NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2004 – 2005

MA1102R CALCULUS

April 2005 – Time Allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO** sections: Section A and Section B. It contains a total of **SIX** questions and comprises **FOUR** printed pages.

2. Answer **ALL** questions in **Section A**. The marks for questions in Section A are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Answer not more than **TWO** questions from Section B. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

*Answer **ALL** questions in this section.*

Question 1 [**20** marks]

Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x^2 - 1, & x < 0 \\ -(x-1)^2\left|\sin\left(\dfrac{\pi}{2(x-1)}\right)\right|, & 0 \le x < 1 \\ x^3 - 3x + 2, & x \ge 1 \end{cases} .$$

(a) Determine all x in **R** at which the function f is *continuous*. Justify your answer.

(b) Find the image of the interval [0, 1] under f, i.e., find $f([0,1])$.

(c) Find the *range* of the function f.

(d) Determine if f is *surjective*.

(e) Determine if f is *differentiable* at x, when $x = 0$ *or* 1. Justify your answer.

Question 2 [**20** marks]

Evaluate, if it exists, each of the following limits.

(a) $\displaystyle\lim_{x\to+\infty} \sqrt{\frac{x^3 + 3x + \sin(x) + 1}{4x^3 + 7x + 1}}$.

(b) $\displaystyle\lim_{x\to 0} \frac{\sin(x^2 + \sin(x^2 + 3x))}{x^2 + 3x}$.

(c) $\displaystyle\lim_{x\to 0^+} x^2 \sin(\ln(x))$.

(d) $\displaystyle\lim_{x\to 0} (e^x + 7x)^{(\frac{1}{x})}$.

(e) $\displaystyle\lim_{x\to 0^+} (\sin(x^4))^{(1/\ln(x))}$.

Question 3 [**20** marks]

(a) Evaluate $\int \frac{6-5x^2-2x}{(x^2+2x+2)(x^2-2x+2)}dx.$

(b) Compute $\int_{-1}^{1} \sqrt{x+2|x|}\, dx.$

(c) Find an antiderivative of $g(x)$, which is defined by

$$g(x) = \begin{cases} x^3+x+7, \ x<1 \\ 3e^{(x-1)} - 6\cos(\pi x), \ x \geq 1 \end{cases}.$$

(d) Evaluate $\int e^{2x} \sin(5x) dx.$

(e) Evaluate $\int \sin^4(5x)\cos^3(5x) dx.$

SECTION B

Answer not more than ***TWO*** *questions from this section. Each question in this section carries 20 marks.*

Question 4 [**20** marks]

(a) (i) State the Extreme Value Theorem.

(ii) Find the critical points of the function g, defined by

$$g(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 4x + 3,$$

in the open interval (0, 5). Determine the absolute maximum and the absolute minimum values of the function in the interval [0, 5].

Hence, or otherwise, prove that there exists a point c in [0, 5] such that $g(c) = c$.

(b) Differentiate each of the following functions.

(i) $h(x) = (1+e^{\sin(2x^2)})^{\cot(x)}$, $x \in (0, \frac{\pi}{2})$.

(ii) $j(x) = \int_{x^2}^{\ln(1+x^2)} \frac{t}{1+t^2+\cos(t^2)}dt.$

(iii) $k(x) = \cos^{-1}\left(\frac{1}{1+x^2}\right)$.

Question 5 [**20** marks]

(a) Differentiate the function k defined on **R** by

$$k(x) = \int_1^x (1 + \frac{t^2}{1+\sin(\pi t)+e^t})dt.$$

(i) Without integrating, show that the function k is injective.

(ii) Determine $(k^{-1})'(0)$.

(b) Suppose f is a continuous function on $[a, b]$ and g is a Riemann integrable function on $[a, b]$. If $g(x) \geq 0$ for any x in $[a, b]$, then show that there exists a point c in $[a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

(c) Find the following limit.

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{i}{n^2} \cdot \sqrt{\frac{7n^2 + i^2}{n^2}}.$$

Question 6 [**20** marks]

Let the function f be defined on **R** by

$$f(x) = \begin{cases} x^5 - 5x^2 + 7, & x \geq 0 \\ x^2 + 7, & x < 0 \end{cases}.$$

(a) Find the intervals on which f is (i) *increasing*, and (ii) *decreasing*.

(b) Find the intervals on which the graph of f is (i) *concave upward*, and (ii) *concave downward*.

(c) Find the *relative extrema* of f, if any.

(d) Find the *points of inflection* of the graph of f.

(e) Sketch the graph of f.

