Answer and Guide To MA1102 Calculus 1997-98 Semester 2 Exam

1. (a) For this part you would want to 'remove' the modulus sign and use all the known properties for inequalities. Note that

$$|x-a| = \begin{cases} x-a, & x-a \geq 0 \\ -(x-a), & x-a < 0 \end{cases}.$$

Thus looking at

$$|x+2| + |x-2| \leq 6 \quad \text{---------------------------} \quad (*)$$

you would want to consider the real line to be the union of the following intervals, $(-\infty,-2), [-2,2), [2,\infty)$ and solve the inequality on each of the intervals.

For $x < -2$, (*) becomes $-(x+2)-(x-2) \leq 6$, i.e., $-2x \leq 6$. Thus $x \geq -3$. Thus the solution for this part is $[-3,-2)$.

For $-2 \leq x < 2$, the inequality becomes $x+2-(x-2) \leq 6$, i.e., $4 \leq 6$, which is always true. Therefore, the solution set for this part is [–2, 2).

For $x \geq 2$, the inequality becomes $x+2+(x-2) \leq 6$, i.e. $2x \leq 6$. This is just $x \leq 3$. Therefore, the solution for this part is [2, 3].

Hence the solution set for (*) is $[-3,-2) \cup [-2,2) \cup [2,3] = [-3,3]$.

(ii) $\frac{1}{x} \leq x$ --- (**)

First note that when $x = 0$, $1/x$ does not make sense and so 0 is not in the solution set of (**). Multiplying (**) by non zero x would have some consequence. It would change the direction of the inequality (**) when x is negative.

For $x > 0$ (**) becomes $1 \leq x^2 = |x|^2$. Taking square root gives $1 \leq |x|$. I.e. $x \geq 1$ or $x \leq -1$. Therefore, the solution set for this part is $(1,\infty)$.

For $x < 0$, (**) becomes $1 \geq x^2 = |x|^2$. Taking square root gives $|x| \leq 1$. I.e. $-1 \leq x \leq 1$. Therefore, the solution set for this part is [–1, 0).

Hence the solution set for (**) is $[-1,0) \cup (1,\infty)$.

(b) (i) Note that $f(1) = [1^2] = 1$ and $f(-1) = [(-1)^2] = 1$. So $f(-1) \neq -f(1)$. Therefore f is not an odd function. Obviously $f(x) = [x^2] = [(-x)^2] = f(-x)$ for all x in **R**. Therefore f is an even function.

(ii) To plot the graph of f over the interval [- 2, 2], we shall have to make the following calculation. Note that $-2 \le x \le 2 \Leftrightarrow |x| \le 2 \Leftrightarrow |x|^2 \le 4 \Leftrightarrow 0 \le x^2 \le 4$. Thus the image of $[-2, 2]$ under the squaring function is [0, 4] . Since the 'bracket' function is constant on the intervals [0, 1), [1,2), [2, 3), [3,4) and the point 4. We shall look for the pre-image of these intervals.

$0 \le x^2 < 1 \Leftrightarrow 0 \le |x|^2 < 1 \Leftrightarrow |x| < 1 \Leftrightarrow -1 < x < 1$. Therefore, we have

$-1 < x < 1 \Rightarrow 0 \le x^2 < 1 \Rightarrow f(x) = [x^2] = 0$.

$1 \le x^2 < 2 \Leftrightarrow 1 \le |x| < \sqrt{2} \Leftrightarrow -\sqrt{2} < x \le -1$ or $1 \le x < \sqrt{2}$. Therefore, we have

$-\sqrt{2} < x \le -1$ or $1 \le x < \sqrt{2} \Rightarrow 1 \le x^2 < 2 \Rightarrow f(x) = [x^2] = 1$. Similarly,

