NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2002 – 2003

MA1102R CALCULUS

November 2002 – Time Allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO** sections: Section A and Section B. It contains a total of **SIX** questions and comprises **FOUR** printed pages.

2. Answer **ALL** questions in **Section A** The marks for questions in Section A are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Answer not more than **TWO** questions from Section B. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

*Answer **ALL** questions in this section.*

Question 1 [**20** marks]

Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x^3 + 1, & x < -1 \\ \sin\left(\frac{\pi}{2x}\right), & -1 \le x \le 1 \text{ and } x \ne 0 \\ 2x^2 - 1, & x > 1 \\ 0, & x = 0 \end{cases}.$$

(a) Find the *range* of the function f .

(b) Determine if f is surjective.

(c) Determine all x in $\mathbf{R}$ at which the function f is *continuous*. Justify your answer.

(d) Is f *differentiable* at $x = 1$? Justify your answer.

(e) Evaluate $\int_{-1}^{1} f(x)dx$.

Question 2 [**20** marks]

Evaluate, if it exists, each of the following limits.

(a) $\lim_{x \to +\infty} \frac{7x^3 + x\sin(x^3) + 1}{2x^2 - 21x^3 + 3}$.

(b) $\lim_{x \to +\infty} \left(\cos(\frac{\pi}{x})\right)^x$.

(c) $\lim_{x \to 0} \frac{\sin(\sin(x))}{x^2 + 2x}$.

(d) $\lim_{x \to +\infty} \sqrt{2 + x + x^2} - \sqrt{2 - x + x^2}$.

(e) $\lim_{x \to 0} \frac{5\sin^{-1}(x)}{2x}$.

Question 3 [**20** marks]

(a) Evaluate $\int \frac{2e^{2x} - \sin(2x)}{e^{2x} + \cos^2(x) + 1} dx.$

(b) Compute $\int_0^3 (|x-1| + |x-2|) dx.$

(c) Compute $\int_1^2 (\ln(5x))^2 dx.$.

(d) Evaluate $\int \cos(\tan(x)) \sec^2(x) dx.$

(e) Evaluate $\int \sqrt{x}\, e^{\sqrt{x}}\, dx.$

SECTION B

Answer not more than ***TWO (2)*** *questions from this section. Each question in this section carries 20 marks.*

Question 4 [**20** marks]

(a) State, but do not prove, the *Mean Value Theorem*.

(b) Prove that $\cot\left(\frac{\pi}{5}\right) - 1 = \frac{\pi}{20} \csc^2(c)$ for some c in $\left(\frac{\pi}{5}, \frac{\pi}{4}\right)$.

(c) Suppose f is a differentiable function defined on **R** such that

1. $f(0) = 0$ and
2. $f'(x) = \frac{x^2}{1+2x^2}$ for all x in **R**.

(i) Show that if $x > 0$, then there exists c in the interval $(0, x)$ such that

$$\frac{f(x)}{x} = \frac{c^2}{1+2c^2}.$$

(ii) Deduce that $0 < f(x) < x$ for $x > 0$ and $0 > f(x) > x$ for $x < 0$.

(d) Show that the equation

$$x^5 + \frac{9}{1+\sin^2(x)} = 3$$

has at least one real root.

Question 5 [**20** marks]

(a) Differentiate the following functions.

(i) $h(x) = (\ln(1+x^2)+1)^{\sin(x)}$.

(ii) $j(x) = \int_x^{x^2} \frac{1}{1+2t^2+\sin(t^2)} dt$.

(b) Let the function k be defined on **R** by

$$k(x) = \int_1^{x^3} e^{-t^2} dt.$$

(i) Without integrating, show that the function k is injective.

(ii) Determine $(k^{-1})'(0)$.

(c) Suppose that the function g defined on **R** satisfies

1. g(x +y) =g(x) g(y) for all x and y in **R**,
2. g(0) = 1 and that
3. g is differentiable at x = 0 and g'(0) = 1.

By considering the limit of the difference quotient

$$\frac{g(x+h)-g(x)}{h},$$

show that g is differentiable at x for all x and that g'(x) = g(x).

Question 6 [**20** marks]

Let the function f be defined on **R** by

$$f(x) = \frac{1+x-x^2}{(x-1)^2} .$$

(a) Show that if $x \neq 1$, then

$$f'(x) = \frac{x-3}{(x-1)^3} \text{ and } f''(x) = \frac{2(4-x)}{(x-1)^4} .$$

(b) Find the intervals on which f is (i) *increasing*, and (ii) *decreasing*.

