NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2003-2004

MA1102R Calculus

April 2004 — Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **NINE (9)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions in **Section A**. Each question in Section A carries 10 marks.

3. Answer not more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1. [10 marks]

Evaluate the following limits.

(a) $\lim_{x\to 0^+} x^x$,

(b) $\lim_{x\to\pi} \dfrac{\cos x + |\cos x|}{(\pi - x)^2}$.

Question 2. [10 marks]

Using the $\epsilon - \delta$ definition of the limit, show that $\lim_{x\to 1}(3-2x) = 1$.

Question 3. [10 marks]

Let C be the curve with equation $x^2 + y^2 = y^3$.

(a) Find an equation of the tangent line to the curve C at the point $(2, 2)$.

(b) Find $\dfrac{d^2y}{dx^2}$ at the point $(2, 2)$.

Question 4. [10 marks]

Let $f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x^2 \sin\frac{1}{x} & \text{if } x > 0 \end{cases}$.

(a) Show that f is continuous at $x = 0$.

(b) Find $f'(x)$.

Question 5. [10 marks]

Find the area of the region bounded by the curve $y = \ln x$ and the line joining the points $(1, 0)$ and $(e^2, 2)$.

Question 6. [10 marks]

Evaluate the following integrals.

(a) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$,

(b) $\displaystyle\int \frac{x^2+x}{(x^2+1)(x-1)}\,dx$.

SECTION B

Answer not more than **two** *questions from this section. Each question in this section carries 20 marks.*

Question 7. [20 marks]

(a) Let $g(x) = x + \arcsin x$. Find the third degree Maclaurin polynomial of $g(x)$.

(b) Let $f : [0, \frac{\pi}{2}) \longrightarrow \mathbb{R}$ be defined by $f(x) = \displaystyle\int_{-\frac{\pi}{6}}^{x} \tan^3 t\,dt$.

(i) Show that f is increasing on $[0, \frac{\pi}{2})$.

(ii) Find $(f^{-1})'(0)$.

(iii) Determine the minimum value of f on $[0, \frac{\pi}{2})$.

Question 8. [20 marks]

Let $f(x) = \dfrac{x^2+x+7}{(2x+1)^{\frac{1}{2}}}$.

(a) Find, if any, the x- and y- intercepts of f.

(b) Show that f has a critical point at $x = 1$.

(c) Find the intervals on which f is (i) increasing, and (ii) decreasing.

(d) Find, if any, the local minima and local maxima of f.

(e) Find, if any, the intervals on which the graph of f is (i) concave upward, and (ii) concave downward.

(f) Find, if any, the points of inflection of the graph of f.

(g) Find, if any, the vertical and horizontal asymptotes of the graph f.

(h) Determine the range of f.

(i) Sketch the graph of f. Indicate clearly the local extrema and points of inflection, if any, on the graph of f.

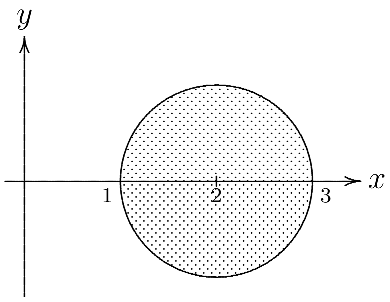
Question 9. [20 marks]

(a) Determine whether the improper integral

$$\int_2^\infty \frac{1}{x(\ln x)^3}\,dx$$

is convergent or not. Find its value if it is convergent.

(b) Find the volume of the solid obtained by revolving the circular disk bounded by the circle $(x-2)^2+y^2=1$ about the y-axis.



(c) Let f be a twice differentiable function defined on $\mathbb{R}$ such that $f''(x)>0$ for all x in (a,b). Suppose that $f(a)=f(b)=0$. Show that $f(x)<0$ for all x in (a,b).

END OF PAPER

Section A

1. (a) Let $y = x^x$. Then $\ln y = x \ln x$. Thus $\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} -x = 0$ by L'Hôpital's Rule.
Therefore, $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1$.

