NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2002 – 2003

MA1102R CALCULUS

April 2003 – Time Allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO** sections: Section A and Section B. It contains a total of **SIX** questions and comprises **FOUR** printed pages.

2. Answer **ALL** questions in **Section A** The marks for questions in Section A are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Answer not more than **TWO** questions from Section B. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

*Answer **ALL** questions in this section*.

Question 1 [**20** marks]

Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x^3 + 1, & x < -1 \\ x\cos\left(\frac{\pi}{2x}\right), & -1 \le x \le 1 \text{ and } x \ne 0 \\ x^2 - 1, & x > 1 \\ 0, & x = 0 \end{cases}.$$

(a) Find the *range* of the function f.

(b) Determine if f is *surjective*.

(c) Determine all x in $\mathbf{R}$ at which the function f is *continuous*. Justify your answer.

(d) Is f *differentiable* at $x = 1$? Justify your answer.

(e) Evaluate $\int_{-1}^{1} f(x)dx$.

Question 2 [**20** marks]

Evaluate, if it exists, each of the following limits.

(a) $\lim_{x \to -\infty} \dfrac{9x^3 + \cos(x^2) + 1}{7x^2 - 29x^3 + 4}$.

(b) $\lim_{x \to 0} \dfrac{\sin^2(5x)}{1 - \cos(3x)}$.

(c) $\lim_{x \to 0} \dfrac{\sin(\sin(x^2) + x^3)}{x^2 + x^3}$.

(d) $\lim_{x \to 27} \dfrac{\sqrt{6 + \sqrt[3]{x}} - 3}{x - 27}$.

(e) $\lim_{x \to \infty} \dfrac{x(5 + \sin(x))}{x^2 + 5}$.

Question 3 [**20** marks]

(a) Evaluate $\int \frac{1}{(x^2+4x+5)(x^2+4x+6)}dx.$

(b) Compute $\int_{-1}^{1} \sqrt{6+|x|}\, dx.$

(c) Find an antiderivative of $g(x)$, which is defined by

$$g(x) = \begin{cases} x^4+1, x \ge 1 \\ 2\sin(\frac{\pi x}{2}), x < 1 \end{cases}.$$

(d) Evaluate $\int \sec^2(\cot(x))\csc^2(x)dx.$

(e) Evaluate $\int x^2 \sin(7x)dx.$

SECTION B

Answer not more than ***TWO*** *questions from this section. Each question in this section carries 20 marks.*

Question 4 [**20** marks]

(a) Find the critical points of the function g, defined by

$$g(x) = \begin{cases} x^2-2x-7, 0 \le x \le 4 \\ 2x^3-33x^2+180x-319, 4 < x \le 7 \end{cases},$$

in the open interval (0, 7). Determine the absolute maximum and the absolute minimum values of the function in the interval [0, 7].

(b) Differentiate the following functions,

(i) $h(x) = (2+\sin^2(x^2))^{\left(\frac{1}{x^2}\right)}.$

(ii) $j(x) = \int_{\sin(x^2)}^{e^{5x}} \frac{t}{1+t^2+\sin(t)}dt.$

(iii) $k(x) = \cos^{-1}\left(\frac{\sin(x)}{2}\right).$

(c) Suppose f is a continuous function defined on the closed and bounded interval [2, 5] such that for all x in [2, 5],

$$2 \le f(x) \le 5.$$

Prove that there exists a point c in [2, 5] such that $f(c) = c$.

Question 5 [**20** marks]

(a) State clearly the *Fundamental Theorem of Calculus.*

Below is the formula for integration by parts.

$$\int_a^b f(x)G(x)dx = [F(x)G(x)]_a^b - \int_a^b F(x)g(x)dx.$$

Here the functions f and g are assumed to be continuous. F is an antiderivative or an indefinite integral of f and G is an antiderivative or an indefinite integral of g.

Prove this formula using the Fundamental Theorem of Calculus and show or explain how the continuity condition on f and g may be replaced by just integrability.

(b) Let the function k be defined on **R** by

$$k(x) = \int_1^{x^5} \frac{1}{t^2 + e^{t^2}} dt.$$

(i) Without integrating, show that the function k is injective.

(ii) Determine $(k^{-1})'(0)$.

(c) Suppose that the function g defined on **R** satisfies

1. $g(x + y) = g(x) + g(y) - 5xy$ for all x and y in **R**,
2. $\lim_{h \to 0} \frac{g(h)}{h} = 7$.