END OF PAPER

Answer To MA1102 Calculus

Question 1

The function f is defined by $f(x)=\begin{cases} x^2-1, & x<0 \\ -(x-1)^2\left|\sin\left(\frac{\pi}{2(x-1)}\right)\right|, & 0\le x<1 \\ x^3-3x+2, & x\ge 1\end{cases}$.

(a) For $x<0$, $f(x)=x^2-1$ is a polynomial function. Therefore, f is continuous on the interval $(-\infty, 0)$ since any polynomial function is continuous on **R** and hence on any interval. Similarly for $x>1$, $f(x) = x^3 - 3x + 2$ is a polynomial function there and so is continuous on $(1, \infty)$.

For $0 < x < 1$, $f(x)=-(x-1)^2\left|\sin\left(\frac{\pi}{2(x-1)}\right)\right|$ and so f is continuous on (0, 1) since $\left|\sin\left(\frac{\pi}{2(x-1)}\right)\right|$ is continuous on **R** $-\{0\}$ and $-(x-1)^2$ is continuous on **R** so that the product of these two functions is continuous on **R** $-\{0\}$ and so on (0, 1). Thus it remains to check the continuity of f at 0 and 1. Note that $f(0) = -1$ and $f(1) = 0$.

Now we determine the left limit at $x = 0$. It is $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2-1=-1$

The right limit at $x = 0$ is $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} -(x-1)^2\left|\sin\left(\frac{\pi}{2(x-1)}\right)\right| = -1\left|\sin\left(-\frac{\pi}{2}\right)\right| = -1.$

Hence, $\lim_{x\to 0} f(x) = -1$, and since $f(0) = -1$ it follows that f is continuous at $x = 0$.

Now consider the left limit of f at $x = 1$,

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} -(x-1)^2\left|\sin\left(\frac{\pi}{2(x-1)}\right)\right| = 0$ by the Squeeze Theorem because for $x\neq 1$

$-(x-1)^2 \le -(x-1)^2\left|\sin\left(\frac{\pi}{2(x-1)}\right)\right| \le 0$ and $\lim_{x\to 1^-} -(x-1)^2 = 0$.

Now $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^3-3x+2=0$ and so $\lim_{x\to 1} f(x) = 0$.

Therefore, $\lim_{x\to 1} f(x) = 0 = f(1)$ and so f is continuous at $x = 1$.

Therefore f is continuous at x for any x in **R**.

(b) To determine the image $f([0, 1])$, first note that $f(0) = -1$ and $f(1) = 0$.

Now observe that for $0 \le x \le 1$, $-1 \le (x-1) \le 0$ so that $-1 \le -(x-1)^2 \le 0$.

Therefore, for $0 \le x < 1$, $-1\le -(x-1)^2 \le -(x-1)^2\left|\sin\left(\frac{\pi}{2(x-1)}\right)\right| = f(x)\le 0$. Hence for all x in [0, 1], $-1 \le f(x) \le 0$. That means $f([0, 1]) \subseteq [-1, 0]$. Since f is continuous on [0, 1] by part (a) and because $f(0) = -1$ and $f(1) = 0$, by the Intermediate Value Theorem, $[-1, 0] \subseteq f([0, 1])$. Therefore, $f([0, 1]) = [-1, 0]$.

(c) For $x < 0$, $x^2 -1 > -1$ and so the image $f((-\infty, 0)) \subseteq (-1, \infty)$. Now for any $y > -1$, $x^2-1= y$ implies that $x^2 = 1+ y$ and so we have a solution $x = -\sqrt{1+y} < 0$ to $x^2-1= y$ in $(-\infty, 0)$. Therefore, $(-1, \infty) \subseteq f((-\infty, 0))$. That means $f((-\infty, 0))= (-1, \infty)$.

Next for $x \ge 1$, $f(x) = x^3 - 3x + 2$ so that $f'(x) = 3x^2 -3 > 0$ for $x > 1$. Therefore, f is strictly increasing on $[1, \infty)$ and so $f(x) \ge f(1) =0$ for $x \ge 1$. Also note that $\lim_{x\to\infty} f(x) = +\infty$ since $\lim_{x\to\infty} x^3-3x+2 = \lim_{x\to\infty} x^3(1-\frac{3}{x^2}+\frac{2}{x^3}) = +\infty$ because $\lim_{x\to\infty} x^3 = +\infty$ and

$\lim_{x\to\infty}(1-\frac{3}{x^2}+\frac{2}{x^3}) = 1 > 0$. Hence, since f is continuous on $[1, \infty)$, by the Intermediate Value Theorem $f([1, \infty)) = [0, \infty)$. Therefore, the range of f is

$$f(\mathbf{R}) = f((-\infty, 0))\cup f([0, 1])\cup f([1, \infty)) = (-1, \infty)\cup[-1, 0]\cup[0, \infty))=[-1, \infty).$$

(d) By part (c) Range(f) $\neq$ **R** = codomain of f. Therefore, f is not surjective.

