NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 2 EXAMINATION 2000 – 2001

MA1102 CALCULUS

April 2001 – Time Allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SIX (6)** questions and comprises **FIVE (5)** printed pages.

2. Answer **ALL** questions in **Section A** The marks for questions in Section A are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Answer not more than **TWO (2)** questions from Section B. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

*Answer **ALL** questions in this section.*

Question 1 [**20** marks]

Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x^2 - 25, & x < -5 \\ 3, & x = -5 \\ 4x + 20, & -5 < x < -1 \\ 2x^2 + 14, & x \geq -1 \end{cases}.$$

(a) Find the *range* of the function f.

(b) Determine if f is surjective.

(c) Determine all x in $\mathbf{R}$ at which the function f is *continuous.*

(d) Find all x in $\mathbf{R}$ at which the function f is *differentiable*. Justify your answer.

(e) Compute $\int_{-6}^{0} f(x)dx$.

Question 2 [**20** marks]

Evaluate, if it exists, each of the following limits.

(a) $\lim\limits_{x \to -\infty} \dfrac{x^2 - 8|x^3| + 7}{24x^3 + 17x + 3}$.

(b) $\lim\limits_{x \to 0} \dfrac{x^2 \sin(\frac{1}{x})}{\sin(x)}$.

(c) $\lim\limits_{x \to \infty} \dfrac{x(2 + \cos(x))}{x^2 + 1}$.

(d) $\lim\limits_{x \to 3} \dfrac{\sqrt{16 + x^2} - 5}{x - 3}$.

(e) $\lim\limits_{x \to 0^+} (\sin(x))^{\sin(x)}$.

Question 3 [**20** marks]

(a) Evaluate $\int \dfrac{xe^{\frac{1}{2}x^2} - \sin(2x)}{e^{\frac{1}{2}x^2} + \cos^2(x) + 2}\,dx.$

(b) Compute $\int_{-1}^{3} \sqrt{7 + |x|}\,dx.$

(c) Find an antiderivative of $g(x)$, which is defined by

$$g(x) = \begin{cases} x^3 + 7, & x \geq 1 \\ 2x^4 + 6, & x < 1 \end{cases}.$$

(d) Evaluate $\int \dfrac{\sqrt{x}}{\sqrt{\sqrt{x} + 3}}\,dx.$

(e) Evaluate $\int x^3 \cos(x)\,dx.$

SECTION B

Answer not more than ***TWO (2)*** *questions from this section. Each question in this section carries 20 marks.*

Question 4 [**20** marks]

(a) Let $g : [-2, 6] \rightarrow \mathbf{R}$ be a function defined by

$$g(x) = \begin{cases} x^2 + 2x - 5, & -2 \leq x < 2 \\ x^3 - 12x^2 + 45x - 47, & 2 \leq x \leq 6 \end{cases}.$$

(i) Find the critical points of the function g in the interval $(-2, 6)$.

(ii) Hence, or otherwise, determine the absolute maximum and the absolute minimum values of the function g.

(b) Let $h : [0, 1] \to \mathbf{R}$ be a function defined by

$$h(x) = \frac{x+1}{e^x + 1}.$$

Prove that there exists a real number c in the interval [0, 1] such that $h(c) = c$.

(c) Find $\frac{d^2y}{dx^2}$ by implicit differentiation if $y^2 + 10x = \cos(5y)$.

Question 5 [**20** marks]

Let the function f be defined on the real numbers **R** by

$$f(x) = \begin{cases} 4x^3 + 3x^2 - 6x - 1, & x \geq 1 \\ 1 - \dfrac{2x}{1+x^2}, & x < 1 \end{cases}.$$

(a) Determine if the function f is *continuous* at $x = 1$.

(b) Find the intervals on which f is (i) *increasing*, and (ii) *decreasing*.

(c) Find the *relative extrema* of f.

(d) Find the intervals on which the graph of f is (i) *concave upward* and (ii) *concave downward*.

(e) Find the *points of inflection* of the graph of f.