   (b) For $x \in (\frac{\pi}{2}, \frac{3\pi}{2}) \setminus \{\pi\}$, we have $\cos x < 0$ so that $\frac{\cos x + |\cos x|}{(\pi - x)^2} = \frac{\cos x - \cos x}{(\pi-x)^2} = 0$.
Thus, $\lim_{x\to\pi} \dfrac{\cos x + |\cos x|}{(\pi - x)^2} = \lim_{x\to\pi} 0 = 0$.

2. Given $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. Thus if $0 < |x-1| < \delta$, then $|(3-2x)-1| = |2-2x| = 2|x-1| < 2\delta = 2 \times \frac{\epsilon}{2} = \epsilon$. Therefore $\lim_{x\to 1}(3-2x) = 1$.

3. (a) By implicit differentiation, we have $2x + 2yy' = 3y^2y'$. Thus at the point $(2,2)$, we have $2(2) + 2(2)y' = 3(2^2)y'$ so that $y' = \frac{1}{2}$. Therefore, an equation of the tangent line to the curve C at the point $(2,2)$ is given by $y - 2 = \frac{1}{2}(x-2)$, or equivalently, $2y = x + 2$.

   (b) Differentiating both sides of $2x + 2yy' = 3y^2y'$ with respect to x, we have $2 + 2(y')^2 + 2yy'' = 6y(y')^2 + 3y^2y''$. At the point $(2,2)$, we have

$$2 + 2(\frac{1}{2})^2 + 2(2)y'' = 6(2)(\frac{1}{2})^2 + 3(2)^2y''.$$

Solving for y'', we obtain $y'' = -\frac{1}{16}$ at the point $(2,2)$.

4. (a) Clearly, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 0 = 0$. As $-x^2 \le x^2 \sin\frac{1}{x} \le x^2$ for all $x \ne 0$ and $\lim_{x\to 0^+} -x^2 = 0$ and $\lim_{x\to 0^+} x^2 = 0$, we have by squeeze theorem that

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 \sin\frac{1}{x} = 0.$$

Consequently, $\lim_{x\to 0} f(x) = 0 = f(0)$ and f is continuous at $x = 0$.

   (b) f is clearly differentiable at any point $x \ne 0$. Let's find $f'(0)$.
First $\lim_{x\to 0^-} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \dfrac{0-0}{x-0} = 0$.
Next we need to compute $\lim_{x\to 0^+} \dfrac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \dfrac{x^2\sin\frac{1}{x} - 0}{x} = \lim_{x\to 0^+} x\sin\frac{1}{x}$.
Note that $-|x| \le x\sin\frac{1}{x} \le |x|$ for all $x \ne 0$. As $\lim_{x\to 0^+} -|x| = 0$ and $\lim_{x\to 0^+} |x| = 0$, we have $\lim_{x\to 0^+} x\sin\frac{1}{x} = 0$ by squeeze theorem. Thus

$f'(0) = 0$. Consequently, $f'(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{if } x > 0 \end{cases}$

5. Note that the points $A(1,0)$ and $B(e^2, 2)$ lie on the curve $y = \ln x$. An equation of the line joining the points $A(1,0)$ and $B(e^2,2)$ is $y = 2(x-1)/(e^2-1)$. Thus the area between the curve $y = \ln x$ and the line AB is

$$\int_1^{e^2} \ln x - \frac{2(x-1)}{(e^2-1)}\,dx = \left[x \ln x - x - \frac{(x^2-2x)}{(e^2-1)}\right]_1^{e^2} = 2e^2 - e^2 + 1 - \frac{e^4 - 2e^2 + 1}{(e^2-1)}$$
$= e^2 + 1 - (e^2 - 1) = 2.$

6. (a) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int 2e^{\sqrt{x}}\,d(\sqrt{x}) = 2e^{\sqrt{x}} + C.$

(b) $\displaystyle\int \frac{x^2+x}{(x^2+1)(x-1)}\,dx = \int \frac{1}{x^2+1} + \frac{1}{x-1}\,dx = \tan^{-1} x + \ln|x-1| + C.$