(e) To check the differentiability of f at $x = 0$ consider the following limits.

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{x^2-1+1}{x} = \lim_{x\to 0^-} x = 0$$

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{-(x-1)^2\left|\sin\left(\frac{\pi}{2(x-1)}\right)\right|+1}{x}$$

$$= \lim_{x\to 0^+}\frac{(x-1)^2\sin\left(\frac{\pi}{2(x-1)}\right)+1}{x}$$

This is because $\left|\sin\left(\frac{\pi}{2(x-1)}\right)\right| = -\sin\left(\frac{\pi}{2(x-1)}\right)$ for $0 < x < ½$. We derive this by observing that $0 < x < ½$ implies that $-1 < x-1 < -1/2$ so that

$$-1 > \frac{1}{x-1} > -2 \text{ and hence } -\frac{\pi}{2} > \frac{\pi}{2(x-1)} > -\pi \text{ and so } \sin\left(\frac{\pi}{2(x-1)}\right) < 0.$$

Therefore, $\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{(x-1)^2\sin\left(\frac{\pi}{2(x-1)}\right)+1}{x}$

$$= \lim_{x\to 0^+}\frac{2(x-1)\sin\left(\frac{\pi}{2(x-1)}\right)-(x-1)^2\cos\left(\frac{\pi}{2(x-1)}\right)\cdot\left(-\frac{\pi}{2(x-1)^2}\right)}{1}$$

by L' Hôpital's Rule.

$$= 2(0-1)\sin\left(-\frac{\pi}{2}\right) = 2.$$

Thus, f is not differentiable at $x = 0$ since $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} \neq \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}$.

$$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+}\frac{x^3-3x+2}{x-1} = \lim_{x\to 1^+}(-3x^2-3) = 0$$ by L' Hôpital's Rule.

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-}\frac{-(x-1)^2\left|\sin\left(\frac{\pi}{2(x-1)}\right)\right|}{x-1} = \lim_{x\to 1^-} = -(x-1)\left|\sin\left(\frac{\pi}{2(x-1)}\right)\right| = 0$$

by the Squeeze Theorem

since $0 \le -(x-1)\left|\sin\left(\frac{\pi}{2(x-1)}\right)\right| \le -(x-1)$ for $x < 1$ and $\lim_{x\to 1^-} -(x-1) = 0$.

Therefore, f is differentiable at $x = 1$ since $\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1} = 0$ and $f'(1) = 0$,.

Question 2

(a)

$$\lim_{x\to+\infty}\sqrt{\frac{x^3+3x+\sin(x)+1}{4x^3+7x+1}} = \lim_{x\to+\infty}\sqrt{\frac{1+\frac{3}{x^2}+\frac{1}{x^3}(\sin(x)+1)}{4+\frac{7}{x^2}+\frac{1}{x^3}}} = \sqrt{\frac{1+0+0}{4+0+0}} = \frac{1}{2}.$$

This is because $\lim_{x\to+\infty}\frac{1}{x^2} = \lim_{x\to+\infty}\frac{1}{x^2} = 0$ and $\lim_{x\to+\infty}\frac{\sin(x)+1}{x^3} = 0$ by the Squeeze Theorem since $-|\frac{2}{x^3}| \le \frac{\sin(x)+1}{x^3} \le |\frac{2}{x^3}|$ for $x > 0$ and $\lim_{x\to+\infty}|\frac{1}{x^3}| = 0$.

(b)

$$\lim_{x\to 0}\frac{\sin(x^2+\sin(x^2+3x))}{x^2+3x} = \lim_{x\to 0}\frac{\sin(x^2+\sin(x^2+3x))}{(x^2+\sin(x^2+3x))}\cdot\frac{x^2+\sin(x^2+3x)}{x^2+3x}$$

$$= \lim_{x\to 0}\frac{\sin(x^2+\sin(x^2+3x))}{(x^2+\sin(x^2+3x))}\cdot\left(\frac{x^2}{x^2+3x}+\frac{\sin(x^2+3x)}{x^2+3x}\right)$$

$$= \lim_{x\to 0}\frac{\sin(x^2+\sin(x^2+3x))}{(x^2+\sin(x^2+3x))}\cdot\lim_{x\to 0}\left(\frac{x}{x+3}+\frac{\sin(x^2+3x)}{x^2+3x}\right)$$

$$= 1\cdot(0+1) = 1$$

because $\lim_{x\to 0}\frac{\sin(x^2+\sin(x^2+3x))}{(x^2+\sin(x^2+3x))} = 1$ and $\cdot\lim_{x\to 0}\frac{\sin(x^2+3x)}{x^2+3x} = 1$.