(f) Sketch the graph of f.

Question 6 [**20** marks]

(a) Differentiate each of the following functions.

(i) $h(x) = (\ln(e + x^2) + e^x)^{\cot(x)}$, $x > 0$.

(ii) $j(x) = \int_{-x}^{\sin(x)} \frac{t}{2+\sin(t^2)}\,dt.$

(b) Let the function f be defined on the real numbers **R** by

$$f(x) = \int_0^x e^{-t^2}\,dt$$

(i) Prove that the function f is an increasing function and that f has an inverse function f^{-1}.

(ii) Hence, or otherwise, find the derivative of f^{-1} at $x = 0$.

(iii) Show that for $x \geq 0$, $e^{x^2} \geq 1 + x^2$. Use this or otherwise show that

$$f(1) \leq \frac{\pi}{4}.$$

(iv) By considering that for $x > 1$, $f(x) = \int_0^1 e^{-t^2}\,dt + \int_1^x e^{-t^2}\,dt$ or otherwise, show that for all $x \geq 0$, $f(x) < 1$. (You may assume that $e > 2.7$ and $\pi < 3.2$.)

END OF PAPER

Answer To MA1102 Calculus

SECTION A (Compulsory)

1. The function f is defined by $f(x)=\begin{cases} x^2-25, & x<-5 \\ 3\,, & x=-5 \\ 4x+20\,, & -5<x<-1 \\ 2x^2+14, & x\geq -1 \end{cases}$.

(a) For $x<-5$, $f(x)=x^2-25>0$. Also, for $x<-5$, $x^2-25>0 \Leftrightarrow x<-5$. Thus f maps $(-\infty,-5)$ onto $(0,\infty)$. (Because for any $y>0$, we can take $x=-\sqrt{y+25}$ (<-5) so that $f(x)=y$) Also, for $-5<x<-1$, $f(x)=4x+20$. Therefore, $0<f(x)<16$. This is because $-5<x<-1 \Leftrightarrow 0<4x+20<16$. For any y with $0<y<16$ we can take $x=\dfrac{y-20}{4}$ and for this value of x, $-5<x<-1$. It follows that f maps $(-5,-1)$ onto $(0, 16)$. Now for $x\geq -1$, $f(x)=2x^2+14\geq 14$. Also for any $y\geq 14$, we can take $x=\sqrt{\dfrac{y-14}{2}}\geq 0>-1$. Therefore f maps $[-1,\infty)$ onto $[14,\infty)$. Hence the range of f is $f((-\infty,-5))\cup\{f(-5)\}\cup f((-5,-1))\cup f([-1,\infty))=(0,\infty)\cup\{3\}\cup(0,16)\cup[14,\infty)=(0,\infty)$.

(b) By part (a) Range(f) = $(0,\infty)\neq \mathbf{R}$ =codomain of f. Therefore, f is not surjective.

(c) When $x<-5$, $f(x)=x^2-25$, which is a polynomial function, therefore f is differentiable on $(-\infty,-5)$, since any polynomial function is differentiable on the real numbers and so is differentiable on any open interval. When $-5<x<-1$, $f(x)=4x+20$ and is a polynomial function and so f is differentiable on (-5, -1). Likewise f is differentiable on $(-1,\infty)$ since $f(x)=2x^2+14$, a polynomial function. Thus we can conclude that f is differentiable at x for $x\neq -5,-1$. Since differentiability implies continuity we conclude that f is continuous at x in $\mathbf{R}$ for $x\neq -5,-1$. Thus it remains to check if f is continuous at $x=-5$ or -1. Consider the left limit at $x=-5$,

$\lim_{x\to -5^-} f(x)=\lim_{x\to -5^-} x^2-25=0$ and the right limit at x = -5

$\lim_{x\to -5^+} f(x)=\lim_{x\to -5^+} 4x+20=0.$

Thus since $\lim_{x\to -5^-} f(x)=\lim_{x\to -5^+} f(x)$, $\lim_{x\to -5} f(x)=0$. But $f(-5)=3$ and so $\lim_{x\to -5} f(x)\neq f(-5)$ and thus f is not continuous at $x=-5$. Now consider the left limit of f at $x=-1$,

$\lim_{x\to(-1)^-} f(x)=\lim_{x\to(-1)^-} 4x+20=16$ and the right limit at $x=-1$,

$\lim_{x\to(-1)^+} f(x)=\lim_{x\to(-1)^+} 2x^2+14=16=f(-1).$

Thus $\lim_{x\to -1} f(x)=f(-1)$ and so f is continuous at $x=-1$.