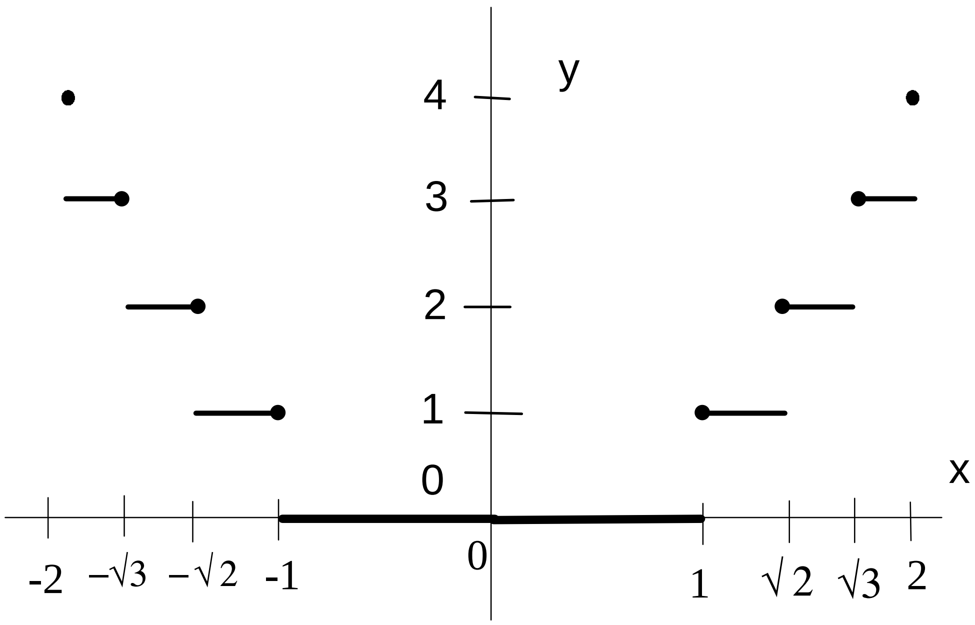
$2 \le x^2 < 3 \Leftrightarrow \sqrt{2} \le |x| < \sqrt{3} \Leftrightarrow -\sqrt{3} < x \le -\sqrt{2}$ or $\sqrt{2} \le x < \sqrt{3}$. Therefore,

$-\sqrt{3} < x \le -\sqrt{2}$ or $\sqrt{2} \le x < \sqrt{3} \Rightarrow 2 \le x^2 < 3 \Rightarrow f(x) = [x^2] = 2$. Likewise,

$3 \le x^2 < 4 \Leftrightarrow \sqrt{3} \le |x| < \sqrt{4} = 2 \Leftrightarrow -2 < x \le -\sqrt{3}$ or $\sqrt{3} \le x < 2$. Therefore,

$-2 < x \le -\sqrt{3}$ or $\sqrt{3} \le x < 2 \Rightarrow 3 \le x^2 < 4 \Rightarrow f(x) = [x^2] = 3$. Finally

$f(2) = f(-2) = [4] = 4$.



The graph of $f(x) = [x^2]$

2. You can use L'Hôpital's Rule here.

(i) $\lim_{x\to 0}\dfrac{\sqrt[3]{8+x}-2}{x}=\lim_{x\to 0}\dfrac{(\sqrt[3]{8+x}-2)((\sqrt[3]{8+x})^2+2\sqrt[3]{8+x}+4)}{x((\sqrt[3]{8+x})^2+2\sqrt[3]{8+x}+4)}$

$$=\lim_{x\to 0}\frac{(8+x-8)}{x((\sqrt[3]{8+x})^2+2\sqrt[3]{8+x}+4)}$$

$$=\lim_{x\to 0}\frac{1}{((\sqrt[3]{8+x})^2+2\sqrt[3]{8+x}+4)}=\frac{1}{4+4+4}=\frac{1}{12}.$$

Or you can use L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{\sqrt[3]{8+x}-2}{x}=\lim_{x\to 0}\frac{\frac{1}{3}(8+x)^{-2/3}}{1}=\frac{1}{3\cdot 8^{2/3}}=\frac{1}{12}.$$

(ii) $100<x<101\Rightarrow 1<\dfrac{x}{100}<\dfrac{101}{100}\Rightarrow\left[\dfrac{x}{100}\right]=1.$ Therefore, $\lim_{x\to 100^+}\left[\dfrac{x}{100}\right]=1.$

$99<x<100\Rightarrow 0<\dfrac{99}{100}<\dfrac{x}{100}<1\Rightarrow\left[\dfrac{x}{100}\right]=0.$ Therefore, $\lim_{x\to 100^-}\left[\dfrac{x}{100}\right]=0.$

Therefore, the limit $\lim_{x\to 100}\left[\dfrac{x}{100}\right]$ does not exist.