(c) Find the intervals on which the graph of f is (i) *concave upward*, and (ii) *concave downward*.

(d) Find the *relative extrema* of f, if any.

(e) Find the *absolute extrema* of f, if any.

(f) Find the *points of inflection* of the graph of f.

(g) Find the horizontal and vertical asymptotes of the graph of f and sketch the graph of f.

END OF PAPER

Answer To MA1102 Calculus

Question 1

The function f is defined by $f(x) = \begin{cases} x^3+1, & x<-1 \\ \sin\left(\frac{\pi}{2x}\right), & -1 \le x \le 1 \text{ and } x \ne 0 \\ 2x^2-1, & x>1 \\ 0, & x=0 \end{cases}$.

(a) For $x<-1$, $f(x)=x^3+1<0$. Also, for $x<-1$, $x^3+1<0 \Leftrightarrow x<-1$
Thus f maps $(-\infty,-1)$ onto $(-\infty, 0)$. (Because for any $y < 0$, we can take $x=\sqrt[3]{y-1}\,(<-1)$ so that $f(x)=y$) For $-1 \le x \le 1$ and $x \ne 0$, $|f(x)| = \left|\sin\left(\frac{\pi}{2x}\right)\right| \le 1$.
Now since $½ \le x \le 1$ if and only if $\frac{\pi}{2} \le \frac{\pi}{2x} \le \pi$, the image of [1/2, 1] under f is the image of [π/2, π] under the sine function and so f ([1/2, 1]) = [0, 1]. Similarly the image of [-1, -1/2] under f is the image of [−π, −π/2] under the sine function and so f ([-1, -1/2]) = [-1, 0]. This is because $-1\le x \le -½$ if and only if $-\pi \le \frac{\pi}{2x} \le -\frac{\pi}{2}$. Thus, with $f(0) = 0$, we conclude that f ([-1, 1]) = [-1, 1].
Another easier way to show this is as follows. For $-1 \le x \le 1$, we observe as above that $-1 \le f(x) \le 1$. This means the image of [-1, 1] under f is contained in [-1, 1].
Next observe that f (1/3) = sin(3π/2) = -1 and f (1) = sin(π/2) = 1. Note that f on the interval [1/3, 1] is given by sin(π/(2x)) and so is continuous on [1/3, 1] because of the fact that the function π/(2x) is continuous on [1/3, 1] and sine is a continuous function implying that the composite sin(π/(2x)) is continuous on [1/3, 1]. Therefore, by the Intermediate Value Theorem any y with $-1 = f(1/3) \le y \le 1 = f(1)$ is in the image of [1/3, 1] under f. This means f ([-1, 1]) contains [-1, 1]. Therefore, f ([-1, 1]) = [-1, 1].
Now for $x > 1$, $f(x) = 2x^2 - 1 > 1$. Also for any $y > 1$, we can take $x = \sqrt{\frac{y+1}{2}} > 1$ such that $f(x) = y$. Therefore, f maps (1, ∞) onto (1, ∞). Hence the range of f is $f((-\infty,-1)) \cup f([-1, 1]) \cup f((1,\infty)) = (-\infty,0)\cup[-1, 1]\cup(1, \infty) = (-\infty, \infty) =$ **R**.

(b) By part (a) Range(f) = **R** =codomain of f . Therefore, f is surjective.

(c) When $x<-1$, $f(x)=x^3+1$, is a polynomial function and so f is continuous on $(-\infty,-1)$, since any polynomial function is continuous on **R** and so is continuous on any open interval. When $-1<x<0$, $f(x)=\sin\left(\frac{\pi}{2x}\right)$ and since sine is a continuous function and the function $\frac{\pi}{2x}$ is a continuous function on $x > 0$, f on (–1, 0) being the composite of these two continuous functions is therefore continuous on (–1, 0). Similarly, f is continuous on the interval (0, 1). f is continuous on (1, ∞) since $f(x) = 2x^2 - 1$, a polynomial function. Thus we can conclude that f is continuous at x for $x \ne -1, 0, 1$. Thus it remains to check if f is continuous at $x = -1$, 0 or 1.
Consider the left limit at $x=-1$,

$\lim_{x\to-1^-} f(x) = \lim_{x\to-1^-} x^3+1 = 0$ and the right limit at $x=-1$

$$\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} \sin\left(\frac{\pi}{2x}\right) = -1.$$

Thus, since $\lim_{x\to -1^-} f(x) \neq \lim_{x\to -1^+} f(x)$, $\lim_{x\to -1} f(x)$ does not exist. It follows that f is not continuous at $x = -1$. Now consider the left limit of f at $x = 1$,

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sin\left(\frac{\pi}{2x}\right) = 1$ and the right limit at $x = 1$,

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x^2 - 1 = 1 = f(1)$.