Hence f is continuous at x for all $x\neq -5$.

(d) From part (c) since f is not continuous at $x=-5$, f is not differentiable at $x=-5$. Now it remains to check differentiability at $x=-1$.

$$\lim_{x\to -1^+}\frac{f(x)-f(-1)}{x+1}=\lim_{x\to -1^+}\frac{2x^2+14-16}{x+1}=\lim_{x\to -1^+}2(x-1)=-4$$

$$\lim_{x\to -1^-}\frac{f(x)-f(-1)}{x+1}=\lim_{x\to -1^-}\frac{4x+20-16}{x+1}=\lim_{x\to -1^-}4=4.$$

Thus f is not differentiable at $x=-1$ since $\lim_{x\to -1^-}\frac{f(x)-f(-1)}{x+1} \neq \lim_{x\to -1^+}\frac{f(x)-f(-1)}{x+1}$.
Therefore, f is differentiable at x for x not equal to -5 or -1.

(e) $\int_{-6}^{0} f(x)dx = \int_{-6}^{-5} f(x)dx + \int_{-5}^{-1} f(x)dx + \int_{-1}^{0} f(x)dx$

$$= \int_{-6}^{-5} (x^2-25))dx + \int_{-5}^{-1} (4x+20)dx + \int_{-1}^{0} (2x^2+14)dx$$
$$= \left[\frac{x^3}{3} - 25x\right]_{-6}^{-5} + \left[2x^2+20x\right]_{-5}^{-1} + \left[\frac{2x^3}{3} + 14x\right]_{-1}^{0}$$
$$= \left[\frac{6^3-5^3}{3} - 25\right] + \left[2(1-25)+20\times 4\right] + \left[\frac{2}{3}+14\right] = 52$$

2. (a) $\lim_{x\to -\infty}\frac{x^2-8|x|^3+7}{24x^3+17x+3} = \lim_{x\to -\infty}\frac{x^2+8x^3+7}{24x^3+17x+3} = \frac{1}{3}$.

(b) $\lim_{x\to 0}\frac{x^2\sin(1/x)}{\sin(x)} = \lim_{x\to 0}\frac{x}{\sin(x)}\lim_{x\to 0} x\sin(1/x) = 1\times 0 = 0$ since

$\lim_{x\to 0}\frac{x}{\sin(x)} = \lim_{x\to 0}\frac{1}{\cos(x)} = 1$ by L' Hôpital's Rule and

$\lim_{x\to 0} x\sin(1/x) = 0$ by the Squeeze Theorem because for $x \neq 0$

$-|x| \le x\sin(1/x) \le |x|$ and $\lim_{x\to 0}|x| = 0$.

(c) $\lim_{x\to \infty}\frac{x(2+\cos(x))}{x^2+1} = \lim_{x\to \infty}\frac{2/x+\cos(x)/x}{1+1/x^2} = \frac{\lim_{x\to\infty} 2/x + \lim_{x\to\infty}\cos(x)/x}{\lim_{x\to\infty} 1+1/x^2} = \frac{0+0}{1+0} = 0$

since $\lim_{x\to\infty} 2/x = 0$ and $\lim_{x\to\infty}\cos(x)/x = 0$ by the Squeeze Theorem because for $x>0$,
$-1/|x| \le \cos(x)/x \le 1/|x|$ and $\lim_{x\to\infty} 1/|x| = \lim_{x\to\infty} -1/|x| = 0$.