7. (a) Given $g(x) = x + \arcsin x$. Then $g'(x) = 1 + (1-x^2)^{-\frac{1}{2}}$, $g''(x) = x(1-x^2)^{-\frac{3}{2}}$ and $g'''(x) = (1-x^2)^{-\frac{3}{2}} + 3x^2(1-x^2)^{-\frac{5}{2}} = (1+2x^2)(1-x^2)^{-\frac{5}{2}}$. Thus $g(0) = 0, g'(0) = 2, g''(0) = 0$ and $g'''(0) = 1$. Consequently, the third degree Maclaurin polynomial of $g(x)$ is

$$0 + \frac{2}{1!}x + \frac{0}{2!}x^2 + \frac{1}{3!}x^3 = 2x + \frac{1}{6}x^3.$$

(b) (i) Given $f(x) = \displaystyle\int_{-\frac{\pi}{6}}^{x} \tan^3 t\,dt$, with $0 \leq x < \frac{\pi}{2}$. By Fundamental Theorem of Calculus, $f'(x) = \tan^3(x) > 0$, for $x \in (0, \frac{\pi}{2})$. Hence f is (strictly) increasing on $[0, \frac{\pi}{2})$.

(ii) Since $\tan^3 x$ is an odd function on $[-\frac{\pi}{6}, \frac{\pi}{6}]$, we have $f(\frac{\pi}{6}) = \displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \tan^3 t\,dt =$ 0. Therefore, $(f^{-1})'(0) = \dfrac{1}{f'(\frac{\pi}{6})} = \dfrac{1}{\tan^3(\frac{\pi}{6})} = 3\sqrt{3}.$

(iii) Since f is increasing on $[0, \frac{\pi}{2})$, it attains its minimum value at $x = 0$.

Thus, Minimum value $= f(0) = \displaystyle\int_{-\frac{\pi}{6}}^{0} \tan^3 t\,dt = \int_{-\frac{\pi}{6}}^{0} (\sec^2 t - 1)\tan t\,dt$

$$= \int_{-\frac{\pi}{6}}^{0} \sec^2 t \tan t\,dt - \int_{-\frac{\pi}{6}}^{0} \tan t\,dt$$
$$= \left[\frac{1}{2}\tan^2 t\right]_{-\frac{\pi}{6}}^{0} - \Big[\ln|\sec t|\Big]_{-\frac{\pi}{6}}^{0}$$
$$= -\tfrac{1}{6} - \ln \tfrac{\sqrt{3}}{2}.$$

8. Note that the domain of f is $(-\frac{1}{2}, \infty)$. f is clearly differentiable and twice differentiable at each point in its domain. Let's first compute $f'(x)$ and $f''(x)$. We have

$$f'(x) = \frac{3(x+2)(x-1)}{(2x+1)^{\frac{3}{2}}} \quad \text{and} \quad f''(x) = \frac{3(x^2+x+7)}{(2x+1)^{\frac{5}{2}}}.$$

(a) When $x = 0$, $y = f(0) = 7$ so that y-intercept is 7. As $x^2+x+7 = (x-\frac{1}{2})^2+\frac{27}{4}$ is always positive, there is no solution for $f(x) = 0$ and thus no x-intercept.

(b) Note that for $x > -\frac{1}{2}$, $f'(x) = \dfrac{3(x+2)(x-1)}{(2x+1)^{\frac{3}{2}}} = 0$ if and only if $x = 1$. Therefore, f has a critical point at $x = 1$.

(c) From the expression of $f'(x)$, we see that $f'(x) > 0$ for x in $(1, \infty)$ and $f'(x) < 0$ for x in $(-\frac{1}{2}, 1)$. Therefore, f is decreasing on $(-\frac{1}{2}, 1]$ and is increasing on $[1, \infty)$.

(d) By the first derivative test, f has a local minimum at $x = 1$ and $f(1) = \frac{9}{\sqrt{3}} = 3\sqrt{3}$.

(e) From the expression of $f''(x)$, we see that $f''(x) > 0$ for all $x > -\frac{1}{2}$ because x^2+x+7 is always positive. Thus the graph of f is concave upward in $(-\frac{1}{2}, \infty)$.