Determine $g(0)$. Prove that g is differentiable at x for all x in **R** and determine $g'(x)$.

Question 6 [**20** marks]

Let the function f be defined on **R** by

$$f(x) = 6x^5 - 15x^4 + 10x^3 - 180x^2 + 1000 \quad .$$

(a) Find the intervals on which f is (i) *increasing*, and (ii) *decreasing*.

(b) Find the intervals on which the graph of f is (i) *concave upward*, and (ii) *concave downward*.

(c) Find the *relative extrema* of f, if any.

(d) Find the *points of inflection* of the graph of f.

(e) sketch the graph of f.

END OF PAPER

Answer To MA1102 Calculus

Question 1

The function f is defined by $f(x) = \begin{cases} x^3+1, & x < -1 \\ x\cos\left(\frac{\pi}{2x}\right), & -1 \le x \le 1 \text{ and } x \ne 0 \\ x^2-1, & x > 1 \\ 0, & x = 0 \end{cases}$.

(a) For $x < -1$, $f(x) = x^3 + 1 < 0$. Also, for $x < -1$, $x^3 + 1 < 0 \Leftrightarrow x < -1$
Thus f maps $(-\infty, -1)$ onto $(-\infty, 0)$. (Because for any $y < 0$, we can take $x = \sqrt[3]{y-1}$ (< -1) so that $f(x) = y$.) Now for $x > 1$, $f(x) = x^2 - 1 > 0$. Also for any $y > 0$, we can take $x = \sqrt{y+1} > 1$ such that $f(x) = y$. Therefore, f maps $(1, \infty)$ onto $(0, \infty)$. Since $f(0) = 0$, the range of f contains $f((-\infty, -1)) \cup \{f(0)\} \cup f((1,\infty)) = \mathbf{R}$ and so is equal to $\mathbf{R}$.

(b) By part (a) Range(f) = $\mathbf{R}$ = codomain of f. Therefore, f is surjective.

(c) When $x < -1$, $f(x) = x^3 + 1$, is a polynomial function and so f is continuous on $(-\infty, -1)$, since any polynomial function is continuous on $\mathbf{R}$ and so is continuous on any open interval. When $-1 < x < 0$, $f(x) = x\cos\left(\frac{\pi}{2x}\right)$ and since cosine is a continuous function and the function $\frac{\pi}{2x}$ is a continuous function on $x > 0$, f on $(-1, 0)$ being the composite of these two continuous functions is therefore continuous on $(-1, 0)$. Similarly, f is continuous on the interval $(0, 1)$. f is continuous on $(1, \infty)$ since $f(x) = x^2 - 1$, a polynomial function. Thus we can conclude that f is continuous at x for $x \ne -1, 0, 1$. Thus it remains to check if f is continuous at $x = -1$, 0 or 1.
Consider the left limit at $x = -1$,

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x^3 + 1 = 0 \text{ and the right limit at } x = -1$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x\cos\left(\frac{\pi}{2x}\right) = -\cos\left(-\frac{\pi}{2}\right) = 0.$$

Thus, since $\lim_{x\to -1} f(x) = 0$, and $f(-1) = 0$ it follows that f is continuous at $x = -1$. Now consider the left limit of f at $x = 1$,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x\cos\left(\frac{\pi}{2x}\right) = 0 \text{ and the right limit at } x = 1,$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 - 1 = 0 = f(1).$$

Therefore, $\lim_{x\to 1} f(x) = f(1)$ and so f is continuous at $x = 1$.
Now $\lim_{x\to 0} f(x) = \lim_{x\to 0} x\cos\left(\frac{\pi}{2x}\right) = 0$ by the Squeeze Theorem since $-|x| \le x\cos\left(\frac{\pi}{2x}\right) \le |x|$ for $x \ne 0$ and $\lim_{x\to 0} |x| = 0$. Since $f(0) = 0$, we conclude that f is also continuous at $x = 0$. Therefore f is continuous at x for any x in $\mathbf{R}$.

