NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2004 – 2005

MA1102R CALCULUS

November 2004 – Time Allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO** sections: Section A and Section B. It contains a total of **SIX** questions and comprises **FOUR** printed pages.

2. Answer **ALL** questions in **Section A**. The marks for questions in Section A are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Answer not more than **TWO** questions from Section B. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

*Answer **ALL** questions in this section.*

Question 1 [**20** marks]

Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} -x^3 + 5x + 3, & x < -1 \\ x^2 \sin\left(\dfrac{\pi}{2x}\right), & -1 \le x \le 1 \text{ and } x \ne 0 \\ x^2 - 7x + 7, & x > 1 \\ 0, & x = 0 \end{cases}.$$

(a) Find the *range* of the function f.

(b) Determine if f is *surjective*.

(c) Determine all x in $\mathbf{R}$ at which the function f is *continuous*. Justify your answer.

(d) Determine if f is *differentiable* at x, when $x = 1$ *or* -1. Justify your answer.

(e) Evaluate $\int_{-1}^{2} f(x)dx$.

Question 2 [**20** marks]

Evaluate, if it exists, each of the following limits.

(a) $\displaystyle \lim_{x \to +\infty} \frac{5x^2 + 7x + \sin(x^2) + 1}{4x^2 + 3x + 5}.$

(b) $\displaystyle \lim_{x \to 0} \frac{1 - \cos(7x)}{\sin^2(x + x^2)}.$

(c) $\displaystyle \lim_{x \to 0^+} \frac{\sin(\sin(x^2 + x))}{x^2 + 3x}.$

(d) $\displaystyle \lim_{x \to 0^+} (\sin(x^3))^{(x^2)}.$

(e) $\displaystyle \lim_{x \to 0^+} (3^x + 5x)^{(1/x)}.$

Question 3 [**20** marks]

(a) Evaluate $\int \frac{1}{(x^2+6x+10)(x^2+6x+11)}dx$.

(b) Compute $\int_{-1}^{1} \cos(5+|x|)dx$.

(c) Find an antiderivative of $g(x)$, which is defined by

$$g(x) = \begin{cases} x^2+2x-2, x<1 \\ \frac{1}{x}+\sin(\pi x), x \geq 1 \end{cases}.$$

(d) Evaluate $\int e^x \sin(6x)dx$.

(e) Evaluate $\int x\sec^2(\tan(x^2))\ \sec^2(x^2)dx$.

SECTION B

*Answer not more than **TWO** questions from this section. Each question in this section carries 20 marks.*

Question 4 [**20** marks]

(a) Find the critical points of the function g, defined by

$$g(x) = x^3 - 9x^2 + 24x + 7,$$

in the open interval (1, 5). Determine the absolute maximum and the absolute minimum values of the function in the interval [1, 5].

(b) Differentiate each of the following functions.

(i) $h(x) = (2+\cos(\sin(x^2)))^{\tan(x)}$, $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

(ii) $j(x) = \int_{x^2}^{\ln(1+x^2)} \frac{t}{1+t^2+\cos(t^2)}dt$.

(iii) $k(x) = \sin^{-1}(\sin^2(x))$.

(c) Suppose f is a continuous function defined on the closed and bounded interval [0, 1] such that $f(0) = f(1)$. Prove that there exists a point c in $[\frac{1}{7}, 1]$ such that $f(c) = f\left(\frac{1}{6}(c-\frac{1}{7})\right)$. Hence, or otherwise, deduce that there exists a point c in $[\frac{1}{7}, 1]$ such that

$$\cos(2\pi c) = \cos(\frac{c\pi}{3} - \frac{\pi}{21}).$$

Question 5 [**20** marks]

(a) State clearly the *Fundamental Theorem of Calculus*.

Use it ,or otherwise, to differentiate the function

$$g(x) = \int_{\ln(x)}^{2x} \frac{t}{1+\cos^2(t)+e^{2t}}\,dt.$$

(b) Let the function k be defined on **R** by

$$k(x) = \int_0^x (1+\tfrac{1}{2}\sin(\cos(t^2)))\,dt.$$

(i) Without integrating, show that the function k is injective.