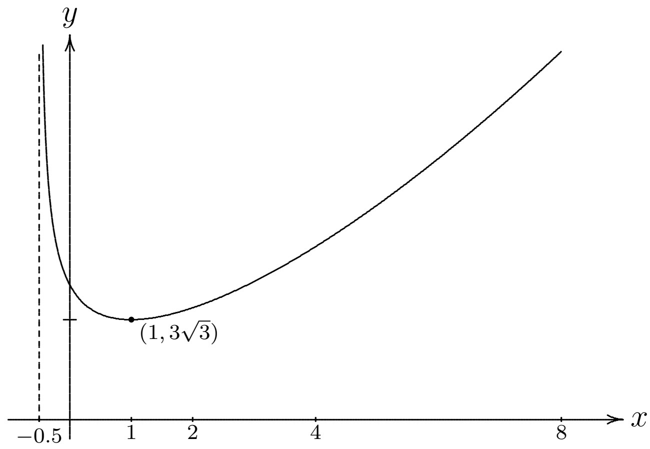
(f) As f is twice differentiable at each point in its domain and $f''(x)$ is never zero, there is no inflection point of the graph of f.

(g) Since $f(x) = \dfrac{x^2+x+7}{(2x+1)^{\frac{1}{2}}} > x^2+x+7$ for $x > 0$, and $\lim\limits_{x\to\infty} x^2+x+7 = \infty$, we have $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x^2+x+7}{(2x+1)^{\frac{1}{2}}} = \infty$. Therefore, there is no horizontal asymptote of the graph of f.

On the other hand, $\lim\limits_{x\to-\frac{1}{2}^+} f(x) = \lim\limits_{x\to-\frac{1}{2}^+} \dfrac{x^2+x+7}{(2x+1)^{\frac{1}{2}}} = \infty$. Thus $x = -\frac{1}{2}$ is a vertical asymptote of the graph of f.

(h) By (c), we know that f attains its absolute minimum value $3\sqrt{3}$ at $x = 1$. Moreover f is continuous in $(-\frac{1}{2}, \infty)$ with $\lim\limits_{x\to\infty} f(x) = \infty$. Thus we conclude that the range of f is $[3\sqrt{3}, \infty)$.

(i) The graph of f is shown below.



The graph of $f(x) = (x^2 + x + 7)/\sqrt{2x+1}$

9. (a) $$\int_2^\infty \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty}\left[-\frac{1}{2(\ln x)^2}\right]_2^b$$
$$= \lim_{b\to\infty}\left[-\frac{1}{2(\ln b)^2} + \frac{1}{2(\ln 2)^2}\right] = \frac{1}{2(\ln 2)^2}.$$

(b) The solid generated is a solid torus. The upper semi-circle has equation $y = \sqrt{1-(x-2)^2}$. Using the Shell Method, we have

$$\begin{aligned}\text{Volume} &= 2\times 2\pi\int_1^3 x\sqrt{1-(x-2)^2}\,dx\\ &= 4\pi\int_{-1}^1 (t+2)\sqrt{1-t^2}\,dt \qquad \text{using a substitution } t = x-2.\\ &= 4\pi\int_{-1}^1 t\sqrt{1-t^2}\,dt + 8\pi\int_{-1}^1 \sqrt{1-t^2}\,dt\\ &= -\frac{4\pi}{3}\left[(1-t^2)^{\frac{3}{2}}\right]_{-1}^1 + 8\pi \times \text{area of the semi-circle of radius 1}\\ &= -\frac{4\pi}{3}\times 0 + 8\pi\times\frac{\pi}{2} = 4\pi^2.\end{aligned}$$

(c) By Rolles' Theorem, there exists $c \in (a,b)$ such that $f'(c) = 0$. Since $f''(x) > 0$, we have $f'(x)$ is increasing on $[a,b]$. Thus, $f'(x) < 0$ for all $x \in [a,c)$ and $f'(x) > 0$ for all $x \in (c,b]$. Therefore, f is decreasing on $[a,c]$ and increasing on $[c,b]$. As $f(a) = f(b) = 0$, we must have $f(x) < 0$ for all $x \in (a,b)$.