OR $$\lim_{x\to 0}\frac{\sin(x^2+\sin(x^2+3x))}{x^2+3x} = \lim_{x\to 0}\frac{\cos(x^2+\sin(x^2+3x))\cdot(2x+\cos(x^2+3x)(2x+3))}{2x+3}$$

by L' Hôpital's Rule

$$= \frac{\cos(0)\cdot(0+\cos(0)\cdot 3)}{3} = 1$$

(c) $\lim_{x\to 0^+} x^2\sin(\ln(x)) = 0$ by the Squeeze Theorem since

$$-x^2 \le x^2\sin(\ln(x)) \le x^2 \text{ for } x > 0 \text{ and } \lim_{x\to 0^+} x^2 = 0.$$

(d) $\lim_{x\to 0}(e^x+7x)^{(1/x)}$. Let $y = (e^x+7x)^{(1/x)}$.

Since $\lim_{x\to 0}\ln(y) = \lim_{x\to 0}\frac{1}{x}\ln(e^x+7x) = \lim_{x\to 0}\frac{\frac{e^x+7}{e^x+7x}}{1} = \lim_{x\to 0^+}\frac{e^x+7}{e^x+7x} = \frac{1+7}{1+0} = 8$

by L' Hôpital's Rule,

Therefore, $\lim_{x\to 0} y = e^{\lim_{x\to 0}\ln(y)} = e^8$

(e) Let $y = (\sin(x^4))^{(1/\ln(x))}$. Then $\ln(y) = \frac{1}{\ln(x)}\ln(\sin(x^4))$.

Now,

$$\lim_{x\to 0^+}\ln(y) = \lim_{x\to 0^+}\frac{\ln(\sin(x^4))}{\ln(x)} = \lim_{x\to 0^+}\frac{\frac{4x^3\cos(x^4)}{\sin(x^4)}}{1/x}$$ by L' Hôpital's Rule,

$$= 4\lim_{x\to 0^+}\frac{x^4}{\sin(x^4)}\cos(x^4) = 4\cdot 1\cdot\cos(0) = 4 \text{ since } \lim_{x\to 0^+}\frac{x^4}{\sin(x^4)} = 1$$

OR by L' Hôpital's Rule,

$$= 4\lim_{x\to 0^+}\frac{4x^3\cos(x^4)-4x^7\sin(x^4)}{4x^3\cos(x^4)} = 4\lim_{x\to 0^+}\frac{\cos(x^4)-x^4\sin(x^4)}{\cos(x^3)} = 4$$

Therefore, $\lim_{x\to 0^+} y = e^{\lim_{x\to 0^+}\ln(y)} = e^4$.

Question 3

(a) $\int \frac{6-5x^2-2x}{(x^2+2x+2)(x^2-2x+2)}dx = \int \frac{2x+2}{(x^2+2x+2)}dx + \int \frac{-2x+1}{(x^2-2x+2)}dx$

by a partial fraction expansion determined as follows.

Writing,

$$\frac{6-5x^2-2x}{(x^2+2x+2)(x^2-2x+2)} = \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{x^2-2x+2}$$

then $(Ax+B)(x^2-2x+2)+(Cx+D)(x^2+2x+2) = 6-5x^2-2x.$

Comparing coefficient of x^3 : A + C = 0 ---------------------------- (1)
Comparing constant terms : 2B +2D = 6, i.e. B + D = 3 ----------------- (2)
Comparing coefficient of x^2 : –2A+B + 2C+D = –5.
Since B+D = 3 by (2) we get –2A +2C = –5 –3 = –8, i.e.,
–A + C = – 4 ----------------------------- (3)
Comparing coefficients of x: 2A– 2B +2C+2D = –2.
Since A + C = 0 we get from above – B + D = –1 ---------------- (4)
Equation (1) + Equation (3) gives 2C = – 4 and so C= –2 and A = –C = 2.
Equation (2) + Equation (4) gives 2D = 2 and so D = 1 and B = 3–D = 2

