NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2005 – 2006

MA1102R CALCULUS

November 2005 – Time Allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO** sections: Section A and Section B. It contains a total of **SIX** questions and comprises **FOUR** printed pages.

2. Answer **ALL** questions in **Section A**. The marks for questions in Section A are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Answer not more than **TWO** questions from Section B. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

*Answer **ALL** questions in this section.*

Question 1 [**20** marks]

Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} -x^2 + 1, & x < 0 \\ (x-1)^2 \sin\left(\dfrac{\pi}{2(x^2+1)}\right), & 0 \le x < 1 \\ x^4 - 4x + 3, & x \ge 1 \end{cases}.$$

(a) Determine all x in $\mathbf{R}$ at which the function f is *continuous*. Justify your answer.

(b) Find the image of the interval [0, 1] under f, i.e., find $f([0,1])$.

(c) Find the *range* of the function f.

(d) Determine if f is *surjective*.

(e) Determine if f is *differentiable* at x, when $x = 0$ *or* 1. Justify your answer.

Question 2 [**20** marks]

Evaluate, if it exists, each of the following limits.

(a) $\lim\limits_{x \to +\infty} \sqrt{\dfrac{x^5 + 3x^2 + \cos(1/x) + 2}{6x^5 + 3x + 1}}$.

(b) $\lim\limits_{x \to 0} \dfrac{\sin(2x + \sin(x + \sin(x)))}{x + \sin(x)}$.

(c) $\lim\limits_{x \to 0^+} x \cos(e^{(1/x)})$.

(d) $\lim\limits_{x \to 0} (1 + 9x^3)^{(1/x^3)}$.

(e) $\lim\limits_{x \to \infty} \dfrac{(\ln(\ln(x)))^3}{\ln(x)}$.

Question 3 [**20** marks]

(a) Evaluate $\int \dfrac{35-9x^2-4x}{(x^2+4x+5)(x^2-4x+5)}dx.$

(b) Compute $\int_1^3 \sqrt{x+2[x]}\,dx$, where $[t]$ denotes the greatest integer $\le t$.

(c) Find an antiderivative of $g(x)$, which is defined by

$$g(x)=\begin{cases} x+x^5+3, & x<1 \\ 3\sin(\frac{\pi x}{2})-\cos(\pi x)e^{\sin(\pi x)}+1, & x\ge 1 \end{cases}.$$

(d) Evaluate $\int \sin^{-1}(2x)dx.$

(e) Evaluate $\int \sin^5(3x)\cos^4(3x)dx.$

SECTION B

Answer not more than ***TWO*** *questions from this section. Each question in this section carries 20 marks.*

Question 4 [**20** marks]

(a) Find the critical points of the function g, defined by

$$g(x)=x^3-12x^2+45x+1,$$

in the open interval (2, 6). Determine the absolute maximum and the absolute minimum values of the function in the interval [2, 6].

(b) Differentiate each of the following functions.

(i) $h(x)=(7+2\sin(x+\cos(x)))^{\sec(x)}$, $x\in(0,\frac{\pi}{2})$.

(ii) $j(x)=\int_{\ln(x)}^{x^3}\dfrac{t^2}{2+\sin(t^2)+e^t}dt,\ x\in(0,\infty)$.

(iii) $k(x)=\tan^{-1}(\sec^2(x)),\ x\in(0,\frac{\pi}{2})$.

(c) Suppose f and g are two differentiable functions defined on the real numbers **R** such that $f'(x)>g'(x)$ for all x in **R** and $f(0)=g(0)$. Prove that for $x>0$, $f(x)>g(x)$ and for $x<0$, $f(x)<g(x)$.

Question 5 [**20** marks]

(a) (i) Suppose f is a continuous function defined on the closed and bounded interval $[a, b]$ such that f is differentiable on $[a, b]$. Give the integral formula for the arc length of the curve $y = f(x)$ from $x = a$ to $x = b$.

(ii) Use this formula or otherwise, find the arc length of the curve $y = \ln(\cos(x))$ from $x = 0$ to $x = \frac{\pi}{3}$.

(b) Differentiate the function k defined on **R** by

$$k(x) = \int_1^{x^3} (2 + \sin(\cos(t)))dt.$$

(i) Without integrating, show that the function k is injective.

(ii) Determine $(k^{-1})'(0)$.

(c) Find the following limit.