Therefore, $\lim_{x\to 1} f(x) = f(1)$ and so f is continuous at $x = 1$.

Now we claim that the limit of f at $x = 0$ cannot be equal to $f(0) = 0$. We shall show that we can find a $\varepsilon > 0$ such that for any $\delta > 0$, we can find a x_δ such that $|x_\delta - 0| < \delta$ but $|f(x_\delta) - f(0)| = |f(x_\delta)| \geq \varepsilon$. We shall take $\varepsilon = 1$. For any $\delta > 0$, since $\lim_{N\to\infty} \frac{1}{2N+1} = 0$, threr exists a positive integer $N > 1$ such that $\frac{1}{2N+1} < \delta$. So we take $0 < x_\delta = \frac{1}{2N+1} < \delta$

Then $|f(x_\delta) - f(0)| = |f(x_\delta)| = \left|\sin\left((2N+1)\frac{\pi}{2}\right)\right| = 1 \geq \varepsilon$. Thus f is not continuous at $x = 0$. Therefore, f is continuous at x for all x except for $x = -1$ or 0.

(d) To check the differentiability of f at $x = 1$ consider the following limits.

$$\lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{2x^2 - 1 - 1}{x - 1} = \lim_{x\to 1^+} 2(x+1) = 4$$

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{\sin\left(\frac{\pi}{2x}\right) - 1}{x - 1} = \lim_{x\to 1^-} \frac{\cos\left(\frac{\pi}{2x}\right)\cdot\left(-\frac{\pi}{2x^2}\right)}{1} = 0$$

by L' Hôpital's Rule.

Therefore, f is not differentiable at $x = 1$ since $\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} \neq \lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1}$.

(e) f is Riemann integrable on [-1,1] since the restriction of f is continuous on $[-1, 1]$ except for $x = 0$. Note that f is an odd function on $[-1, 1]$, i.e. $f(-x) = -f(x)$ for all x in $[-1, 1]$ because $f(-x) = \sin(\pi/(-2x)) = -\sin(\pi/(2x)) = -f(x)$ for $x \neq 0$ and for x = 0, obviously $f(-0) = f(0) = 0 = -0 = -f(0)$.

$\int_{-1}^{0} f(x)dx = \int_{-1}^{0} -f(x)\frac{du}{dx}dx$ where $u = -x$ so that $\frac{du}{dx} = -1$

$= -\int_{1}^{0} f(-u)du$ by the Change of Variable formula,

$= \int_{1}^{0} f(u)du$ since f is an odd function,

$= -\int_{0}^{1} f(u)du = -\int_{0}^{1} f(x)dx$ by renaming the variable.

Therefore, $\int_{-1}^{1} f(x)dx = \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx = -\int_{0}^{1} f(x)dx + \int_{0}^{1} f(x)dx = 0$

Question 2

(a) $$\lim_{x\to+\infty} \frac{7x^3 + x\sin(x^3) + 1}{2x^2 - 21x^3 + 3} = \lim_{x\to+\infty} \frac{7 + \frac{1}{x^2}\sin(x^3) + \frac{1}{x^3}}{\frac{2}{z} - 21 + \frac{3}{x^3}} = \frac{7}{-21} = -\frac{1}{3}$$

This is because $\lim_{x\to\infty} \frac{\sin(x^3)}{x^2} = 0$ by the Squeeze Theorem since

$$-\frac{1}{x^2} \leq \frac{\sin(x^3)}{x^2} \leq \frac{1}{x^2} \text{ for } x > 0 \text{ and } \lim_{x\to\infty} \frac{1}{x^2} = 0$$

(b) $\lim_{x\to\infty} \left(\cos(\frac{\pi}{x})\right)^x$. Let $y = \left(\cos(\frac{\pi}{x})\right)^x$. Since

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} x\ln\left(\cos(\tfrac{\pi}{x})\right) = \lim_{x\to\infty} \frac{\ln(\cos(\frac{\pi}{x}))}{\frac{1}{x}} = \lim_{x\to\infty} \frac{-\frac{\sin(\frac{\pi}{x})}{\cos(\frac{\pi}{x})}\cdot\left(-\frac{\pi}{x^2}\right)}{-\frac{1}{x^2}}$$ by L' Hôpital's Rule,

$$= \lim_{x\to\infty} -\pi\tan(\tfrac{\pi}{x}) = 0,$$

$$\lim_{x\to\infty} y = e^{\lim_{x\to\infty}\ln(y)} = e^0 = 1.$$

(c) $\lim_{x\to 0} \frac{\sin(\sin(x))}{x^2+2x} = \lim_{x\to 0} \frac{\cos(\sin(x))\cos(x)}{2x+2} = \frac{1}{2}$ by L' Hôpital's Rule.

(d) $\lim_{x\to+\infty} \sqrt{2+x+x^2} - \sqrt{2-x+x^2} = \lim_{x\to+\infty} \frac{2x}{\sqrt{2+x+x^2}+\sqrt{2-x+x^2}}$

$\lim_{x\to+\infty} \frac{2}{\sqrt{\frac{2}{x^2}+\frac{1}{x}+1}+\sqrt{\frac{2}{x^2}-\frac{1}{x}+1}} = \frac{2}{2} = 1$

(e) $\lim_{x\to 0} \frac{5\sin^{-1}(x)}{2x} = \lim_{x\to 0} \frac{5\frac{1}{\sqrt{1-x^2}}}{2} = \frac{5}{2}$ by L' Hôpital's rule.

Question 3

(a) $\int \frac{2e^{2x}-\sin(2x)}{e^{2x}+\cos^2(x)+1}dx. = \int \frac{1}{e^{2x}+\cos^2(x)+1}\frac{dy}{dx}dx,$

where $y = e^{2x}+\cos^2(x)+1$, $\frac{dy}{dx} = 2e^{2x}-\sin(2x)$,

$= \int \frac{1}{y}dy$ by substitution or change of variable

$= \ln|y| + C = \ln(e^{2x}+\cos^2(x)+1)+C.$

(b) $\int_0^3(|x-1|+|x-2|)dx$

$= \int_0^1(|x-1|+|x-2|)dx + \int_1^2(|x-1|+|x-2|)dx + \int_2^3(|x-1|+|x-2|)dx.$

$= \int_0^1(3-2x)dx + \int_1^2 1dx + \int_2^3(2x-3)dx.$

$= [3x-x^2]_0^1 + 1 + [x^2-3x]_2^3 = 2+1+2 = 5..$

(c) $\int(\ln(5x))^2dx = x(\ln(5x))^2 - \int 2\ln(5x)dx = x(\ln(5x))^2 - 2x\ln(5x) + \int 2dx$

by integration by parts

$= x(\ln(5x))^2 - 2x\ln(5x) + 2x + C.$

Therefore, $\int_1^2(\ln(5x))^2dx = \left[x(\ln(5x))^2 - 2x\ln(5x) + 2x\right]_1^2$

$= 2(\ln(10))^2 - 4\ln(10) + 4 - (\ln(5))^2 + 2\ln(5) - 2$

. $= 2(\ln(5))^2 + 2(\ln(2))^2 + 4\ln(5)(\ln(2) - 4\ln(10) - (\ln(5))^2 + 2\ln(5) + 2$

. $= (\ln(5))^2 + 2(\ln(2))^2 + 4\ln(5)(\ln(2) - 2\ln(5) - 4\ln(2) + 2$

.

(d) $\int \cos(\tan(x))\sec^2(x)dx = \int \cos(\tan(x))\frac{du}{dx}dx$, where $u = \tan(x)$

$= \int \cos(u)du = \sin(u) + C = \sin(\tan(x)) + C$ by substitution or change of variable.

(e) Evaluate $\int \sqrt{x}\, e^{\sqrt{x}}\, dx$. Let $u = \sqrt{x}$. Then $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$.