(ii) Determine $(k^{-1})'(0)$.

(c) Find the following limit.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2} \cdot \cos\left(1+2\left(\frac{i}{n}\right)^2\right).$$

Question 6 [**20** marks]

Let the function f be defined on **R** by

$$f(x) = 3x^5 - 30x^2 + 1 \quad .$$

(a) Find the intervals on which f is (i) *increasing*, and (ii) *decreasing*.

(b) Find the intervals on which the graph of f is (i) *concave upward*, and (ii) *concave downward*.

(c) Find the *relative extrema* of f, if any.

(d) Find the *points of inflection* of the graph of f.

(e) Sketch the graph of f.

END OF PAPER

Answer To MA1102 Calculus

Question 1

The function f is defined by $f(x) = \begin{cases} -x^3+5x+3, & x<-1 \\ x^2\sin\left(\frac{\pi}{2x}\right), & -1\le x\le 1 \text{ and } x\ne 0 \\ x^2-7x+7, & x>1 \\ 0, & x=0 \end{cases}$.

(a) For $x<-1$, $f(x) = -x^3+5x+3 = -(x+1)^3+3(x+1)^2+2(x+1)-1$
(expressing in terms of $(x+1)$)
$= -(x+1)^3+3((x+1)+\frac{1}{3})^2-\frac{4}{3} \ge -\frac{4}{3}$

Thus, $\lim_{x\to-\infty} f(x) = +\infty$ since $\lim_{x\to-\infty} -(x+1)^3 = +\infty$. Now $f(-1) = -1$. Therefore, by the Intermediate Value Theorem, since $-x^3+5x+3$ is continuous on $(-\infty, -1]$,
$[-1, \infty) \subseteq f((-\infty, -1]) \subseteq [-4/3, \infty)$.
Now for $x > 1$, $f(x) = x^2-7x+7 = (x-\frac{7}{2})^2+7-\frac{49}{4} = (x-\frac{7}{2})^2-\frac{21}{4} \ge -5\frac{1}{4}$.
Since $f(\frac{7}{2}) = -5\frac{1}{4} = -\frac{21}{4}$ and since $\lim_{x\to\infty} x^2-7x+7 = +\infty$, by the Intermediate Value Theorem, $f((1, \infty)) = [-5\frac{1}{4}, \infty)$ because f is continuous on the interval $[7/2, \infty)$. Thus, since $|f(x)| \le 1$ for $|x| \le 1$ and because $[-4/3, \infty) \subseteq [-5\frac{1}{4}, \infty)$, Range $f = f((-\infty, -1]) \cup f([-1,1]) \cup f((1, \infty)) = f((1, \infty)) = [-5\frac{1}{4}, \infty)$.

(b) By part (a) Range(f) $\ne \mathbf{R}$ = codomain of f. Therefore, f is not surjective.

(c) When $x<-1$, $f(x) = -x^3+5x+3$, is a polynomial function and so f is continuous on $(-\infty, -1)$, since any polynomial function is continuous on $\mathbf{R}$ and therefore continuous on any open interval. When $-1<x<0$, $f(x) = x^2\sin\left(\frac{\pi}{2x}\right)$. Since sine is a continuous function and the function $\frac{\pi}{2x}$ is a continuous function on $x \ne 0$, $\sin\left(\frac{\pi}{2x}\right)$ on $(-1, 0)$ being the composite of these two continuous functions is therefore continuous on $(-1, 0)$. Therefore, as x^2 is continuous on $(-1, 0)$, f being the product of two continuous functions on $(-1, 0)$ is continuous on $(-1, 0)$. Similarly, f is continuous on the interval $(0, 1)$. f is continuous on $(1, \infty)$ since $f(x) = x^2-7x+7$, a polynomial function. Thus we can conclude that f is continuous at x for $x \ne -1, 0, 1$. Thus it remains to check if f is continuous at $x = -1, 0$ or 1.