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^3}{n^6} \cdot \sqrt{3n^4 + i^4}\ .$$

Question 6 [**20** marks]

Let the function f be defined on **R** by

$$f(x) = 2x^5 - 5x^2 + 3\ .$$

(a) Find the intervals on which f is (i) *increasing*, and (ii) *decreasing*.

(b) Find the intervals on which the graph of f is (i) *concave upward*, and (ii) *concave downward*.

(c) Find the *relative extrema* of f, if any.

(d) Find the *points of inflection* of the graph of f.

(e) Sketch the graph of f.

END OF PAPER

Answer To MA1102 Calculus

Question 1

The function f is defined by $f(x) = \begin{cases} -x^2+1, & x<0 \\ (x-1)^2 \sin\left(\dfrac{\pi}{2(x^2+1)}\right), & 0 \le x < 1 \\ x^4-4x+3, & x \ge 1 \end{cases}$.

(a) For $x<0$, $f(x) = -x^2+1$ is a polynomial function. Therefore, f is continuous on the interval $(-\infty, 0)$ since any polynomial function is continuous on **R** and hence on any interval. Similarly for $x > 1$, $f(x) = x^4 - 4x + 3$ is a polynomial function there and so is continuous on $(1, \infty)$.

For $0 < x < 1$, $f(x) = (x-1)^2 \sin\left(\frac{\pi}{2(x^2+1)}\right)$ and so f is continuous on (0, 1) since $\sin\left(\frac{\pi}{2(x^2+1)}\right)$ is continuous on **R** and $(x-1)^2$ is continuous on **R** so that the product of these two functions is continuous on **R** and so on (0, 1). Thus it remains to check the continuity of f at 0 and 1. Note that $f(0) = 1$ and $f(1) = 0$.

Now we determine the left limit at $x = 0$. It is $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -x^2+1 = 1$

The right limit at $x = 0$ is $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x-1)^2 \sin\left(\frac{\pi}{2(x^2+1)}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$

Hence, $\lim_{x\to 0} f(x) = 1$, and since $f(0) = 1$ it follows that f is continuous at $x = 0$.

Now consider the left limit of f at $x = 1$,

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x-1)^2 \sin\left(\frac{\pi}{2(x^2+1)}\right) = 0\cdot\sin\left(\frac{\pi}{4}\right) = 0$

Now $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^4-4x+3 = 0$ and so $\lim_{x\to 1} f(x) = 0$.

Therefore, $\lim_{x\to 1} f(x) = 0 = f(1)$ and so f is continuous at $x = 1$.

Therefore f is continuous at x for any x in **R**.

(b) To determine the image $f([0, 1])$, first note that $f(0) = 1$ and $f(1) = 0$.

Now observe that for $0 \le x \le 1$, $1 \le x^2+1 \le 2$ so that $\frac{\pi}{4} \le \frac{\pi}{2(x^2+1)} \le \frac{\pi}{2}$ and so

$1 \ge \sin\left(\frac{\pi}{2(x^2+1)}\right) \ge 0$ Therefore, for $0 \le x \le 1$, $(x-1)^2 \ge (x-1)^2 \sin\left(\frac{\pi}{2(x^2+1)}\right) \ge 0$. For $0 \le x \le 1$, $0 \le (x-1)^2 \le 1$. Thus, for $0 \le x \le 1$,

$$0 \le f(x) = (x-1)^2 \sin\left(\frac{\pi}{2(x^2+1)}\right) \le (x-1)^2 \le 1.$$

That means $f([0, 1]) \subseteq [0, 1]$. Since f is continuous on [0, 1] by part (a) and because $f(0) = 1$ and $f(1) = 0$, by the Intermediate Value Theorem, $[0, 1] \subseteq f([0, 1])$. Therefore, $f([0, 1]) = [0, 1]$.

(c) For $x < 0$, $-x^2 + 1 < 1$ and so the image $f((-\infty, 0)) \subseteq (-\infty, 1)$. Now for any $y < 1$, $-x^2 + 1 = y$ implies that $x^2 = 1 - y$ and so we have a solution $x = -\sqrt{1-y} < 0$ to $-x^2 + 1 = y$ in $(-\infty, 0)$. Therefore, $(-\infty, 1) \subseteq f((-\infty, 0))$. That means $f((-\infty, 0)) = (-\infty, 1)$.