(d) $\lim_{x\to 3}\frac{\sqrt{16+x^2}-5}{x-3} = \lim_{x\to 3}\frac{\frac{1}{2}(16+x^2)^{-1/2}2x}{1} = \frac{3}{\sqrt{25}} = \frac{3}{5}$ by L' Hôpital's rule.

(e) $\lim_{x\to 0^+}\sin(x)\ln(\sin(x)) = \lim_{x\to 0^+}\frac{\ln(\sin(x))}{\csc(x)} = \lim_{x\to 0^+} -\frac{\cot(x)}{\csc(x)\cot(x)} = \lim_{x\to 0^+} -\sin(x) = 0$

by L' Hôpital's rule and the last equality is because $\lim_{x\to 0^+}\sin(x) = \sin(0) = 0$.

Therefore, $\lim_{x\to 0^+}(\sin(x))^{\sin(x)} = e^{\lim_{x\to\infty}\sin(x)\ln(\sin(x))} = e^0 = .1$

3. (a) $\int \frac{xe^{\frac{1}{2}x^2} - \sin(2x)}{e^{\frac{1}{2}x^2}+\cos^2(x)+2}dx = \int \frac{1}{e^{\frac{1}{2}x^2}+\cos^2(x)+2}\frac{dy}{dx}dx$,

where $y = e^{\frac{1}{2}x^2}+\cos^2(x)+2$, $\frac{dy}{dx} = xe^{\frac{1}{2}x^2} - \sin(2x)$,

$= \int \frac{1}{y}dy$ by substitution or change of variable

$= \ln|y| + C = \ln(e^{\frac{1}{2}x^2}+\cos^2(x)+2) + C$.

(b) $\int_{-1}^{3}\sqrt{7+|x|}\,dx = \int_{-1}^{0}\sqrt{7+|x|}\,dx + \int_{0}^{3}\sqrt{7+|x|}\,dx$

$$= \int_{-1}^{0}\sqrt{7-x}\,dx + \int_{0}^{3}\sqrt{7+x}\,dx$$
$$= \left[-\tfrac{2}{3}(7-x)^{\frac{3}{2}}\right]_{-1}^{0} + \left[\tfrac{2}{3}(7+x)^{\frac{3}{2}}\right]_{0}^{3}$$
$$= \frac{2}{3}(8^{\frac{3}{2}} - 7^{\frac{3}{2}}) + \frac{2}{3}(10^{\frac{3}{2}} - 7^{\frac{3}{2}})$$

$$= \frac{2}{3}(16\sqrt{2} + 10\sqrt{10} - 14\sqrt{7}) = \frac{4}{3}(8\sqrt{2} + 5\sqrt{10} - 7\sqrt{7})$$

(c) $g(x) = \begin{cases} x^3 + 7, x \geq 1 \\ 2x^4 + 6, x < 1 \end{cases}$.

Note that $\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (x^3 + 7) = 8 = g(1)$ and

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (2x^4 + 6) = 8.$$

Therefore, $\lim_{x \to 1} g(x) = g(1)$ and so g is continuous at $x = 1$. Since g is a polynomial function on the open interval (-∞, 1) and also on (1, ∞), g is continuous on these two intervals. Thus g is continuous on the whole of **R**. Therefore we can use the Fundamental Theorem of Calculus to obtain an antiderivative. By the FTC,

$G(x) = \int_1^x g(t)dt$ is an antiderivative of g(x).

Now $$G(x) = \int_1^x g(t)dt = \begin{cases} \int_1^x g(t)dt, x \geq 1 \\ \int_1^x g(t)dt, x < 1 \end{cases} = \begin{cases} \int_1^x (t^3 + 7)dt, x \geq 1 \\ \int_1^x (2t^4 + 6)dt, x < 1 \end{cases}$$

$$= \begin{cases} \left[\frac{t^4}{4} + 7t\right]_1^x, x \geq 1 \\ \left[\frac{2t^5}{5} + 6t\right]_1^x, x \geq 1 \end{cases} = \begin{cases} \frac{x^4}{4} + 7x - \frac{29}{4}, x \geq 1 \\ \frac{2x^5}{5} + 6x - \frac{32}{5}, x \geq 1 \end{cases}.$$

(Any antiderivative of g(x) of the form G(x) + C is acceptable.).