(d) To check the differentiability of f at $x = 1$ consider the following limits.

$$\lim_{x\to 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x\to 1^+} \frac{x^2 - 1 - 0}{x-1} = \lim_{x\to 1^+} (x+1) = 2$$

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^-}\frac{x\cos\left(\frac{\pi}{2x}\right)-0}{x-1}=\lim_{x\to 1^-}\frac{\cos\left(\frac{\pi}{2x}\right)-x\sin\left(\frac{\pi}{2x}\right)\cdot\left(-\frac{\pi}{2x^2}\right)}{1}=\frac{\pi}{2}$$

by L' Hôpital's Rule.

Therefore, f is not differentiable at $x = 1$ since $\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}\neq\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$.

(e) f is Riemann integrable on [-1,1] since the restriction of f is continuous on [–1, 1]. Note that f is an odd function on [–1, 1], i.e. $f(-x) = -f(x)$ for all x in [–1, 1] because $f(-x) = -x\cos(\pi/(-2x)) = -x\cos(\pi/(2x)) = -f(x)$ for $x \neq 0$ and for x = 0, obviously $f(-0) = f(0) = 0 = - 0 = -f(0)$.

$\int_{-1}^{0} f(x)dx = \int_{-1}^{0} -f(x)\frac{du}{dx}dx$ where $u = -x$ so that $\frac{du}{dx} = -1$

$= -\int_{1}^{0} f(-u)du$ by the Change of Variable formula,

$= \int_{1}^{0} f(u)du$ since f is an odd function,

$= -\int_{0}^{1} f(u)du = -\int_{0}^{1} f(x)dx$ by renaming the variable.

Therefore, $\int_{-1}^{1} f(x)dx = \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx = -\int_{0}^{1} f(x)dx + \int_{0}^{1} f(x)dx = 0$

Question 2

(a) $$\lim_{x\to-\infty}\frac{9x^3+\cos(x^2)+1}{7x^2-29x^3+4}=\lim_{x\to-\infty}\frac{9+\frac{1}{x^3}\cos(x^2)+\frac{1}{x^3}}{\frac{7}{x}-29+\frac{4}{x^3}}=-\frac{9}{29}$$

This is because $\lim_{x\to-\infty}\frac{\cos(x^2)}{x^3}=0$ by the Squeeze Theorem since

$-|\frac{1}{x^3}|\leq\frac{\cos(x^2)}{x^3}\leq|\frac{1}{x^3}|$ for $x < 0$ and $\lim_{x\to-\infty}|\frac{1}{x^3}|=0$.

(b) $$\lim_{x\to 0}\frac{\sin^2(5x)}{1-\cos(3x)}=\lim_{x\to 0}\frac{2\sin(5x)\cos(5x)5}{3\sin(3x)}=\lim_{x\to 0}\frac{5\sin(10x)}{3\sin(3x)}$$ by L' Hôpital's Rule

$$=\lim_{x\to 0}\frac{50\cos(5x)}{9\cos(3x)}=\frac{50}{9}$$ by L' Hôpital's Rule.

(c) $$\lim_{x\to 0}\frac{\sin(\sin(x^2)+x^3)}{x^2+x^3}=\lim_{x\to 0}\frac{\sin(\sin(x^2)+x^3)}{\sin(x^2)+x^3}\frac{\sin(x^2)+x^3}{x^2}\frac{1}{1+x}$$

$$=\lim_{x\to 0}\frac{\sin(\sin(x^2)+x^3)}{\sin(x^2)+x^3}\left(\frac{\sin(x^2)}{x^2}+x\right)\frac{1}{1+x}=1\ .$$

Or $$\lim_{x\to 0}\frac{\sin(\sin(x^2)+x^3)}{x^2+x^3}=\lim_{x\to 0}\frac{\cos(\sin(x^2)+x^3)(2x\cos(x^2)+3x^2)}{2x+3x^2}$$ by L' Hôpital's Rule

$$=\lim_{x\to 0}\frac{\cos(\sin(x^2)+x^3)(2\cos(x^2)+3x)}{2+3x}=1.$$

(d) $$\lim_{x\to 27}\frac{\sqrt{6+\sqrt[3]{x}}-3}{x-27}=\lim_{x\to 27}\frac{\frac{1}{2}(6+\sqrt[3]{x})^{-\frac{1}{2}}\frac{1}{3}x^{-\frac{2}{3}}}{1}$$ by L' Hôpital's Rule

$= \frac{1}{2}(6+\sqrt[3]{27})^{-\frac{1}{2}}\frac{1}{3}(27)^{-\frac{2}{3}} = \frac{1}{162}$.