(iii) You will need to use the definition of the limit. Recall that we say the limit $\lim_{x\to\infty}g(x)$ exists if and only if we can find a real number L such that $\lim_{x\to\infty}g(x)=L$. Thus the limit $\lim_{x\to\infty}g(x)$ does not exist if and only if for any real number L $\lim_{x\to\infty}g(x)\neq L$. *Therefore, we must know what it means to say* $\lim_{x\to\infty}g(x)\neq L$. Recall $\lim_{x\to\infty}g(x)=L$ if and only if for any $\varepsilon>0$ we can find an integer N such that for all $x>N$ $|g(x)-L|<\varepsilon$. *Negating this statement means to say* $\lim_{x\to\infty}g(x)\neq L$ *if and only if* we can find an $\varepsilon>0$ *such that for any N we can find a particular* $x>N$ *such that* $|g(x)-L|\geq\varepsilon$.

Now for any L we can find an n_0 such that $(4n_0+1)\frac{\pi}{2}\geq L+1$. The ε we take here is 1. Thus for any N take n = max (N+1, n_0) and $x=(4n+1)\frac{\pi}{2}$. Then

$$(4n+1)\tfrac{\pi}{2}\geq(4n_0+1)\tfrac{\pi}{2}\geq L+1\Rightarrow x\sin(x)-L=(4n+1)\tfrac{\pi}{2}-L\geq 1.$$

This shows that $\lim_{x\to\infty}x\sin(x)\neq L$ for any L. Therefore the limit $\lim_{x\to\infty}x\sin(x)$ does not exist.

(iv) $\lim_{x\to\infty}x\sin(\frac{1}{x})=\lim_{x\to\infty}\dfrac{\sin(\frac{1}{x})}{\frac{1}{x}}=\lim_{x\to\infty}\dfrac{\cos(\frac{1}{x})\cdot(-\frac{1}{x^2})}{(-\frac{1}{x^2})}$ by L'Hôpital's Rule

$$=\lim_{x\to\infty}\cos(\tfrac{1}{x})=\cos(0)=1.$$

Or $\lim_{x\to\infty}x\sin(\frac{1}{x})=\lim_{1/x\to 0}\dfrac{\sin(\frac{1}{x})}{\frac{1}{x}}=1.$

(v) $\lim_{x\to 0}\frac{\tan(3x)}{x} = \lim_{x\to 0}\frac{\sin(3x)}{x\cos(3x)} = \lim_{x\to 0}\frac{\sin(3x)}{3x}\cdot\frac{3}{\cos(3x)} = 1\cdot\frac{3}{\cos(0)} = 3$ since $\lim_{x\to 0}\frac{\sin(3x)}{3x} = 1.$

Or $\lim_{x\to 0}\frac{\tan(3x)}{x} = \lim_{x\to 0}\frac{\sec^2(3x)\cdot 3}{1}$ by L'Hôpital's Rule

$$= \frac{\sec^2(0)\cdot 3}{1} = 3.$$

(vi) $g(x) = \begin{cases} \sqrt{x}\,\sin(\frac{1}{x}), & x > 0, \\ x^2\cos(\frac{1}{x}), & x < 0 \end{cases}$.

$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+}\sqrt{x}\,\sin(\frac{1}{x}) = 0$ by the Squeeze Theorem since $-\sqrt{x} \le \sqrt{x}\,\sin(\frac{1}{x}) \le \sqrt{x}$ for $x > 0$ and $\lim_{x\to 0^+}\sqrt{x} = 0.$

$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} x^2\cos(\frac{1}{x}) = 0$ by the Squeeze Theorem since $-x^2 \le x^2\cos(\frac{1}{x}) \le x^2$ for $x < 0$ and $\lim_{x\to 0^-} x^2 = 0.$

Therefore, $\lim_{x\to 0} g(x) = 0.$

3(a) (i) $\lim_{x\to\infty}\frac{2x+5}{\sqrt{x^2-4}} = \lim_{x\to\infty}\frac{(2x+5)/|x|}{\sqrt{x^2-4}\,/|x|} = \lim_{x\to\infty}\frac{(2x+5)/x}{\sqrt{x^2-4}\,/\sqrt{x^2}}$

since $|x| = \sqrt{x^2}$ and for $x > 0$, $|x| = x$

$$= \lim_{x\to\infty}\frac{2+5/x}{\sqrt{1-4/x^2}} = \frac{2+0}{\sqrt{1-0}} = 2.$$