$\int 2xe^{\sqrt{x}} \frac{1}{2\sqrt{x}} dx = \int 2xe^{\sqrt{x}} \frac{du}{dx} dx = 2\int u^2 e^u du$ by substitution

$= 2u^2e^u - 2\int 2ue^u du$ by integration by parts

$= 2u^2e^u - 4ue^u + \int 4e^u du$ by integration by parts

$= 2u^2e^u - 4ue^u + 4e^u + C = 2xe^{\sqrt{x}} - 4\sqrt{x}\, e^{\sqrt{x}} + 4e^{\sqrt{x}} + C$

Question 4.

(a) Mean Value Theorem states that if $f : [a, b] \to \mathbf{R}$ is a function such that
1. f is continuous on $[a, b]$ and
2. f is differentiable on (a, b),

then there exists c in (a, b), such that $\frac{f(b) - f(a)}{b - a} = f'(c)$.

(b) Note that $\cot(\frac{\pi}{4}) = 1$. Since cot is continuous on $[\frac{\pi}{5}, \frac{\pi}{4}]$ and differentiable on $(\frac{\pi}{5}, \frac{\pi}{4})$, by the Mean Value Theorem, there exists c in $(\frac{\pi}{5}, \frac{\pi}{4})$ such that

$$\frac{\cot(\frac{\pi}{5}) - \cot(\frac{\pi}{4})}{\frac{\pi}{5} - \frac{\pi}{4}} = -\csc^2(c).$$

Therefore, $\cot(\frac{\pi}{5}) - \cot(\frac{\pi}{4}) = \left(\frac{\pi}{5} - \frac{\pi}{4}\right) \cdot (-\csc^2(c)) = \frac{\pi}{20}\csc^2(c)$.

(c) (i) Since f is differentiable on $\mathbf{R}$, by the Mean Value Theorem, for $x > 0$,

$$\frac{f(x) - f(0)}{x - 0} = f'(c) \text{ for some c such that } 0 < c < x$$

Therefore, since it is given that $f(0) = 0$ by condition (1) and $f'(c) = \frac{c^2}{1 + 2c^2}$, by Condition 2, we have that $\frac{f(x)}{x} = \frac{c^2}{1 + 2c^2}$.

(ii) Thus for $x > 0$, we have then by (i) that $0 < \frac{f(x)}{x} = \frac{c^2}{1 + 2c^2} < 1$ and so multiplying by x (>0) we get $0 < f(x) < x$.

Similarly for $x < 0$, by the Mean Value Theorem we have that for some d such that $x < d < 0$, $0 < \frac{f(x)}{x} = \frac{d^2}{1 + 2d^2} < 1$. Thus, multiplying this inequality by x (<0), we have that $0 > f(x) > x$

(d) Let $g(x) = x^5 + \frac{9}{1 + \sin^2(x)} - 3$. Then g is a continuous function on $\mathbf{R}$.

$g(-2) = -32 - 3 + \frac{9}{1 + \sin^2(-2)} \le -26 < 0$ and $g(0) = 6 > 0$.

Thus $g(-2) < 0 < g(0)$. Thus since g is continuous on $[-2, 0]$, by the Intermediate Value Theorem, there exists c in $(-2, 0)$ such that $g(c) = 0$. That is to say, c is a root of the equation

$$x^5 + \frac{9}{1+\sin^2(x)} = 3$$

Question 5.

(a) (i) $h(x) = (\ln(1+x^2)+1)^{\sin(x)}$

Taking logarithm on both sides we get $\ln(h(x)) = \sin(x)\ln(\ln(1+x^2)+1)$.
Differentiating both sides we get,

$$\frac{h'(x)}{h(x)} = \cos(x)\ln(\ln(1+x^2)+1) + \sin(x)\frac{\frac{2x}{1+x^2}}{\ln(1+x^2)+1}$$
$$= \cos(x)\ln((\ln(1+x^2)+1) + \frac{2x\sin(x)}{(1+x^2)(\ln(1+x^2)+1)}$$

Therefore,

$$h'(x) = \left(\cos(x)\ln((\ln(1+x^2)+1) + \frac{2x\sin(x)}{(1+x^2)(\ln(1+x^2)+1)}\right)\ln(1+x^2)+1)^{\sin(x)}$$

(ii) $j(x) = \int_x^{x^2} \frac{1}{1+2t^2+\sin(t^2)}dt.$

Therefore, $j(x) = \int_0^{x^2} \frac{1}{1+2t^2+\sin(t^2)}dt - \int_0^{x} \frac{1}{1+2t^2+\sin(t^2)}dt.$