Now $\int \frac{2x+2}{(x^2+2x+2)}dx = \ln|x^2+2x+2| + C$

And $\int \frac{-2x+1}{(x^2-2x+2)}dx = \int \frac{-(2x-2)}{(x^2-2x+2)}dx - \int \frac{1}{(x^2-2x+2)}dx$

$= -\ln|x^2-2x+2| - \int \frac{1}{(x-1)^2+1)}dx = -\ln|x^2-2x+2| - \tan^{-1}(x-1) + C'$

Therefore,

$$\int \frac{6-5x^2-2x}{(x^2+2x+2)(x^2-2x+2)}dx = \ln|x^2+2x+2| - \ln|x^2-2x+2| - \tan^{-1}(x-1) + C''$$

$$= \ln\left|\frac{x^2+2x+2}{x^2-2x+2}\right| - \tan^{-1}(x-1) + C''$$

(b) $\int_{-1}^{1} \sqrt{x+2|x|}\, dx = \int_{-1}^{0} \sqrt{x-2x}\, dx + \int_{0}^{1} \sqrt{x+2x}\, dx = \int_{-1}^{0} \sqrt{-x}\, dx + \int_{0}^{1} \sqrt{3x}\, dx$

$= -\int_{1}^{0} \sqrt{u}\, du + \int_{0}^{1} \sqrt{3}\,\sqrt{x}\, dx = (\sqrt{3}+1)\int_{0}^{1} \sqrt{x}\, dx = (\sqrt{3}+1)\frac{2}{3}[x^{3/2}]_0^1 = \frac{2}{3}(\sqrt{3}+1)$

(c) $g(x) = \begin{cases} x^3+x+7, & x<1 \\ 3e^{(x-1)} - 6\cos(\pi x), & x \ge 1 \end{cases}$.

First note that g is continuous on the interval (-∞ , 1) since it is a polynomial function there and polynomial functions are continuous. Note also that g is continuous on (1, ∞) since $\cos(\pi x)$ is a continuous function because the cosine function is continuous and that e^{x-1} is continuous on (1, ∞). Now the left limit at x = 1 is $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} x^3+x+7 = 9$ and the right limit at x = 1, $\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} 3e^{x-1} - 6\cos(\pi x) = 3-6\cos(\pi) = 9 = g(1)$. Therefore, $\lim_{x\to 1} g(x) = g(1)$. Thus g is continuous at x =1. Therefore, g is continuous on **R** and we can use the Fundamental Theorem of Calculus to obtain an antiderivative $G(x)$ given by the following Riemann integral for each x in **R**.

$$G(x)=\int_1^x g(t)dt=\begin{cases}\int_1^x g(t)dt, x<1\\ \int_1^x g(t)dt, x\geq 1\end{cases}=\begin{cases}\int_1^x (t^3+t+7)dt, x<1\\ \int_1^x (3e^{(t-1)}-6\cos(\pi t))dt, x\geq 1\end{cases}$$

$$=\begin{cases}\left[\frac{1}{4}t^4+\frac{1}{2}t^2+7t)\right]_1^x, x<1\\ \left[3e^{(t-1)}-\frac{6}{\pi}\sin(\pi t)\right]_1^x, x\geq 1\end{cases}=\begin{cases}\frac{1}{4}x^4+\frac{1}{2}x^2+7x-7\frac{3}{4}, x<1\\ 3e^{(x-1)}-\frac{6}{\pi}\sin(\pi x)-3, x\geq 1\end{cases}$$

Thus, any antiderivative is given by $G(x)+C$ for any constant C.

(d) $\int e^{2x}\sin(5x)dx=\frac{1}{2}e^{2x}\sin(5x)-\frac{1}{2}\int e^{2x}\cdot 5\cos(5x)dx$ by integration by parts

$=\frac{1}{2}e^{2x}\sin(5x)-\frac{5}{2}[\frac{1}{2}e^{x}\cos(5x)-\int\frac{1}{2}e^{2x}(-5\sin(5x))dx]$ by integration by parts

$=\frac{1}{2}e^{2x}(\sin(5x)-\frac{5}{2}\cos(5x))-\frac{25}{4}\int e^{2x}\sin(5x))dx.$

Therefore, $\int e^{2x}\sin(5x)dx=\frac{4}{29}\frac{1}{2}e^{2x}(\sin(5x)-\frac{5}{2}\cos(5x))+C.$

