Solution to MA3110 Analysis Semester 2 2005/06

Question 1

(a) Let S be a non-empty set of positive real numbers with the following property: There exists a positive real constant C such that for any positive integer n and any n distinct elements $s_1, s_2, \ldots, s_n$ in S, we have $s_1 + s_2 + \ldots + s_n \le C$.

(i) Show that for every positive integer k, the set $S_k = \{ s \in S : s \ge C/k \}$ has at most k elements.

(ii) Deduce that S is countable.

(b) Let (a_n) be a bounded sequence of real numbers. You may use the fact that $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} b_n$, where $b_n = \sup \{a_n, a_{n+1}, a_{n+2}, \ldots\}$. Denote $\limsup_{n\to\infty} x_n$ by a.

(1) Prove that for every $x > a$, there exists $N \in \mathbf{N}$ such that $a_n < x$ for all $n \ge N$.

(2) Prove that for every $y < a$, there exist infinitely many n in $\mathbf{N}$ such that $y < a_n$

(3) Prove that conversely, if a is any real number satisfying both conditions in (1) and (2) above, then $a = \limsup_{n\to\infty} x_n$.

Answer

(a) This is a countability argument.

S consists of positive real numbers for which any n distinct elements in S has their sum $\le C$ for some constant C.

(i) $S_k = \{ s \in S : s \ge C/k \}$ for each positive integer.

Then S_k has at most k elements.

Suppose on the contrary that S_k has more than k elements, i.e., $|S_k| = n > k$.

Let $S_k = \{ s_1, s_2, \ldots, s_n\}$. Then $s_1 + s_2 + \ldots + s_n > n \times C/k > C$ since $n/k > 1$. But since $S_k \subseteq S$,

$s_1 + s_2 + \ldots + s_n \le C$. This contradicts $s_1 + s_2 + \ldots + s_n > C$ and so S_k has at most k elements.

(ii)

Take any s in S. Then by the Archimedean property of $\mathbf{R}$, there exists a positive integer N such that

$N s > C$. Therefore, $s > C/N$ and so $s \in S_N \subseteq \cup\{ S_k : k \in \mathbf{N} \}$. That means $S \subseteq \cup\{ S_k : k \in \mathbf{N} \}$. Since each $S_k \subseteq S$, $\cup\{ S_k : k \in \mathbf{N} \} \subseteq S$. Therefore, $S = \cup\{ S_k : k \in \mathbf{N} \}$. Thus S is a countable union of sets each of which is finite (and so countable). It follows that S is countable.

(b) This is about the property of limit superior.

Given that (a_n) is a bounded sequence. Now we recall why $\limsup_{n\to\infty} x_n$ exists.

For each positive integer n, let b_n be the *supremum* or the *least upper bound* of the set $\{a_n, a_{n+1}, a_{n+2}, \ldots\}$. This exists because $\{a_n, a_{n+1}, a_{n+2}, \ldots\}$ is bounded since the sequence (a_n) is bounded. Plainly (b_n) is bounded also. (If $L < a_n < K$ for all n, then $L \le b_n \le K$ because L and K are also lower and upper bounds for $\{a_n, a_{n+1}, a_{n+2}, \ldots\}$.) Now (b_n) is a decreasing sequence. This is seen as follows. For $n > m$, $\{a_n, a_{n+1}, a_{n+2}, \ldots\} \subseteq \{a_m, a_{n+1}, a_{m+2}, \ldots\}$. Therefore,

$$b_n = \sup \{a_n, a_{n+1}, a_{n+2}, \ldots\} \le \sup \{a_m, a_{n+1}, a_{m+2}, \ldots\} = b_m .$$

Thus, since (b_n) is bounded below, by the Monotone Convergence Theorem, (b_n) is convergent. The limit of (b_n) is $\limsup_{n\to\infty} x_n$. In particular, the limit is the infimum of $\{b_1, b_2, b_3, \ldots\}$. Let $\lim_{n\to\infty} b_n = a$. Then $\limsup_{n\to\infty} x_n = a$.

