NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2005 – 2006

MA1102R CALCULUS

April 2006 – Time Allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO** sections: Section A and Section B. It contains a total of **SIX** questions and comprises **FOUR** printed pages.

2. Answer **ALL** questions in **Section A**. The marks for questions in Section A are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Answer not more than **TWO** questions from Section B. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

*Answer **ALL** questions in this section.*

Question 1 [**20** marks]

Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x^2, & x \le 0 \\ x^2 \sin\left(\dfrac{\pi}{2x}\right), & 0 < x < 1 \\ -x^3 + 3x - 1, & x \ge 1 \end{cases}.$$

(a) Determine all x in $\mathbf{R}$ at which the function f is *continuous*. Justify your answer.

(b) Find the *range* of the function f.

(c) Determine if f is *surjective*.

(d) Determine if f is *differentiable* at x, when $x = 0$ *or* 1. Justify your answer.

Question 2 [**20** marks]

Evaluate, if it exists, each of the following limits.

(a) $\displaystyle \lim_{x \to +\infty} \sqrt{\frac{x^7 + 3x^2 + \sin(x) + 7}{5x^7 + 6x^3 + 1}}$.

(b) $\displaystyle \lim_{x \to 0} \frac{\sin(4x + \sin(\cos(x) - 1))}{x + \sin(x)}$.

(c) $\displaystyle \lim_{x \to 0^+} \sin(x)\sin(e^{\sin(1/x)})$.

(d) $\displaystyle \lim_{x \to 0} (1 + 17x^3)^{(1/x^3)}$.

(e) $\displaystyle \lim_{x \to \infty} \frac{e^{(x^2)}}{1 + x^3 + x^5}$.

Question 3 [**20** marks]

(a) Evaluate $\int \frac{2x^2 - x + 1}{(x^2 + 3x + 3)(x^2 - 3x + 3)} dx.$

(b) Compute $\int_{-1}^{2} \sin(x + 3|x|) dx$.

(c) Find an antiderivative of $g(x)$, which is defined by

$$g(x) = \begin{cases} x^3 + x + 2, & x < 0 \\ 3\sin(\pi x) + \cos(2x)e^{\sin(2x)} + 1, & x \geq 0 \end{cases} .$$

(d) Evaluate $\int \frac{1}{\sqrt{x^2 - 2x - 1}} dx.$

(e) Evaluate $\int \ln(2 + x^2) dx.$

SECTION B

Answer not more than ***TWO*** *questions from this section. Each question in this section carries 20 marks.*

Question 4 [**20** marks]

(a) Find the critical points of the function g, defined by

$$g(x) = 2x^3 - 15x^2 + 24x + 1,$$

in the open interval (0, 5). Determine the absolute maximum and the absolute minimum values of the function in the interval [0, 5].

(b) Differentiate each of the following functions.

(i) $h(x) = (x^2 + 1 + \cos(\cos(x)))^{\sin(x)}$.

(ii) $j(x) = \int_{x}^{\ln(x)} \frac{e^t}{\sin(t + \sin(t^2)) + 2} dt, \ x \in (0, \infty).$

(iii) $k(x) = \cot^{-1}(\csc^2(x)), \ x \in (0, \frac{\pi}{2})$.

(c) Suppose f and g are two continuous functions defined on the interval $[a, b]$ with $a < b$. Suppose $f(x) \geq 0$ for all x in $[a, b]$.

(i) Show that if m is the absolute minimum of $g(x)$ on $[a, b]$ and M is the absolute maximum of $g(x)$ on $[a, b]$, then

$$m\int_a^b f(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b f(x)dx.$$

(ii) Hence, or otherwise, show that there exists a point c in $[a, b]$ such that

$$\int_a^b f(x)g(x)dx = g(c)\int_a^b f(x)dx.$$

Question 5 [**20** marks]

(a) (i) Suppose f is a continuous function defined on the closed and bounded interval $[a, b]$. Give the integral formula for the volume of solid of revolution obtained by rotating about the x-axis the region bounded by the curve $y = f(x)$, the x-axis the lines $x = a$ and $x = b$.

(ii) Use this formula or otherwise, find the volume of the solid of revolution when the ellipse,

$$\frac{x^2}{3} + \frac{y^2}{5} = 1$$

is rotated about the x-axis through 2π radians.

(b) Differentiate the function k defined on **R** by

$$k(x) = \int_1^{x^5} (1 + t^2 + \cos(\sin(\pi t)))dt.$$

(i) Without integrating, show that the function k is injective.

(ii) Determine $(k^{-1})'(0)$.