(e) $\lim_{x\to\infty}\frac{x(5+\sin(x))}{x^2+5} = \lim_{x\to\infty}\frac{\frac{5}{x}+\frac{\sin(x)}{x}}{1+\frac{5}{x^2}} = \frac{0}{1} = 0$ since by the Squeeze Theorem
$\lim_{x\to\infty}\frac{\sin(x)}{x} = 0$ because $-\frac{1}{x} \le \frac{\sin(x)}{x} \le \frac{1}{x}$ for $x > 0$ and $\lim_{x\to\infty}\frac{1}{x} = 0$

OR ence $-6\frac{x}{x^2+5} \le \frac{x(5+\sin(x))}{x^2+5} \le 6\frac{x}{x^2+5}$ for $x > 0$. Thus by the Squeeze Theorem
$\lim_{x\to\infty}\frac{x(5+\sin(x))}{x^2+5} = 0$ since $\lim_{x\to\infty}\frac{x}{x^2+5} = \lim_{x\to\infty}\frac{1/x}{1+5/x^2} = 0$

Question 3

(a) $$\int \frac{1}{(x^2+4x+5)(x^2+4x+6)}dx = \int \frac{1}{((x+2)^2+1)((x+2)^2+2)}dx$$
$$= \int \frac{1}{((x+2)^2+1)}dx - \int \frac{1}{((x+2)^2+2)}dx$$
$$= \tan^{-1}(x+2) - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+2}{\sqrt{2}}\right)$$

(b) $$\int_{-1}^{1}\sqrt{6+|x|}\,dx = \int_{0}^{1}\sqrt{6+|x|}\,dx + \int_{-1}^{0}\sqrt{6+|x|}\,dx$$
$$= \int_{0}^{1}\sqrt{6+x}\,dx + \int_{-1}^{0}\sqrt{6-x}\,dx.$$

$$= [\tfrac{2}{3}(6+x)^{3/2}]_0^1 + [-\tfrac{2}{3}(6-x)^{3/2}]_{-1}^0 = \tfrac{4}{3}\times(7\sqrt{7} - 6\sqrt{6}) = \frac{28\sqrt{7}}{3} - 8\sqrt{6}$$

(c) $$g(x) = \begin{cases} x^4+1, x \ge 1 \\ 2\sin(\frac{\pi x}{2}), x < 1 \end{cases}.$$

First note that g is continuous on the interval $(1, \infty)$ since it si a polynomial function there and polynomial functions are continuous. Note also that g is continuous on $(-\infty, 1)$ since $2\sin(\frac{\pi x}{2})$ is a continuous function because the sine function is continuous. Now the right limit at $x = 1$ is $\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} x^4+1 = 2 = g(1)$ and the left limit at $x = 1$

$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} 2\sin(\frac{\pi x}{2}) = 2\sin(\frac{\pi}{2}) = 2$. Therefore, $\lim_{x\to 1} g(x) = g(1)$. Thus g is continuous at $x = 1$. Therefore, g is continuous on **R** and we can use the Fundamental Theorem of Calculus to obtain an antiderivative $G(x)$ given by the following Riemann integral for each x in **R**.

$$G(x) = \int_1^x g(t)dt = \begin{cases} \int_1^x g(t)dt, x \ge 1 \\ \int_1^x g(t)dt, x < 1 \end{cases} = \begin{cases} \int_1^x (t^4+1)dt, x \ge 1 \\ \int_1^x 2\sin(\frac{\pi t}{2})dt, x < 1 \end{cases}$$

$$\begin{cases} [\frac{1}{5}t^5+t)]_1^x, x \ge 1 \\ \left[-\frac{4}{\pi}\cos(\frac{\pi t}{2})\right]_1^x, x < 1 \end{cases} = \begin{cases} \frac{1}{5}x^5+x-\frac{6}{5}, x \ge 1 \\ -\frac{4}{\pi}\cos(\frac{\pi x}{2}), x < 1 \end{cases}$$

(d) $\int \sec^2(\cot(x))\ \csc^2(x)dx = -\int \sec^2(\cot(x))\frac{du}{dx}dx$, where $u = \cot(x)$

$= -\int \sec^2(u)du = -\tan(u) + C = -\tan(\cot(x)) + C$ by substitution or change of variable.

(e) Evaluate $\int x^2 \sin(7x)dx$.