Next for $x \ge 1$, $f(x) = x^4 - 4x + 3$ so that $f'(x) = 4x^3 - 4 > 0$ for $x > 1$. Therefore, f is strictly increasing on $[1, \infty)$ and so $f(x) \ge f(1) = 0$ for $x \ge 1$. Also note that

$\lim_{x\to\infty} f(x) = +\infty$ since $\lim_{x\to\infty} x^4 - 4x + 3 = \lim_{x\to\infty} x^4\left(1 - \frac{4}{x^3} + \frac{3}{x^4}\right) = +\infty$ because $\lim_{x\to\infty} x^4 = +\infty$ and

$\lim_{x\to\infty}(1-\frac{4}{x^3}+\frac{3}{x^4})=1>0$. Hence, since f is continuous on $[1,\infty)$, by the Intermediate Value Theorem $f([1,\infty))=[0,\infty)$. Therefore, the range of f is

$$f(\mathbf{R}) = f((-\infty,0))\cup f([0,1])\cup f([1,\infty)) = (-\infty,1)\cup[0,1]\cup[0,\infty))=\mathbf{R}.$$

(d) By part (c) Range(f) = $\mathbf{R}$ = codomain of f. Therefore, f is surjective.

(e) To check the differentiability of f at $x = 0$ consider the following limits.

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{-x^2+1-1}{x}=\lim_{x\to 0^-}-x=0$$

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{(x-1)^2\sin\left(\frac{\pi}{2(x^2+1)}\right)-1}{x}$$

$$=\lim_{x\to 0^+}\frac{2(x-1)\sin\left(\frac{\pi}{2(x^2+1)}\right)+(x-1)^2\cos\left(\frac{\pi}{2(x^2+1)}\right)\cdot\frac{-\pi x}{(x^2+1)^2}}{1}=-2$$

by L' Hôpital's Rule.

Thus, f is not differentiable at $x = 0$ since $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}\neq\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}$.

$$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{x^4-4x+3}{x-1}=\lim_{x\to 1^+}(4x^3-4)=0$$ by L' Hôpital's Rule.

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^-}\frac{(x-1)^2\sin\left(\frac{\pi}{2(x^2+1)}\right)}{x-1}=\lim_{x\to 1^-}(x-1)\sin\left(\frac{\pi}{2(x^2+1)}\right)=0$$

Therefore, f is differentiable at $x = 1$ since $\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=0$ and $f'(1) = 0$.

OR,

$$f'(x)=\begin{cases}-2x, & x<0\\ 2(x-1)\sin\left(\frac{\pi}{2(x^2+1)}\right)+(x-1)^2\cos\left(\frac{\pi}{2(x^2+1)}\right)\cdot\frac{-\pi x}{(x^2+1)^2}, & 0<x<1\\ 4x^3-4, & x>1\end{cases}$$

$$\lim_{x\to 0^-}f'(x)=\lim_{x\to 0^-}-2x=0,$$

$$\lim_{x\to 0^+}f'(x)=\lim_{x\to 0^+}2(x-1)\sin\left(\frac{\pi}{2(x^2+1)}\right)+(x-1)^2\cos\left(\frac{\pi}{2(x^2+1)}\right)\cdot\frac{-\pi x}{(x^2+1)^2}=-2$$

Since both limits $\lim_{x\to 0^-}f'(x)$ and $\lim_{x\to 0^+}f'(x)$ are finite and not the same, f is not differentiable at $x = 0$.

Now $\lim_{x\to 1^+}f'(x)=\lim_{x\to 1^+}(4x^3-4)=0$ and

$\lim_{x\to 1^-}f'(x)=\lim_{x\to 1^-}2(x-1)\sin\left(\frac{\pi}{2(x^2+1)}\right)+(x-1)^2\cos\left(\frac{\pi}{2(x^2+1)}\right)\cdot\frac{-\pi x}{(x^2+1)^2}=0$. Thus both limits $\lim_{x\to 1^-}f'(x)$ and $\lim_{x\to 1^+}f'(x)$ are finite and the same, therefore, f is differentiable at $x = 1$.

Question 2

(a) $\lim_{x\to+\infty}\sqrt{\dfrac{x^5+3x^2+\cos(1/x)+2}{6x^5+3x+1}}=\lim_{x\to+\infty}\sqrt{\dfrac{1+\frac{3}{x^3}+\frac{1}{x^5}(\cos(1/x)+2)}{6+\frac{3}{x^4}+\frac{1}{x^5}}}=\sqrt{\dfrac{1+0+0}{6+0+0}}=\dfrac{1}{\sqrt{6}}.$

This is because $\lim_{x\to+\infty}\frac{1}{x^3}=\lim_{x\to+\infty}\frac{1}{x^4}=\lim_{x\to+\infty}\frac{1}{x^5}=0$ and $\lim_{x\to+\infty}\frac{\cos(1/x)+2}{x^5}=0$ by the Squeeze Theorem since $-|\frac{3}{x^5}|\le\frac{\cos(2/x)+2}{x^5}\le|\frac{3}{x^5}|$ for $x\neq0$ and $\lim_{x\to+\infty}|\frac{3}{x^5}|=0$

or

$$\lim_{x\to+\infty}\frac{\cos(1/x)+2}{x^5}=\lim_{x\to+\infty}\frac{\cos(1/x)}{x^5}+\lim_{x\to+\infty}\frac{2}{x^5}=\lim_{x\to+\infty}\frac{2}{x^5}\lim_{x\to+\infty}\cos(1/x)+0=0\times1=0.$$

(b) $\lim_{x\to0}\dfrac{\sin(2x+\sin(x+\sin(x)))}{x+\sin(x)}=\lim_{x\to0}\dfrac{\sin(2x+\sin(x+\sin(x)))}{2x+\sin(x+\sin(x))}\cdot\dfrac{2x+\sin(x+\sin(x))}{x+\sin(x)}$

$$=\lim_{x\to0}\frac{\sin(2x+\sin(x+\sin(x)))}{2x+\sin(x+\sin(x))}\cdot\left(\frac{2x}{x+\sin(x)}+\frac{\sin(x+\sin(x))}{x+\sin(x)}\right)$$

$$=\lim_{x\to0}\frac{\sin(2x+\sin(x+\sin(x)))}{2x+\sin(x+\sin(x))}\cdot\lim_{x\to0}\left(\frac{2}{1+\sin(x)/x}+\frac{\sin(x+\sin(x))}{x+\sin(x)}\right)=1\cdot(1+1)=2$$

because $\lim_{x\to0}\dfrac{\sin(2x+\sin(x+\sin(x)))}{2x+\sin(x+\sin(x))}=1$ and $\lim_{x\to0}\dfrac{\sin(x+\sin(x))}{x+\sin(x)}=\lim_{x\to0}\dfrac{\sin(x)}{x}=1.$

OR

$$\lim_{x\to0}\frac{\sin(2x+\sin(x+\sin(x)))}{x+\sin(x)}=\lim_{x\to0}\frac{\cos(2x+\sin(x+\sin(x)))\cdot(2+\cos(x+\sin(x))(1+\cos(x)))}{1+\cos(x)}$$

by L' Hôpital's Rule

$$=\frac{\cos(0)\cdot(2+\cos(0)\cdot(1+\cos(0)))}{2}=2$$

(c) $\lim_{x\to0^+}x\cos(e^{(1/x)})=0$ by the Squeeze Theorem since

$-|x|\le x\cos(e^{(1/x)})\le|x|$ for $x>0$ and $\lim_{x\to0^+}|x|=0.$

(d) $\lim_{x\to0}(1+9x^3)^{(1/x^3)}$. Let $y=(1+9x^3)^{(1/x^3)}$. Then $\ln(y)=\frac{1}{x^3}\ln(1+9x^3)$

Since $\lim_{x\to0}\ln(y)=\lim_{x\to0}\frac{1}{x^3}\ln(1+9x^3)=\lim_{x\to0}\dfrac{\frac{27x^2}{1+9x^3}}{3x^2}=\lim_{x\to0}\dfrac{9}{1+9x^3}=9$

by L' Hôpital's Rule,

Therefore, $\lim_{x\to0}y=e^{\lim_{x\to0}\ln(y)}=e^9$

(e) $\lim_{x\to\infty}\dfrac{(\ln(\ln(x)))^3}{\ln(x)}=\lim_{x\to\infty}\dfrac{3(\ln(\ln(x)))^2\frac{1}{x\ln(x)}}{1/x}=\lim_{x\to\infty}\dfrac{3(\ln(\ln(x)))^2}{\ln(x)}$ by L' Hôpital's Rule,

$$=\lim_{x\to\infty}\frac{6(\ln(\ln(x)))^1\frac{1}{x\ln(x)}}{1/x}=\lim_{x\to\infty}\frac{6(\ln(\ln(x)))}{\ln(x)}=\lim_{x\to\infty}\frac{6\frac{1}{x\ln(x)}}{1/x}=\lim_{x\to\infty}6\frac{1}{\ln(x)}=0$$

by repeated use of L' Hôpital's Rule and the fact that $\lim_{x\to\infty}\ln(x)=\infty$.

Question 3

(a) $\int \frac{35-9x^2-4x}{(x^2+4x+5)(x^2-4x+5)}dx.$

$$\int \frac{35-9x^2-4x}{(x^2+4x+5)(x^2-4x+5)}dx = \int \frac{2x+4}{(x^2+4x+5)}dx + \int \frac{-2x+3}{(x^2-4x+5)}dx$$

by a partial fraction expansion determined as follows.

Writing,

$$\frac{35-9x^2-4x}{(x^2+4x+5)(x^2-4x+5)} = \frac{Ax+B}{x^2+4x+5} + \frac{Cx+D}{x^2-4x+5}$$

then $(Ax+B)(x^2-4x+5)+(Cx+D)(x^2+4x+5) = 35-9x^2-4x.$

Comparing coefficients of x^3 : A + C = 0 ---------------------------- (1)
Comparing constant terms : 5B +5D = 35, i.e. B + D = 7 ----------------- (2)
Comparing coefficients of x^2 : –4A+B + 4C+D = –9.
Since B+D = 7 by (2) we get –4A +4C = –9 –7 = –16, i.e.,
–A + C = – 4 ----------------------------- (3)
Comparing coefficients of x: 5A– 4B +5C+4D = –4.
Since A + C = 0 we get from above – B + D = –1 ---------------- (4)
Equation (1) + Equation (3) gives 2C = – 4 and so C= –2 and A = –C = 2.
Equation (2) + Equation (4) gives 2D = 6 and so D = 3 and B = 7–D = 4

Now $\int \frac{2x+4}{(x^2+4x+5)}dx = \ln|x^2+4x+5| + C$

And $\int \frac{-2x+3}{(x^2-4x+5)}dx = \int \frac{-(2x-4)}{(x^2-4x+5)}dx - \int \frac{1}{(x^2-4x+5)}dx$

$$= -\ln|x^2-4x+5| - \int \frac{1}{(x-2)^2+1}dx = -\ln|x^2-4x+5| - \tan^{-1}(x-2) + C'$$

Therefore,

$$\int \frac{35-9x^2-4x}{(x^2+4x+5)(x^2-4x+5)}dx = \ln|x^2+4x+5| - \ln|x^2-4x+5| - \tan^{-1}(x-2) + C''$$

$$= \ln\left|\frac{x^2+4x+5}{x^2-4x+5}\right| - \tan^{-1}(x-2) + C''$$

(b) $\int_1^3 \sqrt{x+2[x]}\,dx = \int_1^2 \sqrt{x+2}\,dx + \int_2^3 \sqrt{x+4}\,dx = \left[\tfrac{2}{3}(x+2)^{3/2}\right]_1^2 + \left[\tfrac{2}{3}(x+4)^{3/2}\right]_2^3$
$= \tfrac{2}{3}(4^{3/2}-3^{3/2}) + \tfrac{2}{3}(7^{3/2}-6^{3/2}) = \tfrac{2}{3}(8-3\sqrt{3}+7\sqrt{7}-6\sqrt{6}).$

(c) $g(x) = \begin{cases} x+x^5+3, & x<1 \\ 3\sin(\frac{\pi x}{2}) - \cos(\pi x)e^{\sin(\pi x)} + 1, & x \ge 1 \end{cases}$.

First note that g is continuous on the interval $(-\infty, 1)$ since it is a polynomial function there and polynomial functions are continuous. Note also that g is continuous on $(1, \infty)$ since $3\sin(\frac{\pi x}{2})$ is a continuous function and the product $\cos(\pi x)e^{\sin(\pi x)}$ is continuous on $(1, \infty)$. Now the left limit at $x = 1$ is $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} x+x^5+3 = 5$ and the right limit at $x = 1$, $\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} 3\sin(\frac{\pi x}{2}) - \cos(\pi x)e^{\sin(\pi x)} + 1 = 3 - \cos(\pi) + 1 = 5 = g(1)$. Therefore, $\lim_{x\to 1} g(x) = g(1)$. Thus g is continuous at x =1. Therefore, g is continuous on **R** and we can use the Fundamental Theorem of Calculus to obtain an antiderivative $G(x)$ given by the following Riemann integral for each x in **R**.