(c) Find the following limit.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^4} \cdot \sqrt[3]{7n^3 + 2i^3}\ .$$

Question 6 [**20** marks]

Let the function f be defined on **R** by

$$f(x) = x^5 - 20x^2 + 7\ .$$

(a) Find the intervals on which f is (i) *increasing*, and (ii) *decreasing*.

(b) Find the intervals on which the graph of f is (i) *concave upward*, and (ii) *concave downward*.

(c) Find the *relative extrema* of f, if any.

(d) Find the *points of inflection* of the graph of f.

(e) Sketch the graph of f.

END OF PAPER

Answer To MA1102 Calculus

Question 1

The function f is defined by $f(x)=\begin{cases} x^2, & x\le 0 \\ x^2\sin\left(\frac{\pi}{2x}\right), & 0<x<1 \\ -x^3+3x-1, & x\ge 1 \end{cases}$.

(a) For $x<0$, $f(x)=x^2$ is a polynomial function. Therefore, f is continuous on the interval $(-\infty, 0)$ since any polynomial function is continuous on **R** and hence on any interval. Similarly for $x>1$, $f(x) = -x^3+3x-1$ is a polynomial function there and so is continuous on $(1, \infty)$.

For $0<x<1$, $f(x)=x^2\sin\left(\frac{\pi}{2x}\right)$ and so f is continuous on (0, 1) since $\sin\left(\frac{\pi}{2x}\right)$ is continuous on (0, 1) and x^2 is continuous on **R** so that the product of these two functions is continuous on (0, 1). Thus it remains to check the continuity of f at 0 and 1. Note that $f(0) = 0$ and $f(1) = 1$.

Now we determine the left limit at $x = 0$. It is $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 = 0$. The right limit at $x = 0$ is $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2\sin\left(\frac{\pi}{2x}\right) = 0$ by the Squeeze Theorem since $-|x|^2 \le x^2\sin\left(\frac{\pi}{2x}\right)\le |x|^2$ for $x\ne 0$ and $\lim_{x\to 0^+}|x|^2=0$. Therefore, $\lim_{x\to 0} f(x) = 0 = f(0)$ and so f is continuous at $x = 0$.

Now consider the left limit of f at $x = 1$,

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2\sin\left(\frac{\pi}{2x}\right) = 1\cdot\sin\left(\frac{\pi}{2}\right)=1$

Now $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} -x^3+3x-1 = -1+3-1=1$ and so $\lim_{x\to 1} f(x) = 1$.

Therefore, $\lim_{x\to 1} f(x) = 1 = f(1)$ and so f is continuous at $x = 1$.

Therefore f is continuous at x for any x in **R**.

(b) For $x\le 0$, $x^2\ge 0$ and so the image $f((-\infty, 0])\subseteq [0,\infty)$. Now for any $y\ge 0$, $x^2 = y$ has a solution $x=-\sqrt{y}\le 0$ to $x^2 = y$ in $(-\infty, 0]$. Therefore, $[0,\infty)\subseteq f((-\infty, 0])$. That means $f((-\infty, 0]=[0,\infty)$.

Next for $x>1$, $f(x) = -x^3+3x-1$ so that $f'(x) = -3x^2+3 = 3(1-x^2)<0$ for $x>1$. Therefore, f is strictly decreasing on $[1,\infty)$ and so $f(x)\le f(1) = 1$ for $x\ge 1$. Hence $f([1,\infty))\subseteq(-\infty, 1]$. Also note that

$\lim_{x\to\infty} f(x) = -\infty$ since $\lim_{x\to\infty} -x^3+3x-1 = \lim_{x\to\infty} -x^3\left(1-\frac{3}{x^2}+\frac{1}{x^3}\right) = -\infty$ because $\lim_{x\to\infty} -x^3 = -\infty$ and $\lim_{x\to\infty}\left(1-\frac{3}{x^2}+\frac{1}{x^3}\right)=1>0$. Hence, since f is continuous on $[1,\infty)$, by the Intermediate Value Theorem $f([1,\infty)) = (-\infty, 1]$. We deduce this as follows. For any y in $(-\infty, 1]$, say $y<1$. Then since $\lim_{x\to\infty} f(x) = -\infty$, there exists a point $K>1$ such that $f(K)<y$. Thus, $f(K)<y<1= f(1)$. Hence since f is continuous on $[1, K]$, by the Intermediate Value Theorem, there is a point k in $[1, K]$, hence in $[1,\infty)$ such that $f(k) = y$. This means $(-\infty, 1]\subseteq f([1,\infty))$ and so $f([1,\infty)) = (-\infty, 1]$. Now observe that $f((-\infty, 0]))\cup f([1,\infty)) = [0,\infty)\cup(-\infty, 1]$ =**R**. Therefore, the range of f is

$f(\mathbf{R}) = f((-\infty, 0])\cup f((0, 1))\cup f([1,\infty)) = f((0, 1))\cup\mathbf{R}=\mathbf{R}$.