Consider the left limit at $x = -1$,
$\lim_{x\to-1^-} f(x) = \lim_{x\to-1^-} -x^3+5x+3 = -1$ and the right limit at $x = -1$
$\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} x^2\sin\left(\frac{\pi}{2x}\right) = \sin\left(-\frac{\pi}{2}\right) = -1$.

Thus, since $\lim_{x\to-1} f(x) = -1$, and $f(-1) = -1$ it follows that f is continuous at $x = -1$.

Now consider the left limit of f at $x = 1$,
$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} x^2\sin\left(\frac{\pi}{2x}\right) = 1$ and the right limit at $x = 1$,
$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} x^2-7x+7 = 1 = f(1)$.

Therefore, $\lim_{x\to1} f(x) = f(1)$ and so f is continuous at $x = 1$.

Now $\lim_{x\to0} f(x) = \lim_{x\to0} x^2\sin\left(\frac{\pi}{2x}\right) = 0$ by the Squeeze Theorem since $-|x|^2 \le x^2\sin\left(\frac{\pi}{2x}\right) \le |x|^2$ for $x \ne 0$ and $\lim_{x\to0} |x|^2 = 0$. Since $f(0) = 0$, we conclude that f is also continuous at $x = 0$. Therefore f is continuous at x for any x in $\mathbf{R}$.

(d) To check the differentiability of f at $x = 1$ consider the following limits.

$$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{x^2-7x+7-1}{x-1}=\lim_{x\to 1^+}(2x-7)=-5$$

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^-}\frac{x^2\sin\left(\frac{\pi}{2x}\right)-1}{x-1}=\lim_{x\to 1^-}\frac{2x\sin\left(\frac{\pi}{2x}\right)-x^2\cos\left(\frac{\pi}{2x}\right)\cdot\left(\frac{\pi}{2x^2}\right)}{1}=2$$

by L' Hôpital's Rule.

Therefore, f is not differentiable at $x = 1$ since $\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}\neq\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$.

$$\lim_{x\to -1^-}\frac{f(x)-f(-1)}{x+1}=\lim_{x\to -1^-}\frac{-x^3+5x+3+1}{x+1}=\lim_{x\to -1^-}(-3x^2+5)=2$$

$$\lim_{x\to -1^+}\frac{f(x)-f(-1)}{x+1}=\lim_{x\to -1^+}\frac{x^2\sin\left(\frac{\pi}{2x}\right)+1}{x+1}=\lim_{x\to -1^+}\frac{2x\sin\left(\frac{\pi}{2x}\right)-x^2\cos\left(\frac{\pi}{2x}\right)\cdot\left(\frac{\pi}{2x^2}\right)}{1}=2$$

by L' Hôpital's Rule.

Therefore, f is differentiable at $x = -1$ since $\lim_{x\to -1^-}\frac{f(x)-f(-1)}{x+1}=\lim_{x\to -1^+}\frac{f(x)-f(-1)}{x+1}$ and $f'(-1) = 2$.

(e) f is Riemann integrable on $[-1,2]$ since the restriction of f is continuous on $[-1, 2]$. Note that f is an odd function on $[-1, 1]$, i.e. $f(-x) = -f(x)$ for all x in $[-1, 1]$ because $f(-x) = x^2\sin(\pi/(-2x)) = -x^2\sin(\pi/(2x)) = -f(x)$ for $x \neq 0$ and for $x = 0$, obviously $f(-0) = f(0) = 0 = -0 = -f(0)$.

$\int_{-1}^{0} f(x)dx=\int_{-1}^{0}-f(x)\frac{du}{dx}dx$ where $u = -x$ so that $\frac{du}{dx}=-1$

$=-\int_{1}^{0} f(-u)du$ by the Change of Variable formula,

$=\int_{1}^{0} f(u)du$ since f is an odd function,

$=-\int_{0}^{1} f(u)du=-\int_{0}^{1} f(x)dx$ by renaming the variable.

Therefore, $\int_{-1}^{1} f(x)dx=\int_{-1}^{0} f(x)dx+\int_{0}^{1} f(x)dx=-\int_{0}^{1} f(x)dx+\int_{0}^{1} f(x)dx=0.$

Thus, $\int_{-1}^{2} f(x)dx=\int_{-1}^{1} f(x)dx+\int_{1}^{2} f(x)dx=\int_{1}^{2}(x^2-7x+7)dx=[\frac{x^3}{3}-\frac{7x^2}{2}+7x]_1^2=-\frac{7}{6}$

Question 2

(a) $$\lim_{x\to+\infty}\frac{5x^2+7x+\sin(x^2)+1}{4x^2+3x+5}=\lim_{x\to+\infty}\frac{5+\frac{7}{x}+\frac{1}{x^2}\sin(x^2)+\frac{1}{x^2}}{4+\frac{3}{x}+\frac{5}{x^2}}=\frac{5}{4}.$$

This is because $\lim_{x\to+\infty}\frac{1}{x^2}=\lim_{x\to+\infty}\frac{1}{x}=0$ and $\lim_{x\to+\infty}\frac{\sin(x^2)}{x^2}=0$ by the Squeeze Theorem since $-|\frac{1}{x^2}|\le\frac{\sin(x^2)}{x^2}\le|\frac{1}{x^2}|$ for $x < 0$ and $\lim_{x\to+\infty}|\frac{1}{x^2}|=0.$

(b) $\lim_{x\to 0}\frac{1-\cos(7x)}{\sin^2(x+x^2)}=\lim_{x\to 0}\frac{7\sin(7x)}{(1+2x)\sin(2x+2x^2)}$ by L' Hôpital's Rule

$=\lim_{x\to 0}\frac{49\cos(7x)}{2\sin(2x+2x^2)+2(1+2x)^2\cos(2x+2x^2)}$ by L' Hôpital's Rule

$=\frac{49}{2}$

(c) $\lim_{x\to 0^+} \dfrac{\sin(\sin(x^2+x))}{x^2+3x} = \lim_{x\to 0^+} \dfrac{\sin(\sin(x^2+x))}{\sin(x^2+x)} \dfrac{\sin(x^2+x)}{x^2+x} \dfrac{x+1}{x+3} = 1\cdot 1\cdot \dfrac{1}{3} = \dfrac{1}{3}$

since $\lim_{x\to 0^+} \dfrac{\sin(\sin(x^2+x))}{\sin(x^2+x)} = \lim_{x\to 0^+} \dfrac{\sin(x^2+x)}{x^2+x} = 1,$

Or $\lim_{x\to 0^+} \dfrac{\sin(\sin(x^2+x))}{x^2+3x} = \lim_{x\to 0^+} \dfrac{\cos(\sin(x^2+x))\cos(x^2+x)(2x+1)}{2x+3}$ by L' Hôpital's Rule

$$= \frac{\cos(0)\cos(0)\cdot 1}{3} = \frac{1}{3}.$$

(d) Let $y = (\sin(x^3))^{(x^2)}$. Then $\ln(y) = x^2 \ln(\sin(x^3))$.
Now,

$$\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} \frac{\ln(\sin(x^3))}{1/x^2} = \lim_{x\to 0^+} \frac{\frac{3x^2\cos(x^3)}{\sin(x^3)}}{-2/x^3} \quad \text{by L' Hôpital's Rule,}$$

$$= -\tfrac{3}{2} \lim_{x\to 0^+} \frac{x^3}{\sin(x^3)} x^2 = -\tfrac{3}{2}\cdot 1\cdot 0 = 0 \text{ since } \lim_{x\to 0^+} \frac{x^3}{\sin(x^3)} = 1$$

OR by L' Hôpital's Rule,

$$= -\tfrac{3}{2} \lim_{x\to 0^+} \frac{5x^4}{3x^2\cos(x^3)} = -\tfrac{5}{2} \lim_{x\to 0^+} \frac{x^2}{\cos(x^3)} = -\tfrac{5}{2}\cdot \tfrac{0}{1} = 0$$

Therefore, $\lim_{x\to 0^+} y = e^{\lim_{x\to 0^+} \ln(y)} = e^0 = 1.$

(e) $\lim_{x\to 0^+} (3^x + 5x)^{(1/x)}$. Let $y = (3^x + 5x)^{(1/x)}$.

Since $\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} \dfrac{1}{x} \ln(3^x + 5x) = \lim_{x\to 0^+} \dfrac{\frac{\ln(3)3^x + 5}{3^x + 5x}}{1} = \lim_{x\to 0^+} \dfrac{\ln(3)3^x + 5}{3^x + 5x} = \ln(3) + 5$

by L' Hôpital's Rule,

Therefore, $\lim_{x\to\infty} y = e^{\lim_{x\to\infty} \ln(y)} = e^{\ln(3)+5} = 3e^5.$

Question 3

(a) $\displaystyle\int \frac{1}{(x^2+6x+10)(x^2+6x+11)}dx = \int \frac{1}{((x+3)^2+1)((x+3)^2+2)}dx$

$$= \int \frac{1}{((x+3)^2+1)}dx - \int \frac{1}{((x+3)^2+2)}dx$$

$$= \tan^{-1}(x+3) - \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C$$

(b) $\displaystyle\int_{-1}^{1} \cos(5+|x|)dx = \int_0^1 \cos(5+x)dx + \int_{-1}^{0} \cos(5-x)dx$

$$= [\sin(5+x)]_0^1 + [-\sin(5-x)]_{-1}^0 = 2(\sin(6)-\sin(5))$$

OR use the fact that $\cos(5 + |x|)$ is an even function,

$$\int_{-1}^{1} \cos(5+|x|)dx = 2\int_0^1 \cos(5+x)dx = 2[\sin(5+x)]_0^1 = 2(\sin(6)-\sin(5))$$

(c) $g(x) = \begin{cases} x^2 + 2x - 2, x < 1 \\ \frac{1}{x} + \sin(\pi x), x \ge 1 \end{cases}$.

First note that g is continuous on the interval (-∞ , 1) since it is a polynomial function there and polynomial functions are continuous. Note also that g is continuous on (1, ∞) since $\sin(\pi x)$ is a continuous function because the sine function is continuous and that 1/x is continuous on (1, ∞). Now the left limit at x = 1 is $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} x^2+2x-2 = 1$ and the right limit at x = 1, $\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} \frac{1}{x}+\sin(\pi x) = 1+0 = 1 = g(1)$. Therefore, $\lim_{x\to 1} g(x) = g(1)$. Thus g is continuous at x =1. Therefore, g is continuous on **R** and we can use the Fundamental Theorem of Calculus to obtain an antiderivative $G(x)$ given by the following Riemann integral for each x in **R**.

$$G(x) = \int_1^x g(t)dt = \begin{cases} \int_1^x g(t)dt, x<1 \\ \int_1^x g(t)dt, x\geq 1 \end{cases} = \begin{cases} \int_1^x (t^2+2t-2)dt, x<1 \\ \int_1^x (\frac{1}{t}+\sin(\pi t))dt, x\geq 1 \end{cases}$$

$$= \begin{cases} \left[\frac{1}{3}t^3+t^2-2t)\right]_1^x, x<1 \\ \ln(x)+\left[-\frac{1}{\pi}\cos(\pi t)\right]_1^x, x\geq 1 \end{cases} = \begin{cases} \frac{1}{3}x^3+x^2-2x+\frac{2}{3}, x<1 \\ \ln(x)-\frac{1}{\pi}\cos(\pi x)-\frac{1}{\pi}, x\geq 1 \end{cases}$$

Thus, any antiderivative is given by $G(x)+C$ for any constant C.

(d) Evaluate $\int e^x \sin(6x)dx$.

$\int e^x \sin(6x)dx = e^x\sin(6x) - \int e^x\cdot 6\cos(6x)dx$ by integration by parts

$= e^x\sin(6x) - 6[e^x\cos(6x) - \int e^x(-6\sin(6x))dx]$ by integration by parts

$= e^x(\sin(6x)-6\cos(6x)) - 36\int e^x\sin(6x))dx.$

Therefore, $\int e^x\sin(6x)dx = \frac{e^x}{37}(\sin(6x)-6\cos(6x)) + C.$

(e) $\int x\sec^2(\tan(x^2))\sec^2(x^2)dx = \int \frac{1}{2}\sec^2(\tan(x^2))\frac{du}{dx}dx$, where $u = \tan(x^2)$

$= \int \frac{1}{2}\sec^2(u)du = \frac{1}{2}\tan(u)+C = \frac{1}{2}\tan(\tan(x^2)) + C$ by substitution or change of variable.

Question 4.

(a) Recall $g(x) = x^3-9x^2+24x+7$

Thus, $g'(x) = 3x^2-18x+24 = 3(x^2-6x+8) = 3(x-2)(x-4)$. Therefore, $g'(x) = 0$ if and only if x = 2 or 4. Hence g has two stationary points in (1, 5), namely 2 and 4. Since g is differentiable, the critical points of g in (1, 5) are 2 and 4. Since g is continuous on the closed and bounded interval [1, 5] and so by the Extreme Value Theorem g has absolute extrema on the interval [1, 5] and they are given respectively by the maximum and minimum of the values of the critical points and the end points under g. Now g(1) =23, g(2) =27 , g(4) = 23, g(5) = 27. Therefore, the absolute maximum of g on [1, 5] is 27 and the absolute minimum of g on [1, 5] is 23.

(b) (i) $h(x) = (2+\cos(\sin(x^2))^{\tan(x)}$.

Taking logarithm on both sides we get $\ln(h(x)) = \tan(x)\ln(2+\cos(\sin(x^2)))$.

Differentiating both sides we get,

$$\frac{h'(x)}{h(x)} = \sec^2(x)\ln(2+\cos(\sin(x^2))) + \tan(x)\frac{-2x\sin(\sin(x^2))\cos(x^2)}{2+\cos(\sin(x^2))}$$
$$= \sec^2(x)\ln(2+\cos(\sin(x^2))) - 2\frac{x\tan(x)\sin(\sin(x^2))\cos(x^2)}{2+\cos(\sin(x^2))}$$

Therefore, $h'(x) =$

$$\left[\sec^2(x)\ln(2+\cos(\sin(x^2))) - 2\frac{x\tan(x)\sin(\sin(x^2))\cos(x^2)}{2+\cos(\sin(x^2))}\right](2+\cos(\sin(x^2)))^{\tan(x)}.$$

(ii) $j(x) = \int_{x^2}^{\ln(1+x^2)} \frac{t}{1+t^2+\cos(t^2)}dt.$

Therefore, $j(x) = \int_0^{\ln(1+x^2)} \frac{t}{1+t^2+\cos(t^2)}dt - \int_0^{x^2} \frac{t}{1+t^2+\cos(t^2)}dt.$

Hence by the Fundamental Theorem of Calculus and the Chain Rule,

$$j'(x) = \frac{2x\ln(1+x^2)}{(1+(\ln(1+x^2))^2+\cos((\ln(1+x^2))^2))(1+x^2)} - \frac{2x^3}{1+x^4+\cos(x^4)}$$

(iii) $k(x) = \sin^{-1}(\sin^2(x))$. Thus by the Chain Rule

$$k'(x) = \frac{1}{\sqrt{1-\sin^4(x)}}\sin(2x) = \frac{\sin(2x)}{\sqrt{1-\sin^4(x)}}.$$

(c) For x in [1/7, 1] define $g(x) = f(x) - f(\frac{1}{6}(x-\frac{1}{7}))$. Since f is defined on [0, 1], g is well defined and since f is continuous on [0, 1], $g(x)$ is continuous on [1/7, 1].

Now g(1/7) = $f(1/7) - f(0)$ and g(1) = $f(1) - f(1/7) = f(0) - f(1/7)$ because $f(1) = f(0)$. Hence g(1) = –g(1/7). Thus 0 lies between g(1) and g(1/7). Therefore, by the Intermediate Value Theorem, there exists a point c in [1/7,1] such that g(c) = 0, i.e., $f(c) - f(\frac{1}{6}(c-\frac{1}{7}))$.

Thus, taking $f(x)$ to be $\cos(2\pi x)$. We have that there exists a point c in [1/7,1] such that $\cos(2\pi c) = \cos(\frac{c\pi}{3} - \frac{\pi}{21})$

Question 5.

(a) **Fundamental Theorem of Calculus**. Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function. Then (i) the function $F : [a, b] \to \mathbf{R}$ defined by $F(x) = \int_a^x f(t)dt$ is differentiable on [a, b] satisfying $F'(x) = f(x)$ for every x in [a, b] and (ii) for any antiderivative G of f the Riemann integral $\int_a^b f(t)dt = G(b) - G(a)$.

$$g(x) = \int_{\ln(x)}^{2x} \frac{t}{1+\cos^2(t)+e^{2t}}dt = \int_0^{2x} \frac{t}{1+\cos^2(t)+e^{2t}}dt + \int_{\ln(x)}^{0} \frac{t}{1+\cos^2(t)+e^{2t}}dt$$
$$= \int_0^{2x} \frac{t}{1+\cos^2(t)+e^{2t}}dt - \int_0^{\ln(x)} \frac{t}{1+\cos^2(t)+e^{2t}}dt.$$
$$= F(2x) - F(\ln(x)) \quad \text{where } F(x) = \int_0^x \frac{t}{1+\cos^2(t)+e^{2t}}dt.$$

Therefore,

$$g'(x) = F'(2x)\cdot 2 - F'(\ln(x))\cdot(\tfrac{1}{x}) \text{ by the } \textit{Chain Rule}$$
$$= \frac{4x}{1+\cos^2(2x)+e^{4x}} - \frac{\ln(x)}{x(1+\cos^2(\ln(x))+e^{2\ln(x)})} \text{ by the FTC.}$$

$$= \frac{4x}{1+\cos^2(2x)+e^{4x}} - \frac{\ln(x)}{x(1+\cos^2(\ln(x)) + x^2)}$$

(b) Now $k(x) = \int_0^x (1 + \frac{1}{2}\sin(\cos(t^2)))dt$.

(i) Thus by the Fundamental Theorem of Calculus and the Chain Rule,

$k'(x) = 1 + \frac{1}{2}\sin(\cos(x^2)) \geq \frac{1}{2} > 0$ for any x. Therefore, k is (strictly) increasing on **R** and hence k is injective.

(ii) Note that $k(0) = 0$ and so $k^{-1}(0) = 0$. Thus,

since $k'(0) = 1 + \frac{1}{2}\sin(\cos(0)) = 1 + \frac{1}{2}\sin(1) = \frac{2+\sin(1)}{2}$

$$(k^{-1})'(0) = \frac{1}{k'(k^{-1}(0))} = \frac{1}{k'(0)} = \frac{2}{2+\sin(1)}$$

(c) Try to write the following as a Riemann sum

$$\sum_{i=1}^{n} \frac{i}{n^2}\cos\left(1+2\left(\frac{i}{n}\right)^2\right) = \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $x_0 < x_1 < \cdots < x_n$ is a regular partition and $\Delta x = \Delta x_i = x_i - x_{i-1}$.

Therefore, we can take $x_i = \frac{i}{n}$ so that $\Delta x = \frac{1}{n}, x_0 = 0$ and $x_n = 1$. Thus by comparing,

$$f(x_i)\Delta x \text{ with } \frac{i}{n^2}\cos\left(1+2\left(\frac{i}{n}\right)^2\right) = \frac{i}{n}\cos\left(1+2\left(\frac{i}{n}\right)^2\right)\cdot\frac{1}{n}$$

we would want $f(x_i) = f(\frac{i}{n}) = \frac{i}{n}\cos\left(1+2\left(\frac{i}{n}\right)^2\right)$. Thus $f(x) = x\cos(1+2x^2)$.

Therefore $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2}\cos\left(1+2\left(\frac{i}{n}\right)^2\right) = \int_0^1 x\cos(1+2x^2)dx = \frac{1}{4}[\sin(1+2x^2)]_0^1$

$= \frac{1}{4}(\sin(3) - \sin(1))$.

Question 6

Recall $f(x) = 3x^5 - 30x^2 + 1$. Note that f is continuous and differentiable on **R**.

$f'(x) = 15x^4 - 60x = 15x(x^3 - 4) = 15x(x - 4^{(1/3)})(x^2 + 4^{(1/3)}x + 4^{(2/3)})$.

Now we know that the cubic $g(x) = x^3 - 4 = 0$ has a real root. (We have used the identity $(a^3 - b^3) = (a - b)(a^2 + ab + b^2)$ to obtain the above factorisation.) Notice that $x^2 + 4^{(1/3)}x + 4^{(2/3)} = (x + 4^{(1/3)}/2)^2 + 4^{(2/3)} - \frac{1}{4}4^{(2/3)}) > 0$.

Therefore, $f'(x) = 15x(x - 4^{(1/3)})((x + 4^{(1/3)}/2)^2 + \frac{3}{4}4^{(2/3)})$ -------------------- (1)

$f''(x) = 60x^3 - 60 = 60(x^3 - 1) = 60(x-1)(x^2 + x + 1) = 60(x-1)((x + \frac{1}{2})^2 + \frac{3}{4})$ ---- (2).

So f'' is given by a cubic polynomial function. Again we know it must have a real root. The root is easily obtained by the above factorisation.

(a) From (1), $f'(x) = 0$ if and only if $x = 0$ and $x = 4^{(1/3)}$. From (1) the sign of $f'(x)$ is the same as the sign of $x(x - 4^{(1/3)})$ because $(x + 4^{(1/3)}/2)^2 + \frac{3}{4}4^{(2/3)} > 0$. Thus we have: $x < 0 \Rightarrow x < 4^{(1/3)} \Rightarrow x - 4^{(1/3)} < 0 \Rightarrow x(x - 4^{(1/3)}) > 0 \Rightarrow f'(x) > 0$ so that f is increasing on $(-\infty, 0]$. Now $0 < x < 4^{(1/3)} \Rightarrow x - 4^{(1/3)} < 0 \Rightarrow x(x - 4^{(1/3)}) < 0 \Rightarrow f'(x) < 0$ so that f is decreasing on $[0, 4^{(1/3)}]$ and $x > 4^{(1/3)} \Rightarrow x(x - 4^{(1/3)}) > 0 \Rightarrow f'(x) > 0$ so that f is increasing on $[4^{(1/3)}, \infty)$. Note that the end points of these intervals are included by virtue of continuity there.

(b) From (2), $f''(x) = 0 \Leftrightarrow x = 1$ and that the sign of $f''(x)$ is the same as that of $x - 1$. Now $x < 1 \Rightarrow x - 1 < 0 \Rightarrow f''(x) < 0$. Therefore, the graph of f is concave downward on the interval $(-\infty, 1)$. Likewise from (2), $x > 1 \Rightarrow x - 1 > 0$ so that $f''(x) > 0$ when $x > 1$. Thus the graph of f is concave upward on $(1, \infty)$.

(c) From part (a), by the first derivative test, $f(0) = 1$ is a relative maximum and $f(4^{(1/3)}) = 1 - 36 * 2^{(1/3)}$ is a relative minimum.

(d) From part b, since at $x = 1$, there is a change of concavity before and after $x = 1$, $(1, f(1)) = (1, -26)$ is a point of inflection of the graph of f. There are no other points of inflection.

(e)