$$G(x) = \int_1^x g(t)dt = \begin{cases} \int_1^x g(t)dt, x < 1 \\ \int_1^x g(t)dt, x \geq 1 \end{cases} = \begin{cases} \int_1^x (t + t^5 + 3)dt, x < 1 \\ \int_1^x (3\sin(\frac{\pi t}{2}) - \cos(\pi t)e^{\sin(\pi t)} + 1)dt, x \geq 1 \end{cases}$$

$$= \begin{cases} \left[\frac{1}{6}t^6 + \frac{1}{2}t^2 + 3t)\right]_1^x, x < 1 \\ \left[-\frac{1}{\pi}e^{\sin(\pi t)} - \frac{6}{\pi}\cos(\frac{\pi t}{2}) + t\right]_1^x, x \geq 1 \end{cases} = \begin{cases} \frac{1}{6}x^6 + \frac{1}{2}x^2 + 3x - 3\frac{2}{3}, x < 1 \\ -\frac{1}{\pi}e^{\sin(\pi x)} - \frac{6}{\pi}\cos(\frac{\pi x}{2}) + x + \frac{1}{\pi} - 1, x \geq 1 \end{cases}$$

Thus, any antiderivative is given by $G(x) + C$ for any constant C.

(d) $\int \sin^{-1}(2x)dx = x\sin^{-1}(2x) - \int x \cdot \frac{2}{\sqrt{1-4x^2}}dx$

by integration by parts

$$= x\sin^{-1}(2x) + \frac{1}{4}\int \frac{-8x}{\sqrt{1-4x^2}}dx = x\sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2} + C$$

(e) $\int \sin^5(3x)\cos^4(3x)dx = -\int \frac{1}{3}\sin^4(3x)\cos^4(3x)\cdot(-3\sin(3x))dx$

$$= -\int \frac{1}{3}(1-\cos^2(3x))^2\cos^4(3x)\cdot(-3\sin(3x))dx = -\frac{1}{3}\int u^4(1-u^2)^2\frac{du}{dx}dx,$$

where $u = \cos(3x)$

$$= -\frac{1}{3}\int u^4(1-2u^2+u^4)du = -\frac{1}{3}\int(u^4 - 2u^6 + u^8)du$$ by substitution or change of variable.

$$= -\frac{1}{3}(\frac{u^5}{5} - 2\frac{u^7}{7} + \frac{u^9}{9}) + C = -\frac{1}{15}\cos^5(3x) + \frac{2}{21}\cos^7(3x) - \frac{1}{27}\cos^9(3x) + C.$$

Question 4.

(a) Recall $g(x) = x^3 - 12x^2 + 45x + 1$.

Thus, $g'(x) = 3x^2 - 24x + 45 = 3(x-3)(x-5)$. Therefore, $g'(x) = 0$ if and only if $x = 3$ or 5. Hence g has two stationary points in (2, 6), namely 3 and 5. Since g is differentiable, the critical points of g in (2, 6) are 3 and 5. Since g is continuous on the closed and bounded interval [2, 6] and so by the Extreme Value Theorem g has absolute extrema on the interval [2, 6] and they are given respectively by the maximum and minimum of the values of the critical points and the end points under g. Now g(2) = 51, g(3) = 55, g(5) =51 and g(6) = 55. Therefore, the absolute maximum of g on [2, 6] is 55 and the absolute minimum of g on [2, 6] is 51.

(b) (i) $h(x) = (7 + 2\sin(x + \cos(x)))^{\sec(x)}$, $x \in (0, \frac{\pi}{2})$.

Taking logarithm on both sides we get $\ln(h(x)) = \sec(x)\ln(7 + 2\sin(x + \cos(x)))$.