(1) Take any $x > a$. Then $x > \inf\{b_1, b_2, b_3, \dots\}$. Thus, there exists an element k in $\{b_1, b_2, b_3, \dots\}$ such that $x > k \geq \inf\{b_1, b_2, b_3, \dots\} = a$ by the definition of infimum or greatest lower bound. Then there exists an integer N such that $k = b_N$. Therefore, for all $n \geq N$,

$$x > b_N = \sup\{a_N, a_{N+1}, a_{N+2}, \dots\} \geq a_n .$$

Hence for all $n \geq N$, $x > a_n$.

(2) Take $y < a$. Then since a is the greatest lower bound or infimum of $\{b_1, b_2, b_3, \dots\}$, y is a lower bound of $\{b_1, b_2, b_3, \dots\}$. That is $y < b_k$ for all positive integer k.

So $y < b_1 = \sup\{a_1, a_2, a_3, \dots\}$. Therefore, by the definition of supremum or least upper bound, there exists $n_1 \geq 1$ such that $y < a_{n_1} \leq b_1$. Now $y < b_{n_1+1}$. And so there exists $n_2 \geq n_1 + 1 > n_1$ such that $y < a_{n_2} \leq b_{n_1+1}$. In this way we inductively define a_{n_k} such that $y < a_{n_k} \leq b_{n_{k-1}+1}$ and $n_k > n_{k-1}$. Hence the set $\{a_{n_1}, a_{n_2}, \dots\}$satisfies $y < a_{n_j}$ for all j in **N** and the set $\{n_j : j \in \mathbf{N}\}$ is an infinite subset of **N** as $n_j \neq n_k$ for $j \neq k$. Therefore, there are infinitely many n in **N** such that $y < a_n$.

(3) Suppose (1) and (2) holds for some real number a.

Take any $\varepsilon > 0$. Then $a + \varepsilon > a$.

Then by part (1) there exists an integer N such that for all $n \geq N$, $a + \varepsilon > a_n$. Therefore, $a + \varepsilon$ is an upper bound for $\{a_N, a_{N+1}, a_{N+2}, \dots\}$. Thus $a + \varepsilon \geq \sup\{a_N, a_{N+1}, a_{N+2}, \dots\} = b_N$.

Therefore, for all $n \geq N$, $b_n \leq b_N \leq a + \varepsilon$. Therefore, $\underline{\lim}_{n\to\infty} b_n \leq a + \varepsilon$. Since ε is arbitrary, $\underline{\lim}_{n\to\infty} b_n \leq a$.

Now consider $a - \varepsilon < a$. By part (2) there exists infinitely many n in N such that $a - \varepsilon < a_n$. Thus for each n in **N**, there exists an integer $m_n \geq n$ such that that $a - \varepsilon < a_{m_n}$. Therefore,

$$a - \varepsilon < a_{m_n} \leq \sup\{a_n, a_{n+1}, \dots\} = b_n .$$

Hence,$a - \varepsilon < b_m$ for each m in **N**. It follows that ,$a - \varepsilon \leq \underline{\lim}_{n\to\infty} b_n$. Since ε is arbitrary, , $a \leq \underline{\lim}_{n\to\infty} b_n$.

Therefore, by the previous inequality, we have $\underline{\lim}_{n\to\infty} b_n = a$ and so $\limsup_{n\to\infty} x_n = a$.

Question 2

(a) Let S be the set $\{1 + \frac{1}{n} : n \in \mathbf{N}\} \cup \{x \in \mathbf{Q}: 2 < x < 3)$, and $F: \mathbf{R} \to \mathbf{R}$ the function defined by

$$F(x) = \begin{cases} 3x \text{ if } x \in S \\ \dfrac{1}{x+2} \text{ if } x \notin S \end{cases}$$

(i) Find the set of all cluster points of S. (You do *not* need to give a proof of your answer.)

(ii) Using the ε - δ definition of a limit, show that $\lim_{x\to 3/2} F(x)$ exists and find its value.