(ii) $\lim_{x\to-\infty}\frac{2x+5}{\sqrt{x^2-4}} = \lim_{x\to-\infty}\frac{(2x+5)/|x|}{\sqrt{x^2-4}\,/|x|} = \lim_{x\to-\infty}\frac{(2x+5)/(-x)}{\sqrt{x^2-4}\,/\sqrt{x^2}}$

since $|x| = \sqrt{x^2}$ and for $x < 0$, $|x| = -x$

$$= \lim_{x\to-\infty}\frac{-2-5/x}{\sqrt{1-4/x^2}} = \frac{-2-0}{\sqrt{1-0}} = -2\ .$$

(b) $f(x) = \begin{cases} -3x+b, & x \le 1, \\ -3x^3+6x+3b, & x > 1 \end{cases}$

(i) For $x < 1$, f is a polynomial function and so f is continuous on $(-\infty, 1)$ since any polynomial function is continuous on the real numbers and so is continuous on any open interval. Similarly, for $x > 1$, f is also given by a polynomial function and so f is continuous on $(1, \infty)$. Now

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} -3x+b = -3+b = f(1)$ and

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} -3x^3+6x+3b = -3+6+3b = 3+3b.$

Therefore, for f to be continuous at $x = 1$, $-3 + b = 3 + 3b$. I.e. $b = -3$. Thus with this value of b, f is a continuous function on the whole of **R**.

(ii) We note that any polynomial function is differentiable on the whole of **R** and so is differentiable on any open interval. Thus our function is differentiable on each of the open intervals, $(-\infty, 1)$ and $(1, \infty)$. Now the limit

$$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{-3\,(1+h)^3 + 6(1+h) - 9 + 6}{h} = \lim_{h\to 0^+} \frac{-3\,h^3 - 9h^2 - 3h}{h} = -3$$

and $\lim_{h\to 0^-} \frac{-3(1+h) - 3 + 6}{h} = \lim_{h\to 0^-} \frac{-3h}{h} = -3$. Therefore, by the definition of derivative f is differentiable at $x = 1$ and f (1) = –3.

4. (a) (i) Refer to your lectures for the statement of the Intermediate Value theorem.

(ii) Let $g(x) = \cos(x) - x$. Then g is continuous on the closed and bounded interval $[0, \frac{\pi}{2}]$. Also $g(0) = \cos(0) - 0 = 1 > 0$ and $g(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) - \frac{\pi}{2} = -\frac{\pi}{2} < 0$. Therefore, by the Intermediate Value Theorem, there is a point c in the interval $(0, \frac{\pi}{2})$ such that $g(c) = 0$, i.e. $\cos(c) = c$.

(b) $f(x) = 2x + \frac{200}{x}$, for $x > 0$. Since f is a rational polynomial function for $x > 0$, f is differentiable for $x > 0$ and

$$f'(x) = 2 - \frac{200}{x^2} = 2\frac{x^2 - 100}{x^2} = \frac{2(x-10)(x+10)}{x^2} \text{ ------------------------(*).}$$

(i) For $0 < x < 10$, $x - 10 < 0$ and $x + 10 > 0$ so that f $(x) < 0$ by (*). Therefore, f is decreasing on (0, 10] since f is continuous at $x = 10$.

For $x > 10$, both $(x - 10)$ and $(x + 10)$ are positive so that f $(x) > 0$ by (*). Therefore, f is increasing on $[10, \infty)$ since f is continuous at $x = 10$.

Thus $f(10) = 2 \cdot 10 + \frac{200}{10} = 40$ is the minimum value of f on the interval $(0, \infty)$.

(ii) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 2x + \frac{200}{x} = \infty$ since $\lim_{x\to\infty} 2x = \infty$ and $\lim_{x\to\infty} \frac{200}{x} = 0$. Therefore f does not have a maximum value over $(0, \infty)$.