(d) $$\int \frac{\sqrt{x}}{\sqrt{\sqrt{x} + 3}} dx = \int 4x \frac{dy}{dx} dx,$$

where $y = (\sqrt{x} + 3)^{\frac{1}{2}}, \frac{dy}{dx} = \frac{1}{2\sqrt{\sqrt{x} + 3}} \frac{1}{2\sqrt{x}}$,

$= 4\int (y^2 - 3)^2 dy$ since $\sqrt{x} = y^2 - 3$,

$= 4\int (y^4 - 6y^2 + 9)dy = 4(\frac{1}{5}y^5 - 2y^3 + 9y) + C$

$= \frac{4}{5}(\sqrt{x} + 3)^{\frac{5}{2}} - 8(\sqrt{x} + 3)^{\frac{3}{2}} + 36(\sqrt{x} + 3)^{\frac{1}{2}} + C.$

(e) $\int x^3 \cos(x)dx = x^3 \sin(x) - \int 3x^2 \sin(x)dx$ by integration by parts

$= x^3 \sin(x) - 3\left(x^2(-\cos(x)) + \int 2x\cos(x)dx\right)$

by integration by parts applied to $\int x^2 \sin(x)dx$

$= x^3 \sin(x) + 3x^2 \cos(x)) - 6\int x\cos(x)dx$

$= x^3 \sin(x) + 3x^2 \cos(x)) - 6\left(x\sin(x) - \int \sin(x)dx\right)$

by integration by parts applied to $\int x\cos(x)dx$

$= x^3 \sin(x) + 3x^2 \cos(x)) - 6x\sin(x) - 6\cos(x) + C.$

Question 4.

(a) $g : [-2, 6] \to \mathbf{R}$ is defined by $g(x) = \begin{cases} x^2 + 2x - 5, & -2 \le x < 2 \\ x^3 - 12x^2 + 45x - 47, & 2 \le x \le 6 \end{cases}$.

(i) Note that g is continuous on the closed interval $[-2, 6]$. For $-2 \le x < 2$, $g(x)$ is given by the polynomial function $x^2 + 2x - 5$ and so g is continuous on $[-2, 2)$. Likewise for $2 < x \le 6$, $g(x)$ is given by the polynomial function $x^3 - 12x^2 + 45x - 47$ and so $g(x)$ is continuous on $(2, 6]$. Now the left limit of g at $x = 2$, $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} x^2 + 2x - 5 = 3$and the right limit $\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} x^3 - 12x^2 + 45x - 47 = 3 = g(2)$. Hence $\lim_{x\to 2} g(x) = g(2)$. Thus g is continuous at $x = 2$. Hence g is continuous on $[-2, 6]$. Therefore, the *Extreme Value Theorem* says that g has an absolute maximum value and an absolute minimum value on $[-2, 6]$.

Now $g'(x) = \begin{cases} 2x + 2, -2 < x < 2 \\ 3x^2 - 24x + 45, 2 < x < 6 \end{cases}$.

$= \begin{cases} 2(x+1), -2 < x < 2 \\ 3(x^2 - 8x + 15), 2 < x < 6 \end{cases} = \begin{cases} 2(x+1), -2 < x < 2 \\ 3(x-3)(x-5), 2 < x < 6 \end{cases}$ --- (1)

Note that g is not differentiable at $x = 2$. This is seen as follows:

$\lim_{x\to 2^-} \frac{g(x) - g(2)}{x-2} = \lim_{x\to 2^-} \frac{g'(x)}{1} = \lim_{x\to 2^-} 2(x+1) = 6$by L' Hôpital's Rule, and