$=\frac{2}{29}e^{2x}(\sin(5x)-\frac{5}{2}\cos(5x))+C.$

Or $=\frac{1}{29}e^{2x}(2\sin(5x)-5\cos(5x))+C.$

(e) $\int\sin^4(5x)\cos^3(5x)dx=\int\frac{1}{5}\sin^4(5x)\cos^2(5x)\cdot 5\cos(5x)dx$

$=\int\frac{1}{5}\sin^4(5x)(1-\sin^2(5x))\cdot 5\cos(5x)dx=\frac{1}{5}\int u^4(1-u^2)\frac{du}{dx}dx,$ where $u=\sin(5x)$

$=\int\frac{1}{5}u^4(1-u^2)du=\frac{1}{5}(\frac{u^5}{5}-\frac{u^7}{7})+C=\frac{1}{25}\sin^5(5x)-\frac{1}{35}\sin^7(5x)+C$ by substitution or change of variable.

Question 4.

(a) (i) Statement of The Extreme Value Theorem.

Suppose $f:[a,b]\to\mathbf{R}$ is a continuous function defined on a closed and bounded interval $[a, b]$. Then there exists elements c and d in the interval $[a, b]$ such that

$f(c)\leq f(x)\leq f(d)$ for all x in $[a, b]$, i.e. $f(c)$ is the absolute ninimum of f and $f(d)$ is the absolute maximum of f.

(ii) Recall $g(x)=\frac{x^3}{3}-\frac{5x^2}{2}+4x+3,$

Thus, $g'(x)=x^2-5x+4=(x-1)(x-4)$. Therefore, $g'(x)=0$ if and only if $x=1$ or 4. Hence g has two stationary points in (0, 5), namely 1 and 4. Since g is differentiable, the critical points of g in (0, 5) are 1 and 4. Since g is continuous on the closed and bounded interval [0, 5] and so by the Extreme Value Theorem g has absolute extrema on the interval [0, 5] and they are given respectively by the maximum and minimum of the values of the critical points and the end points under g. Now g(0) = 3, g(1) = 29/6, g(4) =1/3 and g(5) =13/6. . Therefore, the absolute maximum of g on [0, 5] is 29/6 < 5 and the absolute minimum of g on [0, 5] is 1/3 > 0. Thus g([0,5])=[1/3, 29/6] $\subseteq$ [0,5] and so g maps [0, 5] into [0, 5]. Hence g has a fixed point in [0, 5]. I.e., there exists a point *c* in [0. 5] such that g(*c*) = *c*.

Alternatively, let $h(x) = g(x) - x$. Then h(0) = g(0) = 3 and h(5) = g(5) - 5 = 13/6-5 <0. Since g is continuous on [0, 5], *h* is continuous on [0, 5] and so by the Intermediate Value Theorem, there exists a point *c* in [0, 5] such that $h(c) = 0$, i.e., $g(c) = c$.

(b) (i) $h(x) = (1+e^{\sin(2x^2)})^{\cot(x)}$, $x \in (0, \frac{\pi}{2})$.

Taking logarithm on both sides we get $\ln(h(x)) = \cot(x)\ln(1+e^{\sin(2x^2)})$.

Differentiating both sides we get,

$$\frac{h'(x)}{h(x)} = -\csc^2(x)\ln(1+e^{\sin(2x^2)}) + \cot(x)\frac{4x\cos(2x^2)e^{\sin(2x^2)}}{1+e^{\sin(2x^2)}}$$

Therefore, $h'(x) =$

$$\left[\frac{4x\cos(2x^2)\cot(x)e^{\sin(2x^2)}}{1+e^{\sin(2x^2)}} - \csc^2(x)\ln(1+e^{\sin(2x^2)})\right](1+e^{\sin(2x^2)})^{\cot(x)}.$$

(ii) $j(x) = \int_{x^2}^{\ln(1+x^2)} \frac{t}{1+t^2+\cos(t^2)}dt.$

Therefore, $j(x) = \int_0^{\ln(1+x^2)} \frac{t}{1+t^2+\cos(t^2)}dt - \int_0^{x^2} \frac{t}{1+t^2+\cos(t^2)}dt.$

Hence by the Fundamental Theorem of Calculus and the Chain Rule,

$$j'(x) = \frac{2x\ln(1+x^2)}{(1+(\ln(1+x^2))^2+\cos((\ln(1+x^2))^2))(1+x^2)} - \frac{2x^3}{1+x^4+\cos(x^4)}$$

