Answer and Comment to NUS MA3110 Mathematical Analysis II Sem 1 2010-2011

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Q1 [20 points] Recall that S := lim $\sup_{n\to\infty} a_n$:= $\lim_{n\to\infty}(\sup_{m\geq n} a_m)$ is the largest limit point of the sequence (a_n). Let (a_n) and (b_n) be two sequences.
(i) Prove that $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.
(ii) Construct a concrete example where the inequality in (i) is strict.

Solution.

First some comment about this question.

The first statement about lim sup being the largest limit point of the sequence is not helpful. It needed to be explained what the largest limit point means. First of all there may not be a largest limit point. It begs for the sequence to be bounded above but is certainly not stated in the question. The setter seemed somewhat sloppy in stating the inequality in (i).

Since there is no condition on the sequences (a_n) and (b_n), the inequality in (i) will not make any sense if $\limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$ is of the form $(+\infty)+(-\infty)$ or $(-\infty)+(+\infty)$.

So we will prove (i) under the hypothesis that $\limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$ is not of the form $(+\infty)+(-\infty)$ or $(-\infty)+(+\infty)$.

First of all, if $\limsup_{n\to\infty} a_n = +\infty$ or if $\limsup_{n\to\infty} b_n = +\infty$, then we have nothing to prove.

If $\limsup_{n\to\infty} a_n = -\infty$, since $\sup_{m\geq n} a_m \geq a_n$, $a_n \to -\infty$. By assumption $\limsup_{n\to\infty} b_n \neq +\infty$ and so (b_n) is bounded above, therefore, $a_n + b_n \to -\infty$. This means, $\limsup_{n\to\infty}(a_n + b_n) = -\infty$. And we have nothing to prove.

Similarly, if $\limsup_{n\to\infty} b_n = -\infty$, we have nothing to prove.

So we may assume that both $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$ are real numbers. For any integer $k \geq 1$,

$\alpha_k = \sup\{a_n : n \geq k\} \geq a_n$ and $\beta_k = \sup\{b_n : n \geq k\} \geq b_n$ for all for $n \geq k$. Therefore,

$$a_n + b_n \leq \alpha_k + \beta_k \text{ for all } n \geq k .$$

Hence, $\alpha_k + \beta_k$ is an upper bound for the set $\{a_n + b_n : n \geq k\}$ and so by the definition of supremum,

$$\gamma_k = \sup\{a_n + b_n : n \geq k\} \leq \alpha_k + \beta_k .$$

It follows that $\limsup_{n\to\infty}(a_n + b_n) = \lim_{k\to\infty}\gamma_k \leq \lim_{k\to\infty}\alpha_k + \lim_{k\to\infty}\beta_k = \limsup_{n\to\infty}(a_n) + \limsup_{n\to\infty}(b_n)$.

This completes the proof of part (i)

For a proof of this using the equivalent definition of lim sup a_n being the supremum of all subsequential limits, that is the supremum of the set consisting of all possible limits of subsequence of (a_n), see my article, *All about lim sup and lim inf* .

(ii) It is easy to construct an example giving strict inequality.

Let $a_n = \begin{cases} 1, & n \text{ even,} \\ -1, & n \text{ odd} \end{cases}$ and $b_n = \begin{cases} -\frac{1}{2}, & n \text{ even,} \\ 1, & n \text{ odd} \end{cases}$. Then $a_n + b_n = \begin{cases} \frac{1}{2}, & n \text{ even,} \\ 0, & n \text{ odd} \end{cases}$. And so

$$\limsup_{n\to\infty}(a_n + b_n) = \frac{1}{2} < \limsup_{n\to\infty}(a_n) + \limsup_{n\to\infty}(b_n) = 1 + 1 = 2 .$$

We may also take b_n to be $-a_n$.

Note that question 1 is about the property of lim sup. This is just basic concept.

Q 2 [20 points] Suppose that a function $f : \mathbf{R} \to \mathbf{R}$ has $f(0) = 0$, and $g(x) := |f(x)|$. Prove:
(i) If $g'(0)$ exists, then $g'(0) = 0$, and $f'(0)$ also exists with $f'(0) = 0$;
(ii) If $f'(0)$ exists, then $g'(0)$ exists if and only if $f'(0) = 0$, in which case $g'(0) = 0$.

Solution

Comment.

This question is about limit of function and its properties, testing very basic skill.

(i) Note that $g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} \frac{|f(x)|}{x}$.

If $g'(0)$ exists, let $g'(0) = \alpha$. Hence $\lim_{x\to 0} \frac{|f(x)|}{x} = \alpha$.

If $\alpha > 0$, then there exists $\delta > 0$ such that $|x| < \delta \Rightarrow \left| \frac{|f(x)|}{x} - \alpha \right| < \frac{\alpha}{2} \Rightarrow \frac{|f(x)|}{x} > \frac{\alpha}{2} > 0$.

But for $-\delta < x < 0$, $\frac{|f(x)|}{x} \le 0$. This contradiction shows that $\alpha \le 0$.

If $\alpha < 0$, then there exists $\delta > 0$ such that $|x| < \delta \Rightarrow \left| \frac{|f(x)|}{x} - \alpha \right| < \frac{|\alpha|}{2} \Rightarrow \frac{|f(x)|}{x} < \frac{\alpha}{2} < 0$.

This time for $0 < x < \delta$, $\frac{|f(x)|}{x} \ge 0$. So we have again a contradiction. This means $\alpha \ge 0$.

Hence $g'(0) = \alpha = 0$.

(ii)

Suppose $f'(0)$ exists. Then $\lim_{x\to 0} \frac{f(x)}{x}$ exists.

If $g'(0)$ exists, then by part (i), $f'(0)=0$ and $g'(0)=0$.

Conversely, suppose $f'(0)=0$. Then $\lim_{x\to 0}\frac{f(x)}{x}=0$. Therefore, $\lim_{x\to 0}\left|\frac{|f(x)|}{x}\right|=0$.

This implies that $\lim_{x\to 0}\frac{|f(x)|}{x}=0$. Hence, $g'(0)$ exists and $g'(0)=0$.

Q 3 [20 points] Let $f:[0,1]\to \mathbf{R}$ satisfy the condition that $\left|f(x)-f(y)\right|\le\sqrt{|y-x|}$ for all x, $y\in$ [0, 1]. Use this property to prove that $\lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}=\int_0^1 f(x)dx$, i.e., $\forall\varepsilon>0,\exists N\in\mathbb{N}$ such that $\forall n\ge N$, $\left|\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}-\int_0^1 f(x)dx\right|\le\varepsilon$.

Solution.

Comment.

You can use the Riemann sum convergence property or the inequality.

Observe that the function is continuous on [0.1].

Given any $\varepsilon > 0$, take $\delta=\varepsilon^2$. Then

$$|x-y|<\delta=\varepsilon^2\Rightarrow|f(x)-f(y)|\le\sqrt{|y-x|}<\sqrt{\varepsilon^2}=\varepsilon .$$

This implies that f is uniformly continuous and hence continuous on [0.1]. Thus, f is Riemann integrable.

Note that $\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}$ is a Riemann sum for f with respect to a uniform partition of [0, 1] into n equal subintervals. Plainly the norm of the partition is $1/n$ and the norm tends to 0 as n tends to infinity. Thus, $\lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}=\int_0^1 f(x)dx$. (*This is always true for any Riemann integrable function f on* [0,1].)

Alternatively, consider the integral over each subinterval $[x_{i-1},x_i]$, where $x_i=\frac{i}{n}$. Since we know f is Riemann integrable over [0, 1],

$$\int_0^1 f(x)dx=\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(x)dx .$$

Observe that for all x in $[x_{i-1}, x_i]$, $|f(x) - f(x_i)| \le \sqrt{|x - x_i|} \le \frac{1}{\sqrt{n}}$. Hence, for all x in $[x_{i-1}, x_i]$,

$$f(x_i) - \frac{1}{\sqrt{n}} \le f(x) \le f(x_i) + \frac{1}{\sqrt{n}}.$$

Therefore, taking integrals we have,

$$\int_{x_{i-1}}^{x_i} \left(f(x_i) - \frac{1}{\sqrt{n}} \right) dx \le \int_{x_{i-1}}^{x_i} f(x) dx \le \int_{x_{i-1}}^{x_i} \left(f(x_i) + \frac{1}{\sqrt{n}} \right) dx.$$

Since $x_i - x_{i-1} = \frac{1}{n}$, we get

$$f(x_i)\frac{1}{n} - \frac{1}{n\sqrt{n}} \le \int_{x_{i-1}}^{x_i} f(x) dx \le f(x_i)\frac{1}{n} + \frac{1}{n\sqrt{n}}.$$

Summing from $i = 1$ to n we obtain,

$$\sum_{i=1}^{n} f(x_i)\frac{1}{n} - \frac{1}{\sqrt{n}} \le \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x) dx \le \sum_{i=1}^{n} f(x_i)\frac{1}{n} + \frac{1}{\sqrt{n}},$$

i.e.,

$$\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} - \frac{1}{\sqrt{n}} \le \int_0^1 f(x) dx \le \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} + \frac{1}{\sqrt{n}}.$$

This means

$$\left| \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} - \int_0^1 f(x) dx \right| \le \frac{1}{\sqrt{n}}.$$

Since $\frac{1}{\sqrt{n}} \to 0$, i.e., $\lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} = \int_0^1 f(x) dx$.

(*Remember that this is always true for any Riemann integrable function f and not just for this function with this property. The use of this inequality does require that f be Riemann integrable over each subinterval and so the above proof is not a direct proof of the Riemann integrability of the function f for it uses Riemann integrability. The use of the inequality is but a demonstration of the fact that the Riemann sum converges to the integral of f which we already knew as f is Rieamnn integrable.*

You would not be expected to weave the given inequality into some sort of lower and upper Riemann sums into a proof of integrability of f. This would be too long, technically of little value and time consuming in an exam setting.)

Q 4 [20 points] The p-series test states that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$, and divergent if $p \in [0, 1]$.
Use Taylor expansion, together with the p-series test, to prove that:

(i) If $p > 1$, then $\sum_{n=1}^{\infty} \sin\left(\frac{x^p}{n^p}\right)$ is uniformly convergent on $[-r, r]$ for any $r > 0$.

(ii) If $p \in [0, 1]$, then $\sum_{n=1}^{\infty} \sin\left(\frac{x^p}{n^p}\right)$ is divergent at any $x \neq 0$.

Comment.
For fractional p and not a whole number, x^p may not be defined for negative number x. The setter had overlooked this fact. We shall add in the additional condition for part(i) " whenever x^p is defined in **R"** *and for part (ii) delete "at any $x \neq 0$" and replaced by " for $x > 0$".*

(i) Note that whenever x^p is defined (i.e., x^p is a real number), $\left|\sin\left(\frac{x^p}{n^p}\right)\right| \leq \frac{|x|^p}{n^p}$.

Therefore, for any $r > 0$, $|x| \leq r \Rightarrow \left|\sin\left(\frac{x^p}{n^p}\right)\right| \leq \frac{|x|^p}{n^p} \leq \frac{r^p}{n^p}$. It follows then by the Weierstrass M-test that for $p > 1$, and if x^p is defined also for $x < 0$, $\sum_{n=1}^{\infty} \sin\left(\frac{x^p}{n^p}\right)$ is uniformly convergent on $[-r, r]$ since $\sum_{n=1}^{\infty} \frac{r^p}{n^p} = r^p \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$. If x^p is not defined for $x < 0$, then $\sum_{n=1}^{\infty} \sin\left(\frac{x^p}{n^p}\right)$ is uniformly convergent on $[0, r]$ for $p > 1$.

(ii) We can use the limit comparison test here. We assume that $x > 0$ for x^p to be defined for $p \in [0, 1]$.

Then there exists an integer N such that $N > x$. Consequently, for all $n \geq N$, $\frac{x^p}{n^p} \leq 1$ and so $\sin\left(\frac{x^p}{n^p}\right) > 0$. Note that $\lim_{n \to 0} \sin\left(\frac{x^p}{n^p}\right) \Big/ \left(\frac{x^p}{n^p}\right) = 1 > 0$. And so by the Limit Comparison Test,

$$\sum_{n=N}^{\infty} \sin\left(\frac{x^p}{n^p}\right) \text{ is convergent, if and only if, } \sum_{n=N}^{\infty} \frac{x^p}{n^p} \text{ is convergent.}$$

Since $\sum_{n=N}^{\infty} \frac{x^p}{n^p} = x^p \sum_{n=N}^{\infty} \frac{1}{n^p}$ is divergent for $p \in [0, 1]$, $\sum_{n=N}^{\infty} \sin\left(\frac{x^p}{n^p}\right)$ is divergent for $p \in [0, 1]$.

Hence, $\sum_{n=1}^{\infty} \sin\left(\frac{x^p}{n^p}\right)$ is divergent for $p \in [0, 1]$.

Alternatively, if we decline to use the Limit Comparison Test, we can use a comparison test.

Since for $y > 0$, by Taylor's Theorem, $\sin(y) = y - \frac{y^3}{6} + \frac{1}{120}\cos(\alpha)$ for some α between 0 and y, and so for $0 < y \le \pi/2$, $\sin(y) \ge y - \frac{y^3}{6}$. (This inequality is actually true for all $y \ge 0$ but we don't need this fact here.) Hence, for $\frac{x^p}{n^p} \le 1 < \frac{\pi}{2}$,

$$\sin\left(\frac{x^p}{n^p}\right) \ge \frac{x^p}{n^p} - \frac{1}{6}\frac{x^{3p}}{n^{3p}} = \frac{x^p}{n^p}\left\{1 - \frac{1}{6}\frac{x^{2p}}{n^{2p}}\right\}.$$

Thus, taking any $N > x$, for any integer $m \ge N$,

$$\sum_{n=N}^{m} \sin\left(\frac{x^p}{n^p}\right) \ge \sum_{n=N}^{m} \frac{x^p}{n^p}\left\{1 - \frac{1}{6}\frac{x^{2p}}{n^{2p}}\right\} \ge \frac{1}{2}\sum_{n=N}^{m}\frac{x^p}{n^p}.$$

Since $\frac{1}{2}\sum_{n=N}^{\infty}\frac{x^p}{n^p} = \frac{x^p}{2}\sum_{n=N}^{\infty}\frac{1}{n^p}$ is divergent for $p \in [0, 1]$, $\sum_{n=N}^{\infty}\sin\left(\frac{x^p}{n^p}\right)$ is divergent for $p \in [0, 1]$.

Consequently, $\sum_{n=1}^{\infty}\sin\left(\frac{x^p}{n^p}\right)$ is divergent for $p \in [0, 1]$.

Q 5 [15 points] Prove that $f(x) := \sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{x}{\sqrt{n}}\right)$ is uniformly convergent on **R**, and

$f'(x) = \sum_{n=1}^{\infty} n^{-\frac{3}{2}}\cos\left(\frac{x}{\sqrt{n}}\right)$ for all $x \in$ **R**.

Solution

Comment.

Erroneous question. The setter is apparently unaware that $\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{x}{\sqrt{n}}\right)$ *is not uniformly convergent on **R**. We shall change the question to the following: Prove that* $\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{x}{\sqrt{n}}\right)$ *is uniformly convergent on [–K, K] for any K > 0, converges pointwise but not uniformly on **R**. This is a somewhat more difficult question than usually seen in an examination.*

Take any $K > 0$. For $|x| \le K$, $\left|\frac{1}{n}\sin\left(\frac{x}{\sqrt{n}}\right)\right| \le \frac{|K|}{n^{3/2}}$. Since $\sum_{n=1}^{\infty}\frac{|K|}{n^{3/2}} = |K|\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, being $|K|$ times a convergent p-series, is convergent, by the Weierstrass M Test, $\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{x}{\sqrt{n}}\right)$ converges uniformly on $[-K, K]$. Since each term $\frac{1}{n}\sin\left(\frac{x}{\sqrt{n}}\right)$ is continuous on **R**, $\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{x}{\sqrt{n}}\right)$ converges to a continuous function f on $[-K,\ K]$ for any $K > 0$. This means $\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{x}{\sqrt{n}}\right)$ converges pointwise to a continuous function on **R**.

The derived series of $\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{x}{\sqrt{n}}\right)$ is $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\cos\left(\frac{x}{\sqrt{n}}\right)$. By the same reasoning as above, $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\cos\left(\frac{x}{\sqrt{n}}\right)$ converges uniformly to a continuous function g on $[-K, K]$ for any $K > 0$. Therefore, $f'(x) = g(x)$ for all x in $(-K, K)$. Since K is arbitrary, f is differentiable on **R** and $f'(x) = g(x)$ for all x in **R**.

I have shown that $\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{x}{\sqrt{n}}\right)$ does not converge uniformly on **R** in my article, *Convergence of* $\sum\sin(\sqrt{n}x)/n$ *and other problems,* in My Calculus Web. I reproduce the answer below.

We shall use the inequality $\sin(x) \geq x - \frac{x^3}{6}$ for $x \geq 0$.

Take any integer $N > 1$. Let $x_N = \sqrt{N}$. Then

$$\sin\left(\frac{x_N}{\sqrt{n}}\right) \geq \frac{x_N}{\sqrt{n}} - \frac{1}{6}\left(\frac{x_N}{\sqrt{n}}\right)^3 = \frac{x_N}{\sqrt{n}} - \frac{1}{6}\frac{x_N^3}{n\sqrt{n}} = \frac{\sqrt{N}}{\sqrt{n}} - \frac{1}{6}\frac{N\sqrt{N}}{n\sqrt{n}} \quad \text{------- (1)}$$

Hence,

$$\frac{1}{n}\sin\left(\frac{x_N}{\sqrt{n}}\right) \geq \frac{\sqrt{N}}{n\sqrt{n}} - \frac{1}{6}\cdot\frac{N\sqrt{N}}{n^2\sqrt{n}} \quad \text{---------------------- (2)}$$

Therefore,

$$\sum_{n=N}^{2N}\frac{1}{n}\sin\left(\frac{x_N}{\sqrt{n}}\right) \geq \sum_{n=N}^{2N}\frac{\sqrt{N}}{n\sqrt{n}} - \frac{1}{6}\sum_{n=N}^{2N}\frac{N\sqrt{N}}{n^2\sqrt{n}} \quad \text{---------------------- (3)}$$

Observe that

$$\sum_{n=N}^{2N}\frac{\sqrt{N}}{n\sqrt{n}} \geq \sum_{n=N}^{2N}\frac{\sqrt{N}}{2N\sqrt{2N}} = \frac{1}{2\sqrt{2}}\sum_{n=N}^{2N}\frac{1}{N} \geq \frac{1}{2\sqrt{2}}\cdot\frac{N+1}{N} = \frac{1}{2\sqrt{2}}\left(1+\frac{1}{N}\right) \quad \text{---- (4)}$$

and

$$\frac{1}{6}\sum_{n=N}^{2N}\frac{N\sqrt{N}}{n^2\sqrt{n}} \leq \frac{1}{6}\sum_{n=N}^{2N}\frac{N\sqrt{N}}{N^2\sqrt{N}} = \frac{1}{6}\sum_{n=N}^{2N}\frac{1}{N} = \frac{1}{6}\left(1+\frac{1}{N}\right) \quad \text{--------(5)}$$

It follows from (4) and (5) that (3) becomes

$$\sum_{n=N}^{2N}\frac{1}{n}\sin\left(\frac{x_N}{\sqrt{n}}\right) \geq \frac{1}{2\sqrt{2}}\left(1+\frac{1}{N}\right) - \frac{1}{6}\left(1+\frac{1}{N}\right) = \frac{1}{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{3}\right)\left(1+\frac{1}{N}\right)$$
$$\geq \frac{1}{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{3}\right) > 0.$$

This means that for any $N > 1$,

$$\sum_{n=N}^{\infty}\frac{1}{n}\sin\left(\frac{x_N}{\sqrt{n}}\right) \geq \frac{1}{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{3}\right).$$

Hence, $M_N = \sup\left\{\sum_{n=N}^{\infty}\frac{1}{n}\sin\left(\frac{x}{\sqrt{n}}\right) : x \in \mathrm{R}\right\} \geq \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{3}\right)$.

Therefore, M_N does not tend to 0 as N tends to infinity. Consequently $\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{x}{\sqrt{n}}\right)$ cannot converge uniformly on **R**.

Q 6 [10 points] Prove that the series $f(x) := \sum_{n=1}^{\infty}\frac{\cos(\sqrt{n}x)}{n}$ is divergent at every $x \in$ **R**.

However, show that $g(x) := \sum_{n=1}^{\infty}\frac{\sin(\sqrt{n}x)}{n}$ is pointwise convergent on **R**.

Solution

Comment
This is the most difficult question and requires careful handling of the terms of the series. The key is to lump terms of the same sign together. This requires technical prowess beyond the normal repertoire of undergraduate. Some sloppy, inaccurate or incorrect solutions had been published. A complete detail answer to this question is given in my article, Convergence of $\sum\sin(\sqrt{n}x)/n$ *and other problems, in My Calculus Web. It is Problem 1 and Problem 4 there, where we prove the equivalent statement* $\sum_{n=1}^{\infty}\frac{\sin(\sqrt{n}x\pi)}{n}$ *is pointwise convergent on* **R** *and* $\sum_{n=1}^{\infty}\frac{\cos(\sqrt{n}x\pi)}{n}$ *is divergent at every* $x \in$ **R**.

See
https://my-calculus-web.firebaseapp.com/uniform%20convergence/Problems_uniform%20Convergence.pdf