(There is no need to know what is $f((0, 1))$.)

(c) By part (b) Range(f) = **R** = codomain of f. Therefore, f is surjective.

(e) To check the differentiability of f at $x = 0$ consider the following limits.

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{x^2}{x}=\lim_{x\to 0^-} x=0$$

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{x^2\sin\left(\frac{\pi}{2x}\right)-0}{x}=\lim_{x\to 0^+} x\sin\left(\frac{\pi}{2x}\right)=0$$

by the Squeeze Theorem, since

$$-|x|\le x\sin\left(\frac{\pi}{2x}\right)\le |x| \text{ for } x\ne 0 \text{ and } \lim_{x\to 0^+}|x|=0$$

Thus, f is differentiable at $x = 0$ since $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}$.

Next $\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{-x^3+3x-1-1}{x-1}=-\lim_{x\to 1^+}\frac{x^3-3x+2}{x-1}$

$$=-\lim_{x\to 1^+}\frac{(x+2)(x-1)^2}{x-1}=-\lim_{x\to 1^+}(x+2)(x-1)=0$$

OR $\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{-x^3+3x-1-1}{x-1}=\lim_{x\to 1^+}\frac{-3x^2+3}{1}$ by L' Hôpital's Rule

$= 0.$

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^-}\frac{x^2\sin\left(\frac{\pi}{2x}\right)-1}{x-1}=\lim_{x\to 1^-}\frac{2x\sin\left(\frac{\pi}{2x}\right)-\frac{\pi}{2}\cos\left(\frac{\pi}{2x}\right)}{1}=2$$

by L' Hôpital's Rule

Therefore, f is not differentiable at $x = 1$ since $\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}\ne\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$

OR,
$$f'(x)=\begin{cases} 2x, & x<0 \\ 2x\sin\left(\frac{\pi}{2x}\right)-\frac{\pi}{2}\cos\left(\frac{\pi}{2x}\right), & 0<x<1 \\ -3x^2+3, & x>1 \end{cases}$$

$$\lim_{x\to 1^+} f'(x)=\lim_{x\to 1^+} -3x^2+3=0,$$

$$\lim_{x\to 1^-} f'(x)=\lim_{x\to 1^-} 2x\sin\left(\frac{\pi}{2x}\right)-\frac{\pi}{2}\cos\left(\frac{\pi}{2x}\right)=2$$

Since both limits $\lim_{x\to 1^-} f'(x)$ and $\lim_{x\to 1^+} f'(x)$ are finite and not the same, f is not differentiable at $x = 1$.

Question 2

(a) $\lim_{x\to+\infty}\sqrt{\frac{x^7+3x^2+\sin(x)+7}{5x^7+6x^3+1}}=\lim_{x\to+\infty}\sqrt{\frac{1+\frac{3}{x^5}+\frac{1}{x^7}\sin(x)+\frac{7}{x^7}}{5+\frac{6}{x^4}+\frac{1}{x^7}}}=\sqrt{\frac{1+0+0+0}{5+0+0}}=\frac{1}{\sqrt{5}}.$

This is because $\lim_{x\to+\infty}\frac{1}{x^5}=\lim_{x\to+\infty}\frac{1}{x^4}=\lim_{x\to+\infty}\frac{1}{x^7}=0$ and $\lim_{x\to+\infty}\frac{\sin(x)}{x^7}=0$ by the Squeeze Theorem since $-|\frac{1}{x^7}|\le\frac{\sin(x)}{x^7}\le|\frac{1}{x^7}|$ for $x\ne 0$ and $\lim_{x\to+\infty}|\frac{1}{x^7}|=0$