5(a) The equation is $y^3 + xy^2 + xy + 1 = 0$ -------------------------------------- (1)

Therefore, differentiating implicitly we get

$$3y^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} + y + x\frac{dy}{dx} = 0.$$

Thus $(3y^2+2xy+x)\dfrac{dy}{dx} = -(y+y^2)$. Therefore, if $3y^2+2xy+x \neq 0$, then

$$\frac{dy}{dx} = -\frac{y+y^2}{3y^2+2xy+x} \quad \text{----------------------------} \quad (2)$$

(ii) Note that the point (1, –1) satisfies equation (1). Therefore, the gradient of the tangent line to the curve at the point (1, –1) is given by (2) by substituting 1 for x and –1 for y. It is $-\dfrac{-1+(-1)^2}{3(-1)^2-2+1} = 0$. Therefore, the equation of the tangent line to the curve at the point (1, –1) is $y = -1$.

(b) (i). Refer to your lectures for the statement of Rolle's Theorem.

(ii) For the curve $y = x^2$, the gradient or slope of the tangent line at (x, x^2) is given by

$$\frac{dy}{dx} = 2x .$$

Now f is given to be a differentiable function defined on **R** and so it is continuous on **R**. The slope of the tangent line to the graph of f at $(x,\ f(x))$ is given by $f\ \ (x)$. Therefore, for these two tangent lines to have the same slope, we must have

$$f'(x) = 2x. \text{---------------------------------} (3)$$

We now consider the function $g(x) = f(x) - x^2$. Thus, to find a point c such that (3) holds is equivalent to finding a point c such that $g\ (c) = 0$. Recall that it is given that $f(1) = 1$ and $f(2) = 4$. Observe that g is continuous on the interval [1, 2] since the function f and x^2 are continuous on [1, 2] and that g is differentiable on (1, 2) since both f and x^2 are differentiable on (1, 2). Furthermore $g(1) = f(1) - 1^2 = 1 - 1 = 0$ and $g(2) = f(2) - 2^2 = 4 - 4 = 0$. Thus $g(1) = g(2)$. Therefore, the condition for Rolle's Theorem is satisfied. Hence by Rolle's Theorem there exists a point c in (1, 2) such that $g\ (c) = 0$. I.e., $f'(c) = 2c$.

6. (a) (i) $\displaystyle\lim_{x\to 0} \frac{2-e^x-e^{-x}}{2x^2}$

$\displaystyle = \lim_{x\to 0} \frac{-e^x+e^{-x}}{4x} = \lim_{x\to 0} \frac{-e^x-e^{-x}}{4}$ by two successive use of L'Hôpital's Rule

$\displaystyle = \frac{-1-1}{4} = -2$.

(ii) $\displaystyle\lim_{x\to\infty} x\ln\left(\frac{x-1}{x+1}\right) = \lim_{x\to\infty} x\ln\left(1-\frac{2}{x+1}\right) = \lim_{x\to\infty} \frac{\ln\left(1-\frac{2}{x+1}\right)}{\frac{1}{x}}$

$$=\lim_{x\to\infty}\frac{\frac{2}{(x+1)^2}\Big/\left(1-\frac{2}{x+1}\right)}{-\frac{1}{x^2}} \quad \text{by L'Hôpital's Rule}$$

$$=\lim_{x\to\infty}-\frac{2x^2}{(x+1)^2}\Big/\left(1-\frac{2}{x+1}\right)=-2\lim_{x\to\infty}\frac{1}{(1+\frac{1}{x})^2}\Big/\left(1-\frac{2}{x+1}\right)=-2\tfrac{1}{1}=-2.$$

(b) $$F(x)=\int_{\sin(x^2)}^{\cos(x)}e^{t^2}\,dt=\int_0^{\cos(x)}e^{t^2}\,dt+\int_{\sin(x^2)}^{0}e^{t^2}\,dt$$

$$=\int_0^{\cos(x)}e^{t^2}\,dt+\int_{\sin(x^2)}^{0}e^{t^2}\,dt \quad =\int_0^{\cos(x)}e^{t^2}\,dt-\int_0^{\sin(x^2)}e^{t^2}\,dt$$

$$=G(\cos(x))-G(\sin(x^2))$$

where $G(x)=\int_0^x e^{t^2}\,dt$

$$=G'(\cos(x))(-\sin(x))-G'(\sin(x^2))\cdot\cos(x^2)\cdot 2x$$

by the Chain Rule

$$=-\sin(x)e^{\cos^2(x)}-2x\cos(x^2)e^{\sin^2(x^2)}$$ by the Fundamental Theorem of Calculus.