$\lim_{x\to 2^+} \frac{g(x) - g(2)}{x-2} = \lim_{x\to 2^+} \frac{g'(x)}{1} = \lim_{x\to 2^+} 3(x-3)(x-5) = 9$ and so g is not differentiable at x $=2$ since $\lim_{x\to 2^-} \frac{g(x) - g(2)}{x-2} \ne \lim_{x\to 2^+} \frac{g(x) - g(2)}{x-2}$. From (1), $g'(x) = 0$ in $(-2, 2)$ if and only if $x = -1$ and $g'(x) = 0$ in $(2, 6)$ if and only if $x = 3$ or 5. Hence the critical points in $(-2, 6)$ are -1, 2, 3 and 5.

(ii) Now $g(-2) = -5$, $g(-1) = -6$, $g(2) = 3$ and $g(3) = 7$, $g(5) = 3$ and $g(6) = 7$. Therefore, since g is continuous on $[-2, 6]$, the absolute maximum value of g is 7 and the absolute minimum value of g is -6.

(b) $h : [0, 1] \to \mathbf{R}$ is defined by $h(x) = \frac{x+1}{e^x + 1}$. Then h is continuous on $[0, 1]$.
Define a function $k : [0, 1] \to \mathbf{R}$ by $k(x) = h(x) - x$. Then *plainly*, k is a continuous function on the closed and bounded interval $[0, 1]$.
$k(x) = \frac{x+1}{e^x + 1} - x = \frac{1 - xe^x}{e^x + 1}$. $k(0) = \frac{1}{e^0 + 1} = \frac{1}{2} > 0$ and $k(1) = \frac{1-e}{e+1} < 0$ since e > 1.
Therefore, by the *Intermediate Value Theorem*, there exists a point c in $(0, 1)$ such that $k(c) = 0$, i.e., $h(c) = c$.

Alternative solution:

For any *real number* x, $e^x > x$. *(Why? Obviously,* $x \le 0$ implies $e^x > x$, and for $x > 0$ $e^x > x$ if and only if $x > \ln(x) = \int_1^x \frac{1}{t}\,dt$ which is obviously true.)
Therefore $e^x + 1 > x + 1$ and so
$0 < h(x) = \frac{x+1}{e^x + 1} < 1$ for $0 \le x \le 1$. ---------------------- (1)
Hence h maps $[0, 1]$ into $[0, 1]$.

Define a function $k:[0, 1] \to \mathbf{R}$ by $k(x) = h(x) - x$. Then *plainly*, k is a continuous function on the closed and bounded interval [0, 1].
In particular $k(1) = h(1) - 1 < 0$ by (1) and $k(0) = h(0) > 0$ also by (1). Therefore, by the Intermediate Value Theorem, there exists a point c in (0, 1) such that $k(c) = 0$, i.e., $h(c) = c$.
[Actually, you need only show $h(0) > 0$ and $h(1) < 1$. Obviously, $h(0) = \frac{1}{e^0+1} = \frac{1}{2} > 0$ and $h(1) = \frac{1+1}{e^1+1} = \frac{2}{e+1} < \frac{2}{2} = 1$.]

(c) $y^2 + 10x = \cos(5y)$ -------------------------------- (2)

Differentiating (2) implicitly,

$$2y\frac{dy}{dx} + 10 = -5\sin(5y)\frac{dy}{dx}.$$

Hence $(2y + 5\sin(5y))\frac{dy}{dx} = -10$ ------------------------ (3)

Differentiating (3) implicitly, we get

$$(2 + 25\cos(5y))\left(\frac{dy}{dx}\right)^2 + (2y + 5\sin(5y))\frac{d^2y}{dx^2} = 0.$$

Hence $+\frac{d^2y}{dx^2} = -\frac{2+25\cos(5y)}{2y+5\sin(5y)}\left(\frac{dy}{dx}\right)^2 = -100\frac{2+25\cos(5y)}{(2y+5\sin(5y))^3}$.