(iii) Determine if $\lim_{x\to 2} F(x)$ exists. *Justify your answer.*

(b) Let A and B be non-empty sets, let c in **R** be a cluster point of A and b in B a cluster point of B. Let $g : A \to B$ and $f : B \to \mathbf{R}$ are maps such that f is continuous at b and $\lim_{x\to c} g(x) = b$. Use ε - δ argument to prove that $\lim_{x\to c} f \circ g(x) = f(b)$

Answer

(a) Recall $S = \{1 + \frac{1}{n} : n \in \mathbf{N}\} \cup \{x \in \mathbf{Q}: 2 < x < 3)$ and

$$F(x) = \begin{cases} 3x \text{ if } x \in S \\ \dfrac{1}{x+2} \text{ if } x \notin S \end{cases}$$

(i). Recall the definition of a cluster point. s is a cluster point of S if any open interval containing s also contains infinitely many elements in S.

Thus the set of cluster points of S is $\{1\}\cup[2, 3]$.

Obviously the points in this set are cluster points of S. (Any point in [2, 3] satisfies that any open interval containing it has intersection with [2, 3] as an interval which is a non trivial subinterval of [2, 3] which contains infinitely many rational points in (2, 3). Any open interval containing 1 contains infinitely many points in S of the form $1+ 1/n$. Plainly $(1-\varepsilon, 1+\varepsilon) \supseteq (1-1/N, 1+1/N)$ for some integer N such that $1/N < \varepsilon$. Therefore, for all $n \geq N$, $1+1/n \in S$. Obviously any $x > 3$, has an open neighbourhood that has empty intersection with S. For all $x < 1$, x cannot be a cluster point of S for the same reason. There is only a finite number of members of S in $[y, 2]$ for any $y > 1$. We can deduce this as follows, for any $y > 1$, there exists a positive integer M such that $1/M < y-1$ and so consequently $1+1/M < y$. Thus $[y, 2]\cap S \subseteq \{1+1/n: 1\leq n\leq M\}\cup\{2\}$, which is finite and so $[y, 2]$ can have only a finite number of members of S. Hence (1, 2) does not contain any cluster points of S . (This is because for any x in (1, 2), there exists a $\delta > 0$ such that $x-\delta > 1$ and $[x-\delta, x+\delta] \subseteq (1,2)$ and so $[x-\delta, x+\delta] \subseteq [x-\delta, 2]$, consequently $(x-\delta, x+\delta)$ has only a finite number of members of S .)

(ii) Since 3/2 is not a cluster point of S, there exists a $\delta > 0$ such that $(3/2 - \delta, 3/2+\delta) \cap S = \{3/2\}$. We can take $\delta = 1/2 - 1/3= 1/6$.

Claim that the limit is $1/(3/2+2)= 2/5$.

Give any $\varepsilon > 0$, take $\delta = \min(1/6, 5\varepsilon/2)$.

Then $0< |x -3/2| < \delta$ implies that $x \notin S$ since $0 < |x -3/2| < 1/6$ so that $1+1/3 < x < 1 + 2/3$ and $x \neq 3/2$ and so

$$|F(x) - 2/5| = \left|\frac{1}{x+1} - \frac{2}{5}\right| = \left|\frac{3-2x}{5(x+1)}\right| = \frac{2}{5}\left|\frac{x-\frac{3}{2}}{(x+1)}\right| < \frac{2}{5}\left|x-\frac{3}{2}\right| < \frac{2}{5}\cdot\frac{5\varepsilon}{2} = \varepsilon$$

since $0 < |x -3/2| < 5\varepsilon/2$,

This proves that $\lim_{x\to 3/2} F(x) = \frac{2}{5}$.

(iii) No.