Differentiating both sides we get,

$$\frac{h'(x)}{h(x)} = \sec(x)\tan(x)\ln(7 + 2\sin(x + \cos(x))) + \sec(x)\frac{2\cos(x + \cos(x))(1 - \sin(x))}{7 + 2\sin(x + \cos(x))}$$

Therefore, $h'(x) =$

$$(7+2\sin(x+\cos(x)))^{\sec(x)}\sec(x)\left[\tan(x)\ln(7+2\sin(x+\cos(x))) + \frac{2\cos(x+\cos(x))(1-\sin(x))}{7+2\sin(x+\cos(x))}\right].$$

(ii) $j(x)=\int_{\ln(x)}^{x^3}\frac{t^2}{2+\sin(t^2)+e^t}dt,\ x\in(0,\infty).$

Therefore, $j(x)=\int_0^{x^3}\frac{t^2}{2+\sin(t^2)+e^t}dt-\int_0^{\ln(x)}\frac{t^2}{2+\sin(t^2)+e^t}dt.$

Hence by the Fundamental Theorem of Calculus and the Chain Rule,

$$j'(x)=\frac{3x^8}{2+\sin(x^6)+e^{x^3}}-\frac{(\ln(x))^2\frac{1}{x}}{2+\sin((\ln(x))^2)+x}$$

(iii) $k(x)=\tan^{-1}(\sec^2(x)),\ x\in(0,\frac{\pi}{2}).$ Thus by the Chain Rule

$$k'(x)=(\tan^{-1})'(\sec^2(x))\cdot 2\sec^2(x)\tan(x)=\frac{2\sec^2(x)\tan(x)}{\tan'(\tan^{-1}(\sec^2(x)))}$$

$$=\frac{2\sec^2(x)\tan(x)}{\sec^2(\tan^{-1}(\sec^2(x)))}=\frac{2\sec^2(x)\tan(x)}{1+\sec^4(x)}=\frac{2\cos(x)\sin(x)}{1+\cos^4(x)}=\frac{\sin(2x)}{1+\cos^4(x)}.$$

(c) Define $h(x) = f(x) - g(x)$ for x in **R.** Then since both f and g are differentiable on R, h is also differentiable on **R** and $h'(x) = f'(x) - g'(x) > 0$ because it is given that $f'(x) > g'(x)$ for all x in **R**. Therefore, h is increasing on **R**. Hence for $x > 0$, $h(x) > h(0) = f(0) - g(0) = 0$ since it is given that $f(0) = g(0)$. Thus, for $x > 0$ $f(x) - g(x) = h(x) > 0$ and so $f(x) > g(x)$. Similarly for $x < 0$, $f(x) - g(x) = h(x) < h(0) = f(0) - g(0) = 0$ so that $f(x) < g(x)$.

Question 5.

(a) (i) The arc length of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by,

$$\int_a^b\sqrt{1+(f'(x))^2}\,dx$$

when f is differentiable on $[a, b]$.

(ii) By the above formula the arc length of the curve $y = \ln(\cos(x))$ from $x = 0$ to $x=\frac{\pi}{3}$ is

$$\int_0^{\pi/3}\sqrt{1+\tan^2(x)}\,dx=\int_0^{\pi/3}\sec(x)dx=[\ln(|\sec(x)+\tan(x)|)]_0^{\pi/3}$$

$$=\ln(|\sec(\tfrac{\pi}{3})+\tan(\tfrac{\pi}{3})|)=\ln(2+\sqrt{3})\text{ unit.}$$

(b) Recall $k(x)=\int_1^{x^3}(2+\sin(\cos(t)))dt.$

(i) Therefore, since the integrand $2+\sin(\cos(x))$ is continuous for all x in **R**, k is differentiable on **R** and

$$k'(x)=(2+\sin(\cos(x^3)))\cdot 3x^2$$

by the Fundamental Theorem of Calculus and the Chain Rule.

Thus, for $x \neq 0$, $k'(x) > 0$. Since k is continuous on **R**, because it is differentiable on **R**, k is (strictly) increasing on $(-\infty, 0]$ and on $[0, \infty)$. Therefore, k is (strictly) increasing on **R** and hence k is injective.

(ii) Note that $(k^{-1})'(0) = \dfrac{1}{k'(k^{-1}(0))}$.

$k(1) = \int_1^1 (2+\sin(\cos(t)))dt = 0$ and so since k is injective $k^{-1}(0) = 1$.

From part (i) $k'(1) = 3(2+\sin(\cos(1)))$.

Thus, $(k^{-1})'(0) = \dfrac{1}{k'(k^{-1}(0))} = \dfrac{1}{k'(1)} = \dfrac{1}{3(2+\sin(\cos(1)))}$.

(c) $\lim_{n\to\infty} \sum_{i=1}^{n} \dfrac{i^3}{n^6} \cdot \sqrt{3n^4+i^4}$.