7. (a) Since $f'(x)=xe^{2x}$ for all x in $[-1, 1]$, a candidate for f is given by

$$f(x)=\int xe^{2x}dx=\frac{1}{2}e^{2x}x-\frac{1}{2}\int e^{2x}dx \quad \text{by integration by parts}$$

$$=\frac{1}{2}e^{2x}x-\frac{1}{4}e^{2x}+C \quad \text{-------------------------} \quad (1)$$

Since, $f(0)=-1$, we have

$$f(0)=\frac{1}{2}e^{2\cdot 0}\cdot 0-\frac{1}{4}e^{2\cdot 0}+C=-\frac{1}{4}+C=-1\ .$$

Therefore, $C=-1+\frac{1}{4}=-\frac{3}{4}$. Thus $f(x)=\frac{1}{2}e^{2x}x-\frac{1}{4}e^{2x}-\frac{3}{4}$.

(b) (i) Write the following as a Riemann sum

$$\sum_{i=1}^{n}\frac{2i-n}{n^2}=\sum_{i=1}^{n}\frac{1}{n}\cdot\left(\frac{2i}{n}-1\right)=\sum_{i=1}^{n}f(x_i)\Delta x\ ,$$

where $x_0<x_1<\cdots<x_n$ is a regular partition and $\Delta x=\Delta x_i=x_i-x_{i-1}$. Therefore, we can take $x_i=\frac{i}{n}$ so that $\Delta x=\frac{1}{n}, x_0=0$ and $x_n=1$. Thus by comparing

$f(x_i)\Delta x$ with $\frac{1}{n}\cdot\left(\frac{2i}{n}-1\right)=\Delta x\left(\frac{2i}{n}-1\right)$,

we would want $f(x_i)=\frac{2i}{n}-1=2x_i-1$. Thus $f(x)=2x-1$. Therefore,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2i-n}{n^2}=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\Delta x=\int_0^1(2x-1)dx=[x^2-x]_0^1=0.$$

You do not have to use Riemann sum to find this limit. E.g.,

$$\sum_{i=1}^{n}\frac{2i-n}{n^2}=\frac{2}{n^2}\sum_{i=1}^{n}i-\sum_{i=1}^{n}\frac{1}{n}=\frac{2}{n^2}\sum_{i=1}^{n}i-\frac{1}{n}\sum_{i=1}^{n}1=\frac{2}{n^2}\cdot\frac{n}{2}(n+1)-\frac{1}{n}\cdot n$$

$$=1+\frac{1}{n}-1=\frac{1}{n}.$$

Therefore , $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2i-n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0.$

(ii) Write the following as a Riemann sum

$$\sum_{i=1}^{n} \frac{1}{n} \sin(\pi \cdot \frac{i}{n}) = \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $x_0 < x_1 < \cdots < x_n$ is a regular partition and $\Delta x = \Delta x_i = x_i - x_{i-1}$. Therefore, we can take $x_i = \frac{i}{n}$ so that $\Delta x = \frac{1}{n}, x_0 = 0$ and $x_n = 1$. Thus by comparing

$$f(x_i)\Delta x \text{ with } \frac{1}{n} \sin(\pi \cdot \frac{i}{n}) = \Delta x \sin(\pi \cdot \frac{i}{n}) \ ,$$

we would want $f(x_i) = \sin(\pi \cdot \frac{i}{n}) = \sin(\pi x_i)$. Thus $f(x) = \sin(\pi x)$. Therefore,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \sin(\pi \cdot \frac{i}{n}) = \int_0^1 \sin(\pi x)dx = \left[-\frac{1}{\pi}\cos(\pi x)\right]_0^1$$

$$= -\frac{1}{\pi}\cos(\pi) + \frac{1}{\pi}\cos(0) = -\frac{1}{\pi}\cdot(-1) + \frac{1}{\pi} = \frac{2}{\pi}.$$