(b) $\lim_{x\to 0}\frac{\sin(4x+\sin(\cos(x)-1))}{x+\sin(x)}=\lim_{x\to 0}\frac{\sin(4x+\sin(\cos(x)-1))}{4x+\sin(\cos(x)-1)}\cdot\frac{4x+\sin(\cos(x)-1)}{x+\sin(x)}$

$$=\lim_{x\to 0}\frac{\sin(4x+\sin(\cos(x)-1))}{4x+\sin(\cos(x)-1)}\cdot\left(\frac{4x}{x+\sin(x)}+\frac{\sin(\cos(x)-1)}{x+\sin(x)}\right)$$

$$=\lim_{x\to 0}\frac{\sin(4x+\sin(\cos(x)-1))}{4x+\sin(\cos(x)-1)}\cdot\lim_{x\to 0}\left(\frac{4}{1+\sin(x)/x}+\frac{\sin(\cos(x)-1)}{\cos(x)-1}\frac{\cos(x)-1}{x}\frac{1}{1+\sin(x)/x}\right)$$

$$=1\cdot\left(\tfrac{4}{1+1}+1\cdot 0\cdot\tfrac{1}{1+1}\right)=2$$

because $\lim_{x\to 0}\frac{\sin(4x+\sin(\cos(x)-1))}{4x+\sin(\cos(x)-1)}=1$, $\lim_{x\to 0}\frac{\sin(\cos(x)-1)}{\cos(x)-1}=\lim_{x\to 0}\frac{\sin(x)}{x}=1$and

$\lim_{x\to 0}\frac{\cos(x)-1}{x}=0.$

OR

$$\lim_{x\to 0}\frac{\sin(4x+\sin(\cos(x)-1))}{x+\sin(x)}=\lim_{x\to 0}\frac{\cos(4x+\sin(\cos(x)-1))\cdot(4+\cos(\cos(x)-1)(-\sin(x)))}{1+\cos(x)}$$

by L' Hôpital's Rule

$$=\frac{\cos(0)\cdot(4+\cos(0)\cdot(-\sin(0)))}{2}=2$$

(c) $\lim_{x\to 0^+}\sin(x)\sin(e^{\sin(1/x)})=0$ by the Squeeze Theorem since

$-|\sin(x)|\le\sin(x)\sin(e^{\sin(1/x)})\le|\sin(x)|$ for $x>0$ and $\lim_{x\to 0^+}|\sin(x)|=0.$

(d) $\lim_{x\to 0}(1+17x^3)^{(1/x^3)}$. Let $y=(1+17x^3)^{(1/x^3)}$. Then $\ln(y)=\frac{1}{x^3}\ln(1+17x^3)$

Since $\lim_{x\to 0}\ln(y)=\lim_{x\to 0}\frac{1}{x^3}\ln(1+17x^3)=\lim_{x\to 0}\frac{\frac{51x^2}{1+17x^3}}{3x^2}=\lim_{x\to 0^+}\frac{17}{1+17x^3}=17$

by L' Hôpital's Rule,

Therefore, $\lim_{x\to 0}y=e^{\lim_{x\to 0}\ln(y)}=e^{17}$

(e) $\lim_{x\to\infty}\frac{e^{(x^2)}}{1+x^3+x^5}=\lim_{x\to\infty}\frac{2xe^{(x^2)}}{3x^2+5x^4}=\lim_{x\to\infty}\frac{2e^{(x^2)}}{3x+5x^3}=\lim_{x\to\infty}\frac{4xe^{(x^2)}}{3+15x^2}=\lim_{x\to\infty}\frac{4(1+2x^2)e^{(x^2)}}{30x}$

$=\lim_{x\to\infty}\frac{4(6x+4x^3)e^{(x^2)}}{30}=\infty$ by repeated use of L' Hôpital's Rule,

by repeated use of L' Hôpital's Rule and the fact that $\lim_{x\to\infty}(6x+4x^3)e^{(x^2)}=\infty$.

Question 3

(a) $\int\frac{2x^2-x+1}{(x^2+3x+3)(x^2-3x+3)}dx=\int\frac{-5/18+1/3}{(x^2+3x+3)}dx+\int\frac{5/18x}{(x^2-3x+3)}dx$

by a partial fraction expansion determined as follows.

Writing,

$$\frac{2x^2-x+1}{(x^2+3x+3)(x^2-3x+3)}=\frac{Ax+B}{x^2+3x+3}+\frac{Cx+D}{x^2-3x+3}$$

then $(Ax+B)(x^2-3x+3)+(Cx+D)(x^2+3x+3)=2x^2-x+1.$

Comparing coefficients of x^3 : A + C = 0 ---------------------------- (1)
Comparing constant terms : 3B +3D =1, i.e. B + D = 1/3 ----------------- (2)
Comparing coefficients of x^2 : –3A+B + 3C+D =2.
Since B+D = 1/3 by (2) we get –3A +3C =2 –1/3 = 5/3 , i.e.,
–A + C = 5/9 ---------------------------- (3)
Comparing coefficients of x: 3A– 3B +3C+3D = –1.
Since A + C = 0 we get from above – 3B +3D = –1
and – B +D = –1/3 ---------------- (4)
Equation (1) + Equation (3) gives 2C = 5/9 and so C= 5/18 and A = –C = – 5/18.
Equation (2) + Equation (4) gives 2D = 0 and so D = 0 and B = 1/3–D = 1/3

$$\text{Now} \int \frac{-5/18x+1/3}{(x^2+3x+3)}dx = \int \frac{-5/36(2x+3)+1/3+5/12}{(x^2+3x+3)}dx = \int \frac{-5/36(2x+3)+3/4}{(x^2+3x+3)}dx$$
$$= -\frac{5}{36}\int \frac{2x+3}{(x^2+3x+3)}dx + \frac{3}{4}\int \frac{1}{(x+3/2)^2+3/4}dx$$
$$= -\frac{5}{36}\ln|x^2+3x+3| + \frac{3}{4}\frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x+3/2}{\sqrt{3}/2}\right) + C$$
$$= -\frac{5}{36}\ln|x^2+3x+3| + \frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2x+3}{\sqrt{3}}\right) + C$$

$$\text{And} \int \frac{5/18x}{(x^2-3x+3)}dx = \int \frac{5/36(2x-3)+5/12}{(x^2-3x+3)}dx$$
$$= \frac{5}{36}\ln|x^2-3x+3| + \frac{5}{12}\int \frac{1}{(x-3/2)^2+3/4)}dx$$
$$= \frac{5}{36}\ln|x^2-3x+3| + \frac{5}{6\sqrt{3}}\tan^{-1}\left(\frac{2x-3}{\sqrt{3}}\right) + C'$$

Therefore,
$$\int \frac{2x^2-x+1}{(x^2+3x+3)(x^2-3x+3)}dx = \frac{5}{36}\ln\frac{|x^2-3x+3|}{|x^2+3x+3|} + \frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2x+3}{\sqrt{3}}\right) + \frac{5\sqrt{3}}{18}\tan^{-1}\left(\frac{2x-3}{\sqrt{3}}\right) + C''$$

(b) $$\int_{-1}^{2}\sin(x+3|x|)dx = \int_{-1}^{0}\sin(x-3x)dx + \int_{0}^{2}\sin(x+3x)dx$$
$$= \int_{-1}^{0}\sin(-2x)dx + \int_{0}^{2}\sin(4x)dx = \left[\tfrac{1}{2}\cos(-2x)\right]_{-1}^{0} + \left[-\tfrac{1}{4}\cos(4x)\right]_{0}^{2}$$
$$= \tfrac{3}{4} - \tfrac{1}{2}\cos(2) - \tfrac{1}{4}\cos(8).$$

(c) $$g(x) = \begin{cases} x^3+x+2, \ x<0 \\ 3\sin(\pi x)+\cos(2x)e^{\sin(2x)}+1, \ x\ge 0 \end{cases}.$$

First note that g is continuous on the interval (–∞ , 0) since it is a polynomial function there and polynomial functions are continuous. Note also that g is continuous on (0, ∞) since $3\sin(\pi x)$ is a continuous function and the product $\cos(2x)e^{\sin(2x)}$ is continuous on (0, ∞). Now the left limit at x = 0 is $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} x^3+x+2 = 2$ and the right limit at x = 0, $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} 3\sin(\pi x)+\cos(2x)e^{\sin(2x)}+1 = 0+1+1 = 2 = g(1)$. Therefore, $\lim_{x\to 0} g(x) = g(0)$. Thus g is continuous at x =0. Therefore, g is continuous on **R** and we can use the Fundamental Theorem of Calculus to obtain an antiderivative $G(x)$ given by the following Riemann integral for each x in **R**.

$$G(x) = \int_0^x g(t)dt = \begin{cases} \int_0^x g(t)dt, x<0 \\ \int_0^x g(t)dt, x\ge 0 \end{cases} = \begin{cases} \int_0^x (t^3+t+2)dt, x<0 \\ \int_0^x (3\sin(\pi t)+\cos(2t)e^{\sin(2t)}+1)dt, x\ge 0 \end{cases}$$

$$= \begin{cases} \left[\frac{1}{4}t^4 + \frac{1}{2}t^2 + 2t\right]_0^x, x < 0 \\ \left[\frac{1}{2}e^{\sin(2t)} - \frac{3}{\pi}\cos(\pi t) + t\right]_0^x, x \geq 1 \end{cases} = \begin{cases} \frac{1}{4}x^4 + \frac{1}{2}x^2 + 2x, x < 0 \\ \frac{1}{2}e^{\sin(2x)} - \frac{3}{\pi}\cos(\pi x) + x + \frac{3}{\pi} - \frac{1}{2}, x \geq 0 \end{cases}$$

Thus, any antiderivative is given by $G(x) + C$ for any constant C.