Hence by the Fundamental Theorem of Calculus and the Chain Rule,

$$j'(x) = \frac{2x}{1+2x^4+\sin(x^4)} - \frac{1}{1+2x^2+\sin(x^2)}.$$

(b) Now $k(x) = \int_1^{x^3} e^{-t^2}dt.$

(i) Thus by the Fundamental Theorem of Calculus and the Chain Rule,

$k'(x) = 3x^2e^{-x^6}$. Note that k is continuous since it is differentiable on **R**. Also for $x \neq 0$, $k'(x) > 0$. Therefore, k is (strictly)increasing on $(-\infty, 0]$ and also on $[0, \infty)$. This means k is (strictly) increasing on **R**. Therefore, k is injective.

(ii) Note that $k(1) = 0$ and so $k^{-1}(0) = 1$. Thus since $k'(1) = 3e^{-1} \neq 0$,

$$(k^{-1})'(0) = \frac{1}{k'(k^{-1}(0))} = \frac{1}{k'(1)} = \frac{e}{3}$$

(c) Note that g satisfies the following three properties.

1. $g(x+y) = g(x)\,g(y)$ for all x and y in **R**,
2. $g(0) = 1$ and that 3. g is differentiable at $x = 0$ and $g'(0) = 1$.

For $h \neq 0$, $\frac{g(x+h)-g(x)}{h} = \frac{g(x)g(h)-g(x)}{h}$ by property 1,

$$= g(x)\frac{g(h)-1}{h} = g(x)\frac{g(h)-g(0)}{h}$$ by property 2.

Therefore,

$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} g(x)\frac{g(h)-g(0)}{h} = g(x)\lim_{h\to 0}\frac{g(h)-g(0)}{h} = g(x)g'(0)$$

since g is differentiable at $x = 0$ by Property 3

$$= g(x)\,1 = g(x)$$

Hence g is differentiable at x for all x and that $g'(x) = g(x)$.

Question 6

$$f(x)=\frac{1+x-x^2}{(x-1)^2}=-1-\frac{1}{x-1}+\frac{1}{(x-1)^2} \text{ ---------------- } (*)$$

Note that f is continuous and differentiable on $\mathbf{R}-\{1\}$ since it is a rational function.

(a) Therefore, $f'(x)=\frac{1}{(x-1)^2}-\frac{2}{(x-1)^3}=\frac{x-3}{(x-1)^3}$ ------- (1)

and so $f''(x)=\frac{-2}{(x-1)^3}+\frac{6}{(x-1)^4}=\frac{2(4-x)}{(x-1)^4}$--------- (2).

(b) For $x < 1$, by (1), $f'(x) > 0$. Therefore, f is increasing on the interval $(-\infty, 1)$. For $1 < x < 3$, again by (1), $f'(x) < 0$. Therefore, f is decreasing on the interval $(1, 3]$. Then for $x > 3$, $f'(x) > 0$ again by (1). Thus f is increasing on the interval $[3, \infty)$.

(c) From (2), for $x > 4$, $f''(x) < 0$. Thus the graph of f is concave downward on the interval $(4, \infty)$. Also from (2) for $x < 4$ and $x \neq 1$, $f''(x) > 0$ and so the graph of f is concave upward on the interval $(-\infty, 1)$ and on the interval $(1, 4)$.

(d) From part (b), $f(3) = -5/4$ is a relative minimum and there is no relative maximum.

(e) From (*), we see that for $x < 1$, $f(x) > -1$. From part (d), $f(3) = -5/4$ is the absolute minimum of f on $(1, \infty)$. Thus $-5/4$ is the absolute minimum value of f. And there is no absolute maximum for f.

(f) By part (c), since f is continuous at $x = 4$ and that the graph of f has a change in concavity before and after $x = 4$, $(4, f(4)) = (4, -11/9)$ is a point of inflection.

(g) Note that the limit, $\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty} -1-\frac{1}{x-1}+\frac{1}{(x-1)^2}=-1$. Therefore, the line $y = -1$ is a horizontal asymptote. Note also that $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\frac{1+x-x^2}{(x-1)^2}=+\infty$ since $\lim_{x\to 1^-}\frac{1}{(x-1)^2}=+\infty$ and $\lim_{x\to 1^-} 1+x-x^2=1>0$. Similarly $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\frac{1+x-x^2}{(x-1)^2}=+\infty$. Hence the line $x = 1$ is a vertical asymptote.