$\int x^2 \sin(7x)dx = -x^2 \frac{1}{7}\cos(7x) + \int 2x\frac{1}{7}\cos(7x)dx$ by integration by parts

$= -\frac{x^2}{7}\cos(7x) + \frac{2x}{49}\sin(7x) - \int \frac{2}{49}\sin(7x)dx$ by integration by parts

$= -\frac{x^2}{7}\cos(7x) + \frac{2x}{49}\sin(7x) + \frac{2}{343}\cos(7x) + C$.

Question 4.

(a) Recall $g(x) = \begin{cases} x^2 - 2x - 7, 0 \le x \le 4 \\ 2x^3 - 33x^2 + 180x - 319, 4 < x \le 7 \end{cases}$

Thus, $g'(x) = \begin{cases} 2x - 2, 0 < x < 4 \\ 6x^2 - 66x + 180, 4 < x < 7 \end{cases}$.

Hence g has one stationary point in (0, 4), namely $x = 1$. Since $6x^2 - 66x + 180 = 6(x^2 - 11x + 30) = 6(x - 5)(x - 6) = 0$ if and only if $x = 5$ or 6, there are two sationary points of g in (4, 7), namely $x = 5$, $x = 6$. Now $\lim_{x\to 4^+} g(x) = \lim_{x\to 4} 2x^3 - 33x^2 + 180x - 319 = 1 =$ g(4) and $\lim_{x\to 4^-} g(x) = \lim_{x\to 4} x^2 - 2x - 7 = 1$and so $\lim_{x\to 4} g(x) = g(4)$ and g is continuous at $x = 4$. Now observe that both $x^2 - 2x - 7$ and $2x^3 - 33x^2 + 180x - 319$ are differentiable at $x = 4$ but $\lim_{x\to 4^+} g'(x) = \lim_{x\to 4} 2x - 2 = 6$and is not equal to $\lim_{x\to 4^+} g(x) = \lim_{x\to 4} 6x^2 - 66x + 180 = 12$. We can conclude that g is not differentiable at $x = 4$. Hence the critical points of g in (0, 7) are {1,4,5,6}. Since g is continuous at $x = 4$, g is continuous on the closed and bounded interval [0, 7] and so by the Extreme Value Theorem g has absolute extrema on the interval [0, 2] and they are given respectively by the maximum and minimum of the values of the critical points and the end points under g. Now g(0) = –7, g(1) = –8 , g(4) = 1, g(5) = 6, g(6) = 5, g(7) = 10. Therfore, the absolute maximum of g on [0, 7] is 10 and the absolute minimum of g on [0, 7] is –8.

(b) (i) $h(x) = (2 + \sin^2(x^2))^{\left(\frac{1}{x^2}\right)}$.

Taking logarithm on both sides we get $\ln(h(x)) = \frac{1}{x^2}\ln(2 + \sin^2(x^2))$.

Differentiating both sides we get,

$$\frac{h'(x)}{h(x)} = -\frac{2}{x^3}\ln(2 + \sin^2(x^2)) + \frac{1}{x^2}\frac{2\sin(x^2)\cos(x^2)2x}{2 + \sin^2(x^2)}$$

$$= -\frac{2}{x^3}\ln(2 + \sin^2(x^2)) + \frac{1}{x^2}\frac{2x\sin(2x^2)}{2 + \sin^2(x^2)}$$

$$= \frac{2}{x^3}\left[+\frac{x^2\sin(2x^2)}{2 + \sin^2(x^2)} - \ln(2 + \sin^2(x^2))\right]$$

Therefore,

$$h'(x)=\frac{2}{x^3}\left[+\frac{x^2\sin(2x^2)}{2+\sin^2(x^2)}-\ln(2+\sin^2(x^2))\right](2+\sin^2(x^2))^{\left(\frac{1}{x^2}\right)}.$$

(ii) $j(x)=\int_{\sin(x^2)}^{e^{5x}}\frac{t}{1+t^2+\sin(t)}dt.$

Therefore, $j(x)=\int_0^{e^{5x}}\frac{t}{1+t^2+\sin(t)}dt-\int_0^{\sin(x^2)}\frac{t}{1+t^2+\sin(t)}dt$