5.. $f(x) = \begin{cases} 4x^3 + 3x^2 - 6x - 1, x \ge 1 \\ 1 - \frac{2x}{1+x^2}, x < 1 \end{cases}$.

(a) f is continuous at $x = 1$ if and only if $\lim_{x\to 1} f(x) = f(1)$.

Now $f(1) = 4 + 3 - 6 - 1 = 0$, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1 - \frac{2x}{1+x^2} = 1 - 1 = 0$ and

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 4x^3 + 3x^2 - 6x - 1 = 4 + 3 - 6 - 1 = 0$ Thus $\lim_{x\to 1} f(x) = f(1)$ and so f is continuous at $x = 1$.

(b) $f'(x) = \begin{cases} 12x^2 + 6x - 6, x > 1 \\ -\frac{2(1+x^2) - 2x(2x)}{(1+x^2)^2}, x < 1 \end{cases} = \begin{cases} 6(2x^2 + x - 1), x > 1 \\ 2\frac{x^2-1}{(1+x^2)^2}, x < 1 \end{cases}$

$= \begin{cases} 6(2x-1)(x+1), x > 1 \\ 2\frac{(x-1)(x+1)}{(1+x^2)^2}, x < 1 \end{cases}$ -------------------- (1)

From (1) $f'(x) = 0$ in $(-\infty, 1)$ if and only if $x = -1$ and $f'(x) \neq 0$ for x in $(1, \infty)$.

From (1), for $x < -1$, $f'(x) > 0$ (since $(x - 1) < 0$ and $(x + 1) < 0$)and so f is increasing on $(-\infty, -1]$. Also from (1), for $-1 < x < 1$, $f'(x) < 0$ (since $(x - 1) < 0$ and $(x + 1) > 0$) and so f is decreasing on the interval $[-1, 1]$. The end points are included by continuity. Finally from (1), $f'(x) > 0$ in $(1, \infty)$ and so f is increasing on the interval $[1, \infty)$.

(c) From part (b) or by the 1st derivative test,

$f(-1) = 2$ is a relative maximum of f and $f(1) = 0$ is a relative minimum of f.

(d) $$f''(x)=\begin{cases} 24x+6, x>1 \\ \dfrac{4x(3-x^2)}{(1+x^2)^3}, x<1 \end{cases}$$

$$=\begin{cases} 24x+6, x>1 \\ -\dfrac{4x(x-\sqrt{3})(x+\sqrt{3})}{(1+x^2)^3}, x<1 \end{cases} \qquad \text{------------------- ----------} \ (2)$$

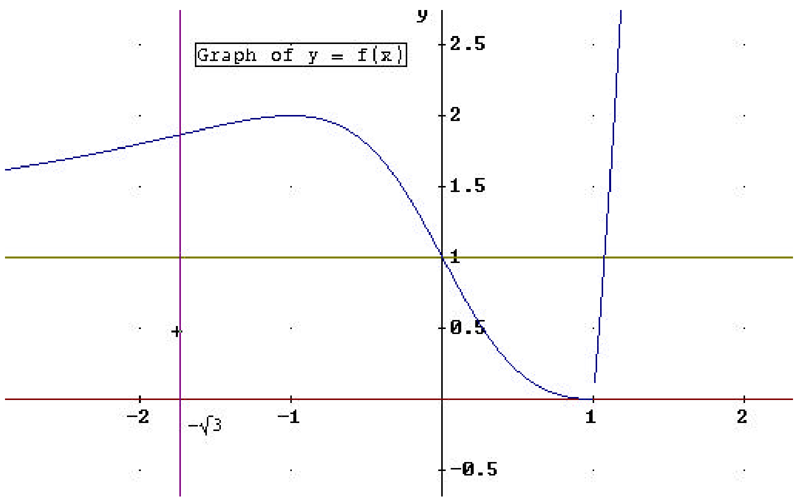
Note that f is not differentiable at $x = 1$. This is deduced as follows.