Let a be a real number. Take $\varepsilon = 1$. Then for any $\delta > 0$, let $\delta_1 = \min(1/2, \delta)$. Then consider the interval $(2 - \delta_1, 2 + \delta_1)$. Pick a rational number x and an irrational number y in this $(2, 2 + \delta_1)$. We can do this by the density of the rational numbers and also that of the irrational numbers. Then $x \in S$ and $y \notin S$. Therefore, $F(x) = 3x > 6$ by the definition of F. Similarly, $F(y) = 1/(y+1) < 1/3$. Now by the triangular inequality,

$$|F(x) - a | + |F(y) - a| \geq |F(x)-F(y)| = F(x)-F(y) > 6 - 1/3 > 2.$$

Then at least one of $|F(x) - a |$ or $F(y) - a|$ must be greater than or equal to 1. Let $x_\delta = x$ if $|F(x) - a | \geq 1$. If $|F(x) - a | < 1$, then let $x_\delta = y$.

Thus for any $\delta > 0$, we can always find an element x_δ in $(2 - \delta_1, 2 + \delta_1)$ such that $|F(x_\delta) - a | \geq 1$. This means $\lim_{x\to 2} F(x) \neq a$ for any a. Therefore, $\lim_{x\to 2} F(x)$ does not exist.

Alternatively, take a sequence (a_n) of rational numbers in (2,3) which converges to 2 and also a sequence (b_n) of irrational numbers in (2,3) also converging to 2. (We can do this by the density of the rational numbers and that of the irrational numbers. For eacn n in **N**, there is a rational number a_n and an irrational number b_n such that $2 < a_n , b_n < 2+ 1/n$. Obviously, by the Comparison Theorem, both (a_n) and (b_n) converge to 2.) Now since a_n is rational and between 2 and 3, a_n is in S. Therefore, the sequence $(F(a_n)) = (3a_n)$ and converges to $3\times2=6$. Now b_n is irrational and so $b_n \notin S$. Thus $F (b_n) = 1/(b_n +1)$. Since (b_n) converges to 2, $(F (b_n))$ converges to $1/(2+1) = 1/3$. Therefore, $\lim_{n\to\infty} F(a_n) \neq \lim_{n\to\infty} F(b_n)$ while $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 2$ and so we conclude that $\lim_{x\to 2} F(x)$ does not exist.

(b) This is just the chain rule for limit. Recall that A and B are non-empty sets, c in **R** is a cluster point of A and b in B is a cluster point of B. $g : A \to B$ and $f : B \to \mathbf{R}$ are maps such that f is continuous at b and $\lim_{x\to c} g(x) = b$.

f is continuous at b means given $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that

for all y in B, $|y - b| < \delta_1$ implies that $|f(y) - f(b)| < \varepsilon$ -------------------- (1)

Now $\lim_{x\to c} g(x) = b$ means that for the same $\delta_1 > 0$ given by (1), there exists $\delta > 0$ such that

for all x in A, $0 < |x - c| < \delta$ implies that $|g(x) - b| < \delta_1$ ---------------------- (2).

Therefore, putting (1) and (2) together, given $\varepsilon > 0$, there exists $\delta > 0$ (given by (2)) such that

for all x in A, $0 < |x - c| < \delta$ implies that $|g(x) - b| < \delta_1$ which in turn implies by (1) that

$|f \circ g(x) - f(b)| = |f(g(x)) - f(b)| < \varepsilon$. This means $\lim_{x\to c} f \circ g(x) = f(b)$.

Question 3.

(a) Let $f : [a, b] \to \mathbf{R}$ be continuous such that for every x in $[a, b]$, there exists a y in $[a, b]$ such that $|f(y)| \le 1/3\, |f(x)|$. Prove that $f(c) = 0$ for some c in $[a, b]$.

(b) Suppose $g : [0, +\infty) \to \mathbf{R}$ is continuous on $[0, +\infty)$, and that there exist positive constants a and K such that $|g(x) - g(y)| \le K|x - y|$ for all x and y in $[a, +\infty)$. Prove that g is uniformly continuous on $[0, +\infty)$.

Answer.

(a) This will involve Bolzano Weierstrass Theorem and a charaterization of continuity by sequences.

Recall $f : [a, b] \to \mathbf{R}$ is continuous such that for every x in $[a, b]$, there exists a y in $[a, b]$ such that $|f(y)| \le 1/3\, |f(x)|$. We shall show that $f(c) = 0$ for some c in $[a, b]$.

We shall construct a sequence in $[a, b]$. Use Bolzano Weierstrass Theorem to obtain a convergent subsequence. The limit of this sequence is the required element c with $f(c) = 0$.

Start with x_0 in $[a, b]$. Then by the property of f there exists an element which we called x_1 in $[a, b]$ such that $|f(x_1)| \le 1/3\, |f(x_0)|$. Again using the property of f there exists an element x_2 in $[a, b]$ such that $|f(x_2)| \le 1/3\, |f(x_1)| \le 1/3^2\, |f(x_0)|$. Repeating this process we get a sequence (x_n) in $[a, b]$ such that $|f(x_n)| \le 1/3^n\, |f(x_0)|$. Then by the Bolzano Weierstrass Theorem (x_n) has a convergent subsequence (x_{n_k}) which converges to an element c in $[a, b]$. (This is the same thing as saying that the closed and bounded interval $[a, b]$ is sequentially compact.) Therefore, since f is continuous at c, for any sequence (a_n) that converges to c, the sequence $(f(a_n))$ converges to $f(c)$. Therefore, the sequence $(f(x_{n_k}))$ converges to $f(c)$. But since $|f(x_n)| \le 1/3^n\, |f(x_0)|$,

$$\lim_{k\to\infty} \left| f(x_{n_k}) \right| \le \lim_{k\to\infty} 1/3^{n_k} |f(x_0| = 0$$

Therefore, $\lim_{k\to\infty} \left| f(x_{n_k}) \right| = 0$ and so by the Squeeze Theorem $\lim_{k\to\infty} f(x_{n_k}) = 0$. It follows that $f(c) = 0$.

(b) This is about uniform continuity and Lipschitz condition.

Recall that $g : [0, +\infty) \to \mathbf{R}$ is continuous on $[0, +\infty)$ and that there exist positive constants a and K such that $|g(x) - g(y)| \le K|x - y|$ for all x and y in $[a, +\infty)$. We shall show that g is uniformly continuous.

Now since $[0, a]$ is a closed and bounded interval and so is compact and since g is continuous on $[0, a]$, g is uniformly continuous on $[0, a]$. Therefore, for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

for all x and y in $[0, a]$, $|x - y| < \delta_1 \Rightarrow |g(x) - g(y)| < \varepsilon/2$ ---------------------- (1)

Also note that taking $\delta_2 = \varepsilon/(2K)$, by the above property of the function g

for all x and y in $[a, +\infty)$, $|x - y| < \delta_2 \Rightarrow |g(x) - g(y)| \le K|x - y| < K\,\varepsilon/(2K) = \varepsilon/2$

---------------------- (2)

Thus given any $\varepsilon > 0$, let $\delta = \min(\delta_1, \delta_2)$. For any x and y in $[0,+\infty)$, if

$|x - y| < \delta$, we proceed as follows.

(i) If $\max(x, y) \le a$, then by (1), since $|x - y| < \delta = \min(\delta_1, \delta_2) \le \delta_1$, $|g(x) - g(y)| < \varepsilon/2 < \varepsilon$ or

(ii) If $\min(x, y) \ge a$, then by (2), since $|x - y| < \delta = \min(\delta_1, \delta_2) \le \delta_2$, $|g(x) - g(y)| < \varepsilon/2 < \varepsilon$ or

(iii) Either (i) $x < a < y$ when $x < y$ or (ii) $y < a < x$ when $y < x$.

For case (i) since $|x - a| < |x - y| < \delta_1$, we have by (1) $|g(x) - g(a)| < \varepsilon/2$ and also since $|a - y| < |x - y| < \delta_2$, by (2) $|g(a) - g(y)| < \varepsilon/2$. Therefore, by the triangular inequality,

$|g(x) - g(y)| \le |g(x) - g(a)| + |g(a) - g(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Similarly for case (ii) when $y < a < x$, we can show that $|g(x) - g(y)| < \varepsilon$.

Hence, by (i) (ii) and (iii) above given $\varepsilon > 0$, there exists $\delta > 0$ such that

for all x and y in $[0, +\infty)$, $|x - y| < \delta \Rightarrow |g(x) - g(y)| < \varepsilon$.

Thus, g is uniformly continuous on $[0, +\infty)$,

Question 4.

(a) Let $f : (0, +\infty) \to \mathbf{R}$ be differentiable $(0, +\infty)$ and $\lim_{x \to +\infty} f'(x) = 0$. Prove that . $\lim_{x \to +\infty} \frac{f(x)}{x} = 0$.

(b) Let g: $\mathbf{R} \to \mathbf{R}$ is a continuous function such that the derivatives $g', \ldots, g^{(999)}$ exist and are continuous on $\mathbf{R}$. Suppose that $g'(x_0) = \ldots = g^{(998)}(x_0) = 0$ and $g^{(999)}(x_0) = 1$ for some x_0 in $\mathbf{R}$. Does g have a relative maximum, or a relative minimum, or neither at x_0 ? *Justify your answer.*

Answer.

(a) This is about how one can handle the infinity question and a simple application of the Mean Value Theorem.

Start with what we are given $\lim_{x \to +\infty} f'(x) = 0$.

Then given $\varepsilon > 0$, there exists a positive integer N such that $x > N$ implies that $|f'(x)| < \varepsilon/2$

------------------------------ (1)

Now focus on the interval $[N, +\infty)$.

For any $x > N$, since f is differentiable, by the Mean Value Theorem, there exists a c such that $x > c > N$ and

$$\frac{f(x) - f(N)}{x - N} = f'(c)$$

This means $f(x) - f(N) = f'(c)(x - N)$.

Dividing by x, we get $\frac{f(x)}{x} = f'(c)(1 - \frac{N}{x}) + \frac{f(N)}{x}$ ----------------- (2)

Next, chose a positive integer M such that

$$x > M \Rightarrow \left|\frac{f(N)}{x}\right| < \frac{\varepsilon}{2}$$ ------------------------------- (3)

(We can find M since $\lim_{x\to+\infty}\frac{f(N)}{x}=0$.)
Now take $K = \max(N, M)$. Then $x > K$ implies that

$$\left|\frac{f(x)}{x}\right| = \left|f'(c)(1-\frac{N}{x})+\frac{f(N)}{x}\right| \text{ by (2) since } x > N$$

$$\le \left|f'(c)(1-\frac{N}{x})\right| + \left|\frac{f(N)}{x}\right| \le |f'(c)| + \left|\frac{f(N)}{x}\right| \text{ since } \left|(1-\frac{N}{x})\right| < 1$$

$$< \frac{\varepsilon}{2} + \left|\frac{f(N)}{x}\right| \text{ by (1) since } c > N$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ by (3) since } x > K = \max(N, M)\ge M.$$

This means $\lim_{x\to+\infty}\frac{f(x)}{x}=0$.

(b) This is a simple application of the Taylor Polynomial expansion with remainder.
Recall that g: $\mathbf{R} \to \mathbf{R}$ is a continuous function such that the derivatives g', … , $g^{(999)}$ exist and are continuous on **R**. It is given that $g'(x_0) = \ldots = g^{(998)}(x_0)=0$ and $g^{(999)}(x_0)=1$.
Then the Lagrange form of the Taylor expansion about x_0 up to degree 998 gives for any x in R,

$$g(x) = g(x_0) + \frac{g^{(999)}(c)}{999!}(x-x_0)^{999} \text{ ------------- (1)}$$

for some c strictly between x and x_0 .
Note that $g^{(999)}$ is continuous and so is continuous at x_0 . Thus, since $g^{(999)}(x_0)=1$, there exists a $\delta > 0$ such that $|x - x_0| < \delta$ implies that $g^{(999)}(x) > 0$. (Take $\varepsilon = 1/2$ and so by continuity there exists a $\delta > 0$ such that $|x - x_0| < \delta$ implies that $1/2 = g^{(999)}(x_0)-1/2< g^{(999)}(x) < g^{(999)}(x_0)+1/2$.)