(iii) $k(x) = \cos^{-1}\left(\frac{1}{1+x^2}\right)$. Thus by the Chain Rule

$$k'(x) = (\cos^{-1})'\left(\frac{1}{1+x^2}\right)\cdot\frac{-2x}{(1+x^2)^2} = \frac{1}{\cos'(\cos^{-1}(\frac{1}{1+x^2}))}\cdot\frac{-2x}{(1+x^2)^2}$$

$$= \frac{1}{\sin(\cos^{-1}(\frac{1}{1+x^2}))}\cdot\frac{2x}{(1+x^2)^2} = \frac{1}{\sqrt{1-\cos^2(\cos^{-1}(\frac{1}{1+x^2}))}}\cdot\frac{2x}{(1+x^2)^2}$$

$$= \frac{1}{\sqrt{1-(\frac{1}{1+x^2})^2}}\cdot\frac{2x}{(1+x^2)^2} = \frac{1}{\sqrt{x^4+2x^2}}\cdot\frac{2x}{(1+x^2)} = 2\frac{x}{|x|}\cdot\frac{1}{\sqrt{2+x^2}\,(1+x^2)}$$

$$= \frac{2sign(x)}{\sqrt{2+x^2}\,(1+x^2)}$$

Question 5.

(a) Recall $k(x) = \int_1^x (1+\frac{t^2}{1+\sin(\pi t)+e^t})dt.$

(i) Therefore, for all *x* in **R**,

$k'(x) = 1 + \frac{x^2}{1+\sin(\pi x)+e^x}$ by the Fundamental Theorem of Calculus

$\geq 1 > 0$,

since $1 + \sin(\pi x) + e^x \geq e^x > 0$ so that $\frac{x^2}{1+\sin(\pi x)+e^x} \geq 0$.

Thus, *k* is (strictly) increasing on **R** and hence *k* is injective.

(ii) Note that $(k^{-1})'(0) = \frac{1}{k'(k^{-1}(0))}$.

$k(1) = \int_1^1 (1 + \frac{t^2}{1+\sin(\pi t)+e^t})dt = 0$ and so since k is injective $k^{-1}(0) = 1$.

From part (i) $k'(1) = 1 + \frac{1}{1+\sin(\pi)+e^1} = 1 + \frac{1}{1+e} = \frac{2+e}{1+e}$.

Thus, $(k^{-1})'(0) = \frac{1}{k'(k^{-1}(0))} = \frac{1}{k'(1)} = \frac{1+e}{2+e}$.

(b) Since f is continuous on $[a, b]$, by the Extreme Value Theorem, there exists points α and β in $[a, b]$ such that $f(\alpha) \le f(x) \le f(\beta)$ for all x in $[a, b]$. It follows then that for all x in $[a, b]$,

$$f(\alpha)\, g(x) \le f(x)g(x) \le f(\beta)g(x) \quad \text{---------------------} \quad (1)$$

because $g(x) \ge 0$ for all x in $[a, b]$.

Now f is Riemann integrable on $[a, b]$, because it is continuous on $[a, b]$ and g is given to be Riemann integrable on $[a, b]$. Therefore, the product $f(x)g(x)$ is Riemann integrable on $[a, b]$. It then follows from (1) that

$$f(\alpha)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le f(\beta)\int_a^b g(x)dx$$

Thus, there exists m such that $f(\alpha) \le m \le f(\beta)$ and $\int_a^b f(x)g(x)dx = m\int_a^b g(x)dx$. Since f is continuous on $[a, b]$, by the Intermediate Value Theorem, there exists a point c in $[\alpha, \beta]$ (if $\alpha \le \beta$) or $[\beta, \alpha]$ (if $\beta \le \alpha$) hence in $[a, b]$, such that $f(c) = m$. Therefore, $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$

(c) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2} \cdot \sqrt{\frac{7n^2+i^2}{n^2}}$.

We seek to write the summation $\sum_{i=1}^{n} \frac{i}{n^2} \cdot \sqrt{\frac{7n^2+i^2}{n^2}}$ as a Riemann sum

$$\sum_{i=1}^{n} \frac{i}{n^2} \cdot \sqrt{\frac{7n^2+i^2}{n^2}} = \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $x_0 < x_1 < \cdots < x_n$ is a regular partition and $\Delta x = \Delta x_i = x_i - x_{i-1}$.
Therefore, we can take $x_i = \frac{i}{n}$ so that $\Delta x = \frac{1}{n}$, $x_0 = 0$ and $x_n = 1$. Thus by comparing,

$$f(x_i)\Delta x \text{ with } \frac{i}{n^2} \cdot \sqrt{\frac{7n^2+i^2}{n^2}} = \frac{i}{n}\sqrt{7+\left(\frac{i}{n}\right)^2} \cdot \frac{1}{n}$$

we would want $f(x_i) = \frac{i}{n}\sqrt{7+\left(\frac{i}{n}\right)^2} = x_i\sqrt{7+x_i^2}$. Thus $f(x) = x\sqrt{7+x^2}$.