(d) $\int \frac{1}{\sqrt{x^2 - 2x - 1}}dx = \int \frac{1}{\sqrt{(x-1)^2 - 2}}dx = \int \frac{1}{\sqrt{2}\sqrt{(\frac{x-1}{\sqrt{2}})^2 - 1}}dx$

using trigonometric substitution:
$\sec(\theta) = \frac{x-1}{\sqrt{2}}$ so that $dx = \sqrt{2}\sec(\theta)\tan(\theta)d\theta$

$$= \int \frac{1}{\sqrt{2}\tan(\theta)}\sqrt{2}\sec(\theta)\tan(\theta)d\theta = \int \sec(\theta)d\theta = \ln|\sec(\theta) + \tan(\theta)| + C$$

$$= \ln\left|\frac{x-1}{\sqrt{2}} + \sqrt{(\frac{x-1}{\sqrt{2}})^2 - 1}\right| + C = \ln|x - 1 + \sqrt{x^2 - 2x - 1}| - \frac{1}{2}\ln(2) + C$$

$$= \ln|x - 1 + \sqrt{x^2 - 2x - 1}| + C'.$$

(e) $\int \ln(2 + x^2)dx = x\ln(2 + x^2) - \int x \cdot \frac{2x}{2 + x^2}dx$ by integration by parts

$$= x\ln(2 + x^2) - \int(2 - \frac{4}{2 + x^2})dx = x\ln(2 + x^2) - 2x + \frac{4}{\sqrt{2}}\tan^{-1}(\frac{x}{\sqrt{2}}) + C$$

$$OR \quad = x\ln(2 + x^2) - 2x + 2\sqrt{2}\tan^{-1}(\frac{\sqrt{2}x}{2}) + C$$

OR part of the integral above is given by:

$$\int \frac{4}{2 + x^2}dx = \int \frac{2}{(1 + (\frac{x}{\sqrt{2}})^2)}dx = \int 2\sqrt{2}\,d\theta = 2\sqrt{2}\,\theta + C' = 2\sqrt{2}\tan^{-1}(\frac{x}{\sqrt{2}}) + C'$$

using trigonometric substitution: $\tan(\theta) = \frac{x}{\sqrt{2}}$ where $dx = \sqrt{2}\sec^2(\theta)d\theta$

Question 4.

(a) Recall $g(x) = 2x^3 - 15x^2 + 24x + 1$

Thus, $g'(x) = 6x^2 - 30x + 24 = 6(x-4)(x-1)$. Therefore, $g'(x) = 0$ if and only if $x = 1$ or 4. Hence g has two stationary points in (0, 5), namely 1 and 4. Since g is differentiable, the critical points of g in (0, 5) are 1 and 4. Since g is continuous on the closed and bounded interval [0, 5] and so by the Extreme Value Theorem g has absolute extrema on the interval [0, 5] and they are given respectively by the maximum and minimum of the values of the critical points in (0, 5) and the end points 1 and 4 under g. Now g(0) = 1, g(1) = 12, g(4) =–15 and g(5) =–4. Therefore, the absolute maximum of g on [0, 5] is 12 and the absolute minimum of g on [0, 5] is –15.

(b) (i)
$h(x) = (x^2 + 1 + \cos(\cos(x)))^{\sin(x)}$.
Taking logarithm on both sides we get $\ln(h(x)) = \sin(x)\ln(x^2 + 1 + \cos(\cos(x)))$.
Differentiating both sides we get,

$$\frac{h'(x)}{h(x)} = \cos(x)\ln(x^2 + 1 + \cos(\cos(x))) + \sin(x)\frac{2x + \sin(\cos(x))\sin(x)}{x^2 + 1 + \cos(\cos(x))}$$

Therefore, $h'(x) =$

$$((x^2+1+\cos(\cos(x)))^{\sin(x)}\left[\cos(x)\ln(x^2+1+\cos(\cos(x))) + \sin(x)\frac{2x+\sin(\cos(x))\,\sin(x)}{x^2+1+\cos(\cos(x))}\right].$$

