NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER EXAMINATION FOR THE DEGREE OF B.SC.

SEMESTER 2

MA1102 CALCULUS

April/May 1998 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.
2. Answer not more than **FIVE (5)** questions.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

1. (a) Solve the following inequalities:

   (i) $|x+2|+|x-2| \leq 6$;

   (ii) $\dfrac{1}{x} \leq x$.

   (b) Consider the function $f(x) = [x^2]$, where $[t]$ denotes the greatest integer less than or equal to t.

   (i) Is f an odd function, an even function, or neither?

   (ii) Sketch the graph of f over the interval $[-2, 2]$.

2. Evaluate the following limits. If the limits do not exist, state why not.

   (i) $\displaystyle\lim_{x\to 0} \frac{\sqrt[3]{8+x}-2}{x}$;

   (ii) $\displaystyle\lim_{x\to 100} \left[\frac{x}{100}\right]$, where $[t]$ denotes the greatest integer less than or equal to $t \in \mathbb{R}$;

   (iii) $\displaystyle\lim_{x\to +\infty} x \sin x$;

   (iv) $\displaystyle\lim_{x\to +\infty} x \sin \frac{1}{x}$;

   (v) $\displaystyle\lim_{x\to 0} \frac{\tan 3x}{x}$;

   (vi) $\displaystyle\lim_{x\to 0} g(x)$ where $g(x) = \begin{cases} \sqrt{x}\,\sin\frac{1}{x} & x > 0, \\ x^2 \cos\frac{1}{x} & x < 0. \end{cases}$

3. (a) Evaluate the following:

   (i) $\displaystyle\lim_{x\to +\infty} \frac{2x+5}{\sqrt{x^2-4}}$;

   (ii) $\displaystyle\lim_{x\to -\infty} \frac{2x+5}{\sqrt{x^2-4}}$.

   (b) Let b denote a real constant, and suppose that

$$f(x) = \begin{cases} -3x+b & x \leq 1, \\ -3x^3+6x+3b & x > 1. \end{cases}$$

   (i) For what value of b will f be continuous for all real values of x? Explain.

   (ii) For the value of b found in (i), will f be a differentiable function for all real values of x? Explain.

4. (a) (i) State, without proof, the Intermediate Value Theorem.

(ii) Prove that there is a value $c \in (0, \frac{\pi}{2})$ such that

$$\cos c = c.$$

(b) Consider the function

$$f(x) = 2x + \frac{200}{x} \quad \text{for all} \quad x > 0.$$

(i) Find the minimum value of f over the interval $(0, \infty)$.

(ii) Does f have a maximum value over $(0, \infty)$? Explain.

5. (a) Suppose that

$$y^3 + xy^2 + xy + 1 = 0.$$

(i) Find $\dfrac{dy}{dx}$ in terms of x and y.

(ii) What is the equation of the tangent line to the curve passing through the point $(1, -1)$?

(b) (i) State, without proof, Rolle's Theorem.

(ii) Let f be a differentiable function on $\mathbb{R}$ such that $f(1) = 1$ and $f(2) = 4$. Show that there exists $c \in (1, 2)$ such that the tangent lines to the graph of f and the graph of $y = x^2$ at the point $x = c$ have the same slope.

6. (a) Evaluate the following limits with the use of L'Hôpital's Rule, or otherwise:

(i) $\displaystyle\lim_{x \to 0} \frac{2 - e^x - e^{-x}}{2x^2}$;

(ii) $\displaystyle\lim_{x \to +\infty} x \ln\left(\frac{x-1}{x+1}\right)$.

(b) Let $F(x) = \displaystyle\int_{\sin(x^2)}^{\cos x} e^{-t^2}\, dt$.

Find $F'(x)$.

7. (a) Suppose that f is a function defined on $[-1,1]$ such that

$$f'(x) = xe^{2x} \ \forall \ x \in [-1,1],$$

and $f(0) = -1$.

Find $f(x)$.

(b) Evaluate the following limits using Riemann sums, or otherwise:

(i) $\lim_{x\to\infty} \sum_{i=1}^{n} \frac{2i-n}{n^2}$;

(ii) $\lim_{x\to\infty} \sum_{i=1}^{n} \frac{1}{n} \sin \frac{\pi i}{n}$.

END OF PAPER