Therefore, $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2}\sqrt{\frac{7n^2+i^2}{n^2}} = \int_0^1 x\sqrt{7+x^2}\,dx$

$= \frac{1}{2}\int_0^1 2x\sqrt{7+x^2}\,dx = \frac{1}{2}\int_0^1 \sqrt{u}\,\frac{du}{dx}dx$, where $u = 7 + x^2$

$= \frac{1}{2}\int_7^8 \sqrt{u}\,du$ by a Change of Variable

$= \frac{1}{2} \cdot \frac{2}{3}[u^{3/2}]_7^8 = \frac{1}{3}(8^{3/2} - 7^{3/2})$.

Question 6

Recall $f(x)=\begin{cases} x^5-5x^2+7, & x\ge 0 \\ x^2+7, & x<0 \end{cases}$

(a) Observe that

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^5-5x^2+7 = 7 = \lim_{x\to 0^-} x^2+7 = \lim_{x\to 0^-} f(x) = f(0).$$

Hence, f is continuous at $x = 0$ and so is continuous on **R** since it is continuous on $(-\infty, 0)$ and on $(0, \infty)$ because $f(x)$ is equal separately to a polynomial function there.

Now

$$f'(x)=\begin{cases} 5x^4-10x, & x>0 \\ 2x, & x<0 \end{cases} = \begin{cases} 5x(x^3-2), & x>0 \\ 2x, & x<0 \end{cases}$$

$$= \begin{cases} 5x(x-2^{1/3})(x^2+2^{1/3}x+2^{2/3}), & x>0 \\ 2x, & x<0 \end{cases} \quad \begin{cases} 5x(x-2^{1/3})((x+\frac{1}{2}2^{1/3})^2+\frac{3}{4}2^{2/3}), & x>0 \\ 2x, & x<0 \end{cases}$$

---------------------------- (1)

Therefore, for $x < 0$, $f'(x) = 2x < 0$ and so f is decreasing on the interval $(-\infty, 0]$.

From (1), for $0 < x < 2^{1/3}$, $f'(x) < 0$ and so f is decreasing on $[0, 2^{1/3}]$. Hence f is decreasing on the interval $(-\infty, 2^{1/3}]$. From (1), for $x > 2^{1/3}$, $f'(x) > 0$ and so f is increasing on $[2^{1/3}, \infty)$.

(b) $$f''(x)=\begin{cases} 20x^3-10, & x>0 \\ 2, & x<0 \end{cases} = \begin{cases} 20(x^3-\frac{1}{2}), & x>0 \\ 2, & x<0 \end{cases}$$

$$= \begin{cases} 20(x-\frac{1}{2^{1/3}})(x^2+\frac{1}{2^{1/3}}x+\frac{1}{2^{2/3}}), & x>0 \\ 2, & x<0 \end{cases}$$ ------------------------ (2)

Thus, $f''(x) < 0$ for $0 < x < 1/2^{1/3}$. Therefore, the graph of f is concave downward on the interval $(0, 1/2^{1/3})$. Also, for $x < 0$, $f''(x) = 2 > 0$. Thus, the graph of f is concave upward on the interval $(-\infty, 0)$. From (2), for $x > 1/2^{1/3}$, $f''(x) > 0$ and so the graph of f is concave upward on the interval $(1/2^{1/3}, \infty)$.

(c) By part (a) $f(2^{1/3}) = 2^{5/3}-5\cdot 2^{2/3}+7 = 7-3\cdot 2^{2/3}$ is a relative minimum. This is also the absolute minimum. There are no relative maximum values for f.

(d) From part (b), there is a change of concavity before and after $x = 0$ and $x = 1/2^{1/3}$.

Now, $f(0) = 7$ and $f(\frac{1}{2^{1/3}}) = \frac{1}{2^{5/3}} - 5\frac{1}{2^{2/3}} + 7 = \frac{1}{2^{2/3}}(\frac{1}{2}-5)+7 = 7-\frac{9}{2\cdot 2^{2/3}}$.

Hence, the points of inflection of the graph of f are

$$(0,7) \text{ and } (\frac{1}{2^{1/3}}, 7-\frac{9}{2\cdot 2^{2/3}}).$$

(e)