We shall write the summation $\sum_{i=1}^{n} \dfrac{i^3}{n^6} \cdot \sqrt{3n^4+i^4}$ as a Riemann sum

$$\sum_{i=1}^{n} \frac{i^3}{n^6} \cdot \sqrt{3n^4+i^4} = \sum_{i=1}^{n} \frac{i^3}{n^3} \sqrt{3+\left(\frac{i}{n}\right)^4} \cdot \frac{1}{n} = \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $x_0 < x_1 < \cdots < x_n$ is a regular partition and $\Delta x = \Delta x_i = x_i - x_{i-1}$.
Therefore, we can take $x_i = \dfrac{i}{n}$ so that $\Delta x = \dfrac{1}{n}, x_0 = 0$ and $x_n = 1$. Thus by comparing,

$$f(x_i)\Delta x \text{ with } \frac{i^3}{n^3} \sqrt{3+\left(\frac{i}{n}\right)^4} \cdot \frac{1}{n}$$

we would want $f(x_i) = \dfrac{i^3}{n^3} \sqrt{3+\left(\dfrac{i}{n}\right)^4} = x_i^3 \sqrt{3+x_i^4}$. Thus $f(x) = x^3\sqrt{3+x^4}$.

Therefore, $\lim_{n\to\infty} \sum_{i=1}^{n} \dfrac{i^3}{n^6} \cdot \sqrt{3n^4+i^4} . = \int_0^1 x^3\sqrt{3+x^4}\,dx$

$$= \frac{1}{4}\int_0^1 4x^3\sqrt{3+x^4}\,dx = \frac{1}{4}\int_0^1 \sqrt{u}\,\frac{du}{dx}dx, \text{ where } u = 3 + x^4$$

$$= \frac{1}{4}\int_3^4 \sqrt{u}\,du \text{ by Change of Variable}$$

$$= \frac{1}{4} \cdot \frac{2}{3}[u^{3/2}]_3^4 = \tfrac{1}{6}(8 - 3\sqrt{3}) = \frac{4}{3} - \frac{\sqrt{3}}{2}.$$

Question 6

Recall $f(x) = 2x^5 - 5x^2 + 3$.

(a) Note that f is continuous on **R** since it is a polynomial function.

Now

$$f'(x) = 10x^4 - 10x = 10x(x^3-1) = 10x(x-1)(x^2+x+1)$$

$$= 10x(x-1)((x+\tfrac{1}{2})^2 + \tfrac{3}{4}) \qquad \text{---------------------------} \quad (1)$$

Therefore, for $x < 0$, $f'(x) > 0$ and so f is increasing on the interval $(-\infty, 0]$.

From (1), for $0 < x < 1$, $f'(x) < 0$ and so f is decreasing on $[0, 1]$. From (1), for $x > 1$, $f'(x) > 0$ and so f is increasing on $[1, \infty)$.

(b) $f''(x) = 40x^3 - 10 = 40(x^3 - \tfrac{1}{4})$

$$= 40(x - \tfrac{1}{4^{1/3}})(x^2 + \tfrac{1}{4^{1/3}}x + \tfrac{1}{4^{2/3}})$$

$$= 40(x - \tfrac{1}{4^{1/3}})((x + \tfrac{1}{2}\cdot\tfrac{1}{4^{1/3}})^2 + \tfrac{3}{4}\tfrac{1}{4^{2/3}}) \text{--------------------} (2)$$

Thus, $f''(x) < 0$ for $x < 1/4^{1/3}$. Therefore, the graph of f is concave downward on the interval $(-\infty, 1/4^{1/3})$. From (2), for $x > 1/4^{1/3}$, $f''(x) > 0$ and so the graph of f is concave upward on the interval $(1/4^{1/3}, \infty)$.

(c) By part (a) $f(0) = 3$ is a relative maximum and $f(1) = 0$ ia a relative minimum.

(d) From part (b), there is a change of concavity before and after $x = 1/4^{1/3}$.

Now $f(\frac{1}{4^{1/3}}) = 2\frac{1}{4^{5/3}} - 5\frac{1}{4^{2/3}} + 3 = \frac{1}{4^{2/3}}(\frac{1}{2} - 5) + 3 = 3 - \frac{9}{2 \cdot 4^{2/3}}$.

Hence, the only point of inflection of the graph of f is

$$(\frac{1}{4^{1/3}}, 3 - \frac{9}{2 \cdot 4^{2/3}}).$$

(e)