(ii) $j(x) = \int_x^{\ln(x)} \frac{e^t}{\sin(t+\sin(t^2))+2}dt,\ x \in (0,\infty).$

Therefore, $j(x) = \int_1^{\ln(x)} \frac{e^t}{\sin(t+\sin(t^2))+2}dt - \int_1^{x} \frac{e^t}{\sin(t+\sin(t^2))+2}dt.$

Hence by the Fundamental Theorem of Calculus and the Chain Rule,

$$j'(x) = \frac{1}{\sin(\ln(x)+\sin(\ln(x)^2))+2} - \frac{e^x}{\sin(x+\sin(x^2))+2}$$

(iii) $k(x) = \cot^{-1}(\csc^2(x)),\ x \in (0, \frac{\pi}{2}).$ Thus by the Chain Rule

$$k'(x) = -(\cot^{-1})'(\csc^2(x)) \cdot 2\csc^2(x)\cot(x) = -\frac{2\csc^2(x)\cot(x)}{\cot'(\cot^{-1}(\csc^2(x)))}$$

$$= \frac{2\csc^2(x)\cot(x)}{\csc^2(\cot^{-1}(\csc^2(x)))} = \frac{2\csc^2(x)\cot(x)}{1+\csc^4(x)} = \frac{2\cos(x)\sin(x)}{1+\sin^4(x)} = \frac{\sin(2x)}{1+\sin^4(x)}.$$

(c) (i) Since m is the absolute minimum of g on $[a, b]$ and M is the absolute maximum of g on $[a,b]$, we have

$$m \le g(x) \le M \quad \text{-------------} \quad (1)$$

for all x in $[a, b]$.

Therefore, since $f(x) \ge 0$ for all x in $[a, b]$, multiplying (1) by $f(x)$ we get

$$mf(x) \le f(x)g(x) \le Mf(x) \quad \text{-------------} \quad (2)$$

for all x in $[a, b]$.

Hence taking integral we get:

$$m\int_a^b f(x)dx \le \int_a^b f(x)g(x)dx \le M\int_a^b f(x)dx.$$

(ii) Therefore, $\int_a^b f(x)g(x)dx = k\int_a^b f(x)dx$ for some k in $[m, M]$. By the Extreme Value Theorem, since g is continuous on $[a, b]$, $m = g(d)$ and $M = g(e)$ for some points d and e in $[a, b]$, and so by the Intermediate Value Theorem there is a point c between d and e and so in $[a, b]$, such that $g(c) = k$. Thus $\int_a^b f(x)g(x)dx = g(c)\int_a^b f(x)dx.$

Question 5.

(a) (i) The volume of the solid of revolution obtained by rotating about the x-axis the region bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by the Riemann integral $\int_a^b \pi(f(x))^2dx.$

(ii) The equation of the ellipse is $\frac{x^2}{3} + \frac{y^2}{5} = 1.$

Thus the curve required is the part of the ellipse above the x- axis. It is of course given by

$$f(x) = \sqrt{5(1-\tfrac{x^2}{3})} \quad \text{for } -\sqrt{3} \le x \le \sqrt{3}.$$

Thus by the formula in (i) the volume of the solid of revolution obtained by rotating the ellipse is given by

$$\int_{-\sqrt{3}}^{\sqrt{3}} 5\pi(1-\frac{x^2}{3})dx = 5\pi\left[x - \frac{x^3}{9}\right]_{-\sqrt{3}}^{\sqrt{3}} = 10\pi(\sqrt{3} - \frac{\sqrt{3}}{3}) = \frac{20\sqrt{3}}{3}\pi$$

(b) Recall $k(x) = \int_1^{x^5}(1+t^2+\cos(\sin(\pi t)))dt.$

(i) Therefore, since the integrand $1+t^2+\cos(\sin(\pi t))$ is continuous for all x in **R**, k is differentiable on **R** and

$$k'(x) = (1+x^{10}+\cos(\sin(\pi x^5)))\cdot 5x^4$$

by the Fundamental Theorem of Calculus and the Chain Rule.

Hence, for $x \neq 0$, $k'(x) > 0$ because $5x^4 > 0$ and $1+x^{10}+\cos(\sin(\pi x^5)) \geq x^{10} > 0$. Since k is continuous on **R**, because it is differentiable on **R**, k is (strictly) increasing on $(-\infty, 0]$ and on $[0, \infty)$. Therefore, k is (strictly) increasing on **R** and hence k is injective.

(ii) Note that $(k^{-1})'(0) = \dfrac{1}{k'(k^{-1}(0))}$.