Hence by the Fundamental Theorem of Calculus and the Chain Rule,

$$j'(x)=\frac{e^{5x}5e^{5x}}{1+e^{10x}+\sin(e^{5x})}-\frac{\sin(x^2)2x\cos(x^2)}{1+\sin^2(x^2)+\sin(\sin(x^2))}$$

$$=\frac{5e^{10x}}{1+e^{10x}+\sin(e^{5x})}-\frac{x\sin(2x^2)}{1+\sin^2(x^2)+\sin(\sin(x^2))}$$

(iii) $k(x)=\cos^{-1}\left(\frac{\sin(x)}{2}\right)$. Thus by the Chain Rule

$$k'(x)=-\frac{1}{\sqrt{1-\sin^2(x)/4}}\frac{\cos(x)}{2}=-\frac{\cos(x)}{\sqrt{4-\sin^2(x)}}=-\frac{\cos(x)}{\sqrt{3+\cos^2(x)}}.$$

(c) Let g(x) = $f(x)-x$. Since f is continuous on [2, 5], g is a continuous function on [2, 5]. Now since $2\le f(x)\le 5$ for all x in [2, 5], g(2) = $f(2)-2\ge 0$ and g(5) = $f(5)-5\le 0$. Therefore, by the Intermediate Value Theorem, there exists a point c in [2, 5] such that g(c) = 0, i.e., $f(c)=c$.

Question 5.

(a) **Fundamental Theorem of Calculus**. Suppose $f:[a,b]\to\mathbf{R}$ is a contunuus function. Then (i) the function $F:[a,b]\to\mathbf{R}$ defined by $F(x)=\int_a^x f(t)dt$ is differentiable on $[a, b]$ satisfying $F'(x)=f(x)$ for every x in $[a, b]$ and (ii) for any antiderivative G of f the Riemann integral $\int_a^b f(t)dt=G(b)-G(a)$.

Proof of the formula for integration by parts.

Suppose f and g are continuous function with antiderivative F and G respectively. Then since F and G are differentiable, they are continuous functions. By the Product Rule $(F\cdot G)(x)=F(x)G(x)$ is differentiable and is of course the antiderivative of its derivative $(F\cdot G)'(x)=F'(x)G(x)+F(x)G'(x)=f(x)G(x)+F(x)g(x)$ which is continuous since f, g, F and G are all continuous functions and that product of continuous functions is contunuous.

Thus by the Fundamental Theorem of Calculus, since $(F\cdot G)'(x)$ is *continuous*

$$\int_a^b(F\cdot G)'(x)dx=(F\cdot G)(b)-(F\cdot G)(a)=F(b)G(b)-F(a)G(a)=[F(b)G(b)]_a^b$$

----------------------------- (1)

But the left hand side is also equal to $\int_a^b f(x)G(x)dx+\int_a^b F(x)g(x)dx.$ Therefore, from (1) we get $\int_a^b f(x)G(x)dx+\int_a^b F(x)g(x)dx=[F(x)G(x)]_a^b$ and so it follows that

$$\int_a^b f(x)G(x)dx=[F(x)G(x)]_a^b-\int_a^b F(x)g(x)dx.$$

The critical part of the proof is (1). If $(F\cdot G)'(x)$ is *Riemann integrable,* then (1) still holds.

Now by the Product Rule, $(F\cdot G)'(x)= F'(x)G(x) + F(x)G'(x) = f(x)G(x)+ F(x)g(x)$. For this to be Riemann integrable just note that product of two (Riemann) integrable functions is (Riemann) integrable. Therefore if f is Riemann integrable, then because G is continuous being differentiable and so is Riemann integrable the product $f(x)G(x)$ is Riemann integrable. Similarly, we deduce that if g is Riemann integrable, then $F(x)g(x)$ is Riemann integrable. Therefore, $(F\cdot G)'(x)= f(x)G(x)+ F(x)g(x)$ being the sum of two Riemann integrable functions is Riemann integrable. Therefore, if f and g are both Riemann integrable functions, then (1) holds and as above we obtain the integration by parts formula.

(b) Now $k(x)=\int_1^{x^5}\frac{1}{t^2+e^{t^2}}dt.$

(i) Thus by the Fundamental Theorem of Calculus and the Chain Rule,

$k'(x)=\frac{5x^4}{x^{10}+e^{x^{10}}}$. Note that k is continuous since it is differentiable on **R**. Also for $x \neq 0$, $k'(x) > 0$. Therefore, k is (strictly)increasing on $(-\infty, 0]$ and also on $[0, \infty)$. This means k is (strictly) increasing on **R**. Therefore, k is injective.