Hence for any x in $(x_0 - \delta, x_0 + \delta)$, $g^{(999)}(x) > 0$. Therefore for any x in $(x_0 - \delta, x_0 + \delta)$, in the Taylor expansion (1), the Lagrange remainder term has the factor $\frac{g^{(999)}(c)}{999!} > 0$ since the c so obtained is between x and x_0. Thus, by (1) for $x > x_0$ in $(x_0 - \delta, x_0 + \delta)$,

$$g(x) = g(x_0) + \frac{g^{(999)}(c)}{999!}(x-x_0)^{999} > g(x_0) \text{ since } (x-x_0)^{999} > 0$$

and that also by (1) for $x < x_0$ in $(x_0 - \delta, x_0 + \delta)$,

$$g(x) = g(x_0) + \frac{g^{(999)}(c)}{999!}(x-x_0)^{999} < g(x_0) \text{ since } (x-x_0)^{999} < 0$$

Thus g cannot have a relative maximum nor relative minimum at x_0.

Question 5.

(a) Let $h:[0, 1] \to \mathbf{R}$ be defined by $h(x) = \begin{cases} -x, & \text{if } x \text{ is rational,} \\ 2x, & \text{if } x \text{ is irrational} \end{cases}$.

Determine if h is integrable on [0, 1]. *Justify your answer*.

(b) Suppose that $f:[a, b] \to \mathbf{R}$ is increasing and differentiable on $[a, b]$ and its derivative f' is Riemann integrable on $[a, b]$, and suppose that $g:[a, b] \to \mathbf{R}$ is continuous on $[a, b]$. Prove that there exists c in $[a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx.$$

(Hint: Use integration by parts.)

Answer.

(a) Recall $h:[0, 1] \to \mathbf{R}$ is defined by $h(x) = \begin{cases} -x, & \text{if } x \text{ is rational,} \\ 2x, & \text{if } x \text{ is irrational} \end{cases}$.

Then h is discontinuous at every irrational points in [0, 1].

Let x be an irrational point in [0, 1]. Then $0 < x < 1$. Then there exists a positive integer N such that $0 < 1/N < x$. Therefore, by the density of the rational numbers, for each $n \geq N$ there exists a rational number a_n such that $0 < x - 1/n < a_n < x$, i.e. $| a_n - x| = x - a_n < 1/n$. Therefore, by the Comparison Theorem, since $1/n$ tends to 0 as n tends to infinity the sequence $(a_n)_{n\geq N}$ tends to x.
Now, $h(a_n) = - a_n$ because a_n is rational. Therefore, the sequence $(h(a_n))_{n\geq N} = (-a_n)_{n\geq N}$ converges to $-x$. Also, by the density of the irrational numbers for each $n \geq N$ there exists an irrational number b_n such that $0 < x - 1/n < b_n < x$, i.e. $| b_n - x| = x - b_n < 1/n$. Similarly we deduce that $(b_n)_{n\geq N}$ tends to x. But since each b_n is irrational $(h(b_n))_{n\geq N} = (2a_n)_{n\geq N}$ and so the sequence $(h(b_n))_{n\geq N}$ converges to $2x$. It follows that because $2x \neq -x$, $(h(a_n))_{n\geq N}$ and $(h(b_n))_{n\geq N}$ do not converge to the same limit while both $(a_n)_{n\geq N}$ and $(b_n)_{n\geq N}$ converge to the same limit x. Consequently, h is not continuous at x. Therefore, h is discontinuous at every irrational point in [0,1]. Therefore, by Lebesgue Theorem h is not Riemann integrable on [0,1] as the set of irrational points in [0, 1] has non zero measure.