$k(1) = \int_1^1 (1+t^2+\cos(\sin(\pi t)))dt = 0$ and so since k is injective $k^{-1}(0) = 1$.

From part (i) $k'(1) = 5(2+\cos(\sin(\pi))) = 15$.

Thus, $(k^{-1})'(0) = \dfrac{1}{k'(k^{-1}(0))} = \dfrac{1}{k'(1)} = \dfrac{1}{15}$.

(c) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \dfrac{i^2}{n^4}\cdot\sqrt[3]{7n^3+2i^3}$.

We shall write the summation $\sum\limits_{i=1}^{n} \dfrac{i^2}{n^4}\cdot\sqrt[3]{7n^3+2i^3}$ as a Riemann sum

$$\sum_{i=1}^{n} \frac{i^2}{n^4}\cdot\sqrt[3]{7n^3+2i^3} = \sum_{i=1}^{n} \frac{i^2}{n^2}\sqrt[3]{7+2\left(\frac{i}{n}\right)^3}\cdot\frac{1}{n} = \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $x_0 < x_1 < \cdots < x_n$ is a regular partition and $\Delta x = \Delta x_i = x_i - x_{i-1}$.

Therefore, we can take $x_i = \frac{i}{n}$ so that $\Delta x = \frac{1}{n}, x_0 = 0$ and $x_n = 1$. Thus by comparing,

$$f(x_i)\Delta x \text{ with } \frac{i^2}{n^2}\sqrt[3]{7+2\left(\frac{i}{n}\right)^3}\cdot\frac{1}{n}$$

we would want $f(x_i) = \dfrac{i^2}{n^2}\sqrt[3]{7+2\left(\frac{i}{n}\right)^3} = x_i^2\sqrt[3]{7+2x_i^3}$. Thus $f(x) = x^2\sqrt[3]{7+2x^3}$.

Therefore, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \dfrac{i^2}{n^4}\cdot\sqrt[3]{7n^3+2i^3} . = \int_0^1 x^2\sqrt[3]{7+2x^3}\,dx$

$$= \frac{1}{6}\int_0^1 6x^2\sqrt[3]{7+2x^3}\,dx = \frac{1}{6}\int_0^1 \sqrt[3]{u}\,\frac{du}{dx}dx, \text{ where } u = 7+2x^3$$

$$= \frac{1}{6}\int_7^9 \sqrt[3]{u}\,du \text{ by Change of Variable}$$

$$= \frac{1}{6}\cdot\frac{3}{4}[u^{4/3}]_7^9 = \frac{1}{8}(9\sqrt[3]{9} - 7\sqrt[3]{7}).$$

Question 6

Recall $f(x) = x^5 - 20x^2 + 7$.

(a) Note that f is continuous on **R** since it is a polynomial function.

Now

$$f'(x) = 5x^4 - 40x = 5x(x^3-8) = 5x(x-2)(x^2+2x+4)$$

$$= 5x(x-2)((x+1)^2+3) \quad \text{--------------------------} \quad (1)$$

Therefore, for $x < 0$, $f'(x) > 0$ and so f is increasing on the interval $(-\infty, 0]$.

From (1), for $0 < x < 2$, $f\,'(x) < 0$ and so f is decreasing on [0, 2]. From (1), for $x > 2$, $f\,'(x) > 0$ and so f is increasing on $[2, \infty)$.

(b) $f''(x) = 20x^3 - 40 = 20(x^3 - 2)$

$$= 20(x - 2^{1/3})(x^2 + 2^{1/3}x + 2^{2/3})$$

$$= 20(x - 2^{1/3})((x + \tfrac{1}{2}\cdot 2^{1/3})^2 + \tfrac{3}{4}2^{2/3}) \text{-------------------- (2)}$$

Thus, $f''(x) < 0$ for $x < 2^{1/3}$. Therefore, the graph of f is concave downward on the interval $(-\infty, 2^{1/3})$. From (2), for $x > 2^{1/3}$, $f''(x) > 0$ and so the graph of f is concave upward on the interval $(2^{1/3}, \infty)$.

(c) By part (a) $f(0) = 7$ is a relative maximum and $f(2) = -41$ is a relative minimum.

(d) From part (b), there is a change of concavity before and after $x = 2^{1/3}$.

Now $f(2^{1/3}) = 2^{5/3} - 20\cdot 2^{2/3} + 7 = 7 - 18\cdot 2^{2/3}$.

Hence, the only point of inflection of the graph of f is

$$(2^{1/2}, 7 - 18\cdot 2^{2/3}).$$

(e) The graph of f .