function. To do so, let $x_2 > x_1 > 1$. Then $f(x_2) - f(x_1) = \int_{x_1}^{x_2} \frac{1}{\ln t} dt > \frac{1}{\ln x_2}(x_2 - x_1) > 0$. Therefore, $f(x_2) > f(x_1)$. That is f is increasing on $(1, \infty)$.

(c) Since f is increasing, it is injective. Therefore, its inverse f^{-1} exists. Note that f is differentiable by the fundamental theorem of calculus and $f'(x) = \frac{1}{\ln x} > 0$ for all $x > 1$. Hence, f^{-1} is differentiable at all points in its domain and

$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$. Note that $f(e) = 0$. Consequently, $(f^{-1})'(0) = \frac{1}{f'(e)} = \frac{1}{\frac{1}{\ln e}} = 1$.

(d) Let $\alpha(t) = t - \ln t$. For $t > e$, $\alpha'(t) = 1 - \frac{1}{t} > 0$. Hence, α is an increasing function on (e, ∞). As $\alpha(e) = e - 1 > 0$, we have $\alpha(t) > 0$ for all $t > e$. Thus, $t > \ln t$ for $t > e$. Therefore, for all $t > e$, we have $\frac{1}{\ln t} > \frac{1}{t}$.

(e) From (d), it follows that for $x > e$,

$$f(x) = \int_e^x \frac{1}{\ln t} dt > \int_e^x \frac{1}{t} dt = \ln x - 1.$$

As $\lim_{x\to\infty} (\ln x - 1) = \infty$, we have $\lim_{x\to\infty} f(x) = \infty$.