Alternatively, we can use the upper and lower Darboux sums.
Let Δ: $x_0 = 0 < x_1 < x_2 \ldots < x_n = 1$ be a partition for [0, 1].
Then the upper Darboux sum with respect to Δ is

$$U(\Delta) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} M_i (x_i - x_{i-1}),$$

where $M_i = \sup\{h(x) : x \in [x_{i-1}, x_i]\} = 2x_i$
Similarly, the lower Darboux sum with respect to Δ is

$$L(\Delta) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} m_i (x_i - x_{i-1}),$$

where $m_i = \inf\{h(x) : x \in [x_{i-1}, x_i]\} = -x_i$.
Therefore, $U(\Delta) - L(\Delta) = 3\sum_{i=1}^{n} x_i(x_i - x_{i-1}) = \frac{3}{2}\sum_{i=1}^{n}[x_i^2 - x_{i-1}^2 + (x_i - x_{i-1})^2]$

$$> \frac{3}{2}\sum_{i=1}^{n}[x_i^2 - x_{i-1}^2] = \frac{3}{2}.$$

Therefore, for any partition Δ, $U(\Delta) - L(\Delta) > 3/2$. Hence h is not Riemann integrable.

(b) This is sometimes called the Third Mean Value Theorem for Integral.
Suppose $f : [a, b] \to \mathbf{R}$ is increasing and differentiable on $[a, b]$ and its derivative f' is Riemann integrable on $[a, b]$, and suppose that $g:[a, b] \to \mathbf{R}$ is continuous on $[a, b]$. Then there exists c in $[a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx.$$

Follow the hint.
Let $G(x) = \int_a^x g(t)dt$. Then since g is continuous, $G(x)$ is an anti-derivative of $g(x)$ by the Fundamental Theorem of Calculus. Then using integration by parts,

$$\int_a^b f(x)g(x)dx = [f(x)G(x)]_a^b - \int_c^b G(x)f'(x)dx$$

$$= f(b)G(b) - \int_c^b G(x)f'(x)dx \quad \text{----------------------} \quad (1)$$

since $G(a) = 0$.
Now, since $G(x)$ is continuous on $[a, b]$, by the Extreme Value Theorem, there exists d and e in $[a, b]$ such that for all x in $[a, b]$,

$$G(d) \leq G(x) \leq G(e). \quad \text{----------------------------} \ (2)$$

Note that since f is increasing and differentiable on $[a, b]$, $f'(x) \geq 0$ for all x in $[a, b]$. Therefore, multiplying (2) by $f'(x)$ we get for all x in $[a, b]$,

$$G(d) f'(x) \leq G(x) f'(x) \leq G(e) f'(x).$$

Thus, taking integrals,

$$G(d)\int_a^b f'(x) \le \int_a^b G(x) f'(x)dx \le G(e)\int_c^b f'(x)dx$$

Hence by the Intermediate Value Theorem, there exists c between d and e and hence in $[a, b]$ such that

$$\int_a^b G(x) f'(x)dx = G(c)\int_c^b f'(x)dx \quad \text{----------} \quad (3)$$

Now since f' is Riemann integrable on $[a, b]$, by Darboux Theorem,

$$\int_a^b f'(x)dx = f(b) - f(a).$$

It follows then from (3) that

$$\int_a^b G(x) f'(x)dx = G(c)(f(b) - f(a)) \quad \text{-------------------} \quad (4)$$

Thus substituting (4) in (1) we obtain,

$$\int_a^b f(x)g(x)dx = f(b)G(b) - \int_c^b G(x) f'(x)dx = f(b)G(b) - G(c)(f(b) - f(a))$$
$$= f(b)(G(b) - G(c)) + f(a)G(c)$$
$$= f(b)\left[\int_a^b g(x)dx - \int_a^c g(x)dx\right] + f(a)\int_a^c g(x)dx$$
$$= f(b)\int_c^b g(x)dx + f(a)\int_a^c g(x)dx$$

This completes the proof.