(ii) Note that $k(1) =0$ and so $k^{-1}(0) = 1$. Thus since $k'(1)=\frac{5}{1+e}\neq 0$.

$$(k^{-1})'(0)=\frac{1}{k'(k^{-1}(0))}=\frac{1}{k'(1)}=\frac{1+e}{5}$$

(c) Note that g satisfies the following.

1. $g(x+y) = g(x) + g(y) - 5xy$ for all x and y in **R**,
2. $\lim_{h\to 0}\frac{g(h)}{h}=7$

Thus, for any fixed x in **R** and for $h \neq 0$,

$$\frac{g(x+h)-g(x)}{h}=\frac{g(x)+g(h)-5xh-g(x)}{h}=\frac{g(h)-5xh}{h}=\frac{g(h)}{h}-5x \text{ by property 1,}$$

Therefore,

$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\lim_{h\to 0}\frac{g(h)}{h}-5x=7-5x \text{ by Property 2}$$

Hence g is differentiable at x for all x and that $g'(x) = 7 - 5x$.

Question 6

5. $f(x)=6x^5-15x^4+10x^3-180x^2+1000$. Note that sincce f is a polynomial function, f is continuous and differentiable on **R**.

$f'(x)=30x^4-60x^3+30x^2-360x=30x(x^3-2x^2+x-12)$

Now we know that the cubic $g(x) = x^3 - 2x^2 + x - 12 = 0$ has a real root. We can try to use the Intermediate Value Theorem to locate the root. (Of Course we can use Cardano's formula for the cubic but we can do this first.) We compute some value of g. $g(1) = 1 - 2 + 1 - 12 = -12 < 0$, $g(2) = 8 - 8 + 2 - 12 = -10 < 0$ and $g(3) = 27 - 18 + 3 - 12 = 0$. Therefore, $(x - 3)$ is a factor of $x^3 - 2x^2 + x - 12$. Perform a long division to obtain x^3

$-2x^2 + x - 12 = (x-3)(x^2 + x + 4)$. Therefore,

$f'(x) = 30x(x^3 - 2x^2 + x - 12) = 30x(x-3)(x^2 + x + 4)$

$= 30x(x-3)((x+\frac{1}{2})^2 - \frac{1}{4} + 4) = 30x(x-3)((x+\frac{1}{2})^2 + \frac{15}{4})$ ------------ (1)

$f''(x) = 120x^3 - 180x^2 + 60x - 360 = 60(2x^3 - 3x^2 + x - 6)$

So f'' is given by a cubic polynomial function. Again we know it must have a real root. So we shall try to locate the root by the use of the Intermediate Value Theorem. Consider the function $k(x) = 2x^3 - 3x^2 + x - 6$. First compute some values of k to narrow down the range of the root. $k(0) = -6 < 0$, $k(2) = 0$. Thus factoring out $(x-2)$, we have

$k(x) = (x-2)(2x^2 + x + 3) = (x-2)(2(x+\frac{1}{4})^2 - \frac{1}{8} + 3)$ and so

$f''(x) = 60k(x) = 60(x-2)(2(x+\frac{1}{4})^2 + \frac{23}{8}))$ ----------------------- (2)

a. From (1), $f'(x) = 0$ if and only if $x = 0$ and $x = 3$. Note that $x^2 + x + 4 > 0$ and so from (1) we have: $x < 0 \Rightarrow f'(x) > 0$ so that f is increasing on $(-\infty, 0]$;
$0 < x < 3 \Rightarrow f'(x) < 0$ so that f is decreasing on $[0, 3]$ and $x > 3 \Rightarrow f'(x) > 0$ so that f is increasing on $[3, \infty)$.

b. From (2), $f''(x) = 0 \Leftrightarrow x = 2$. Now $x < 2 \Rightarrow f''(x) < 0$. Therefore, the graph of f is concave downward on the interval $(-\infty, 2)$. Likewise from (2), $x > 2 \Rightarrow x - 2 > 0$ so that $f''(x) > 0$ when $x > 2$. Thus the graph of f is concave upward on $(2, \infty)$.

c. From part a, by the first derivative test, $f(0) = 1000$ is a relative maximum and $f(3) = -107$ is a relative minimum.

d. From part b, since there is a change of concavity before and after $x = 0$, the point $(2, f(2)) = (2, 312)$ is a point of inflection of the graph of f.

e.