$\lim\limits_{x\to 1^+}\dfrac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1^+}\dfrac{f'(x)}{1} = \lim\limits_{x\to 1^+} 6(2x-1)(x+1) = 12$ by L' Hôpital's Rule, and $\lim\limits_{x\to 1^-}\dfrac{f(x)-f(2)}{x-1} = \lim\limits_{x\to 1^-}\dfrac{f'(x)}{1} = \lim\limits_{x\to 1^-} 2\dfrac{x^2-1}{(1+x^2)^2} = 0$ and so f is not differentiable at $x = 1$ since $\lim\limits_{x\to 1^-}\dfrac{f(x)-f(1)}{x-1} \neq \lim\limits_{x\to 1^+}\dfrac{f(x)-f(1)}{x-1}$.

From (2) for $x < -\sqrt{3}$, $f''(x) > 0$ since $x < 0$, $x < \sqrt{3}$ and $x < -\sqrt{3}$ and so the graph of f is concave upward on the interval $(-\infty, -\sqrt{3})$. For $-\sqrt{3} < x < 0$, $x + \sqrt{3} > 0$, $x < 0$ and $x - \sqrt{3} < 0$ and so $f''(x) = -\dfrac{4x(x-\sqrt{3})(x+\sqrt{3})}{(1+x^2)^3} < 0$ and therefore the graph of f is concave downward on the interval $(-\sqrt{3}, 0)$. For $0 < x < 1$, $x + \sqrt{3} > 0$, $x > 0$ and $x - \sqrt{3} < 0$ and so $f''(x) = -\dfrac{4x(x-\sqrt{3})(x+\sqrt{3})}{(1+x^2)^3} > 0$ and the graph of f is concave upward on the interval $(0, 1)$. Finally for $x > 1$, from (2), $f''(x) = 24x + 6 > 0$ and so the graph of f is concave upward on the interval $(1, \infty)$.

(e) From part (d), there is a change of concavity of the graph of f before and after the points $x = -\sqrt{3}$ and $x = 0$. Therefore the points of inflection of the graph of f are $(-\sqrt{3}, f(-\sqrt{3})) = (-\sqrt{3}, 1 + \dfrac{\sqrt{3}}{2})$ and $(0, f(0)) = (0, 1)$.

(f) The graph of f :



6. (a)

(i) $h(x) = \left(\ln(e+x^2)+e^x\right)^{\cot(x)}$

Then $\ln(h(x)) = \cot(x)\ln(\ln(e+x^2)+e^x)$ ---------------- (1)

Differentiating on both sides, we obtain

$$\frac{1}{h(x)}h'(x) = -\csc^2(x)\ln(\ln(e+x^2)+e^x)+\cot(x)\frac{1}{\ln(e+x^2)+e^x}\left(\frac{2x}{e+x^2}+e^x\right)$$

Therefore, $\dfrac{h'(x)}{h(x)} = \dfrac{\cot(x)(2x+e^{x+1}+x^2e^x)}{(e+x^2)(\ln(e+x^2)+e^x)} - \dfrac{\ln(\ln(e+x^2)+e^x)}{\sin^2(x)}$.

$$h'(x) = \left(\frac{\cot(x)(2x+e^{x+1}+x^2e^x)}{(e+x^2)(\ln(e+x^2)+e^x)} - \frac{\ln(\ln(e+x^2)+e^x)}{\sin^2(x)}\right)(\ln(e+x^2)+e^x)^{\cot(x)}.$$

.

(ii) $j(x) = \int_{-x}^{\sin(x)} \dfrac{t}{2+\sin(t^2)}dt = \int_0^{\sin(x)} \dfrac{t}{2+\sin(t^2)}dt - \int_0^{-x} \dfrac{t}{2+\sin(t^2)}dt$

$= F(\sin(x)) - F(-x)$, where $F(x) = \int_0^x \dfrac{t}{2+\sin(t^2)}dt$

But, by the *Fundamental Theorem of Calculus,*

$$F'(x) = \frac{x}{2+\sin(x^2)}.$$

Therefore, $j'(x) = F'(\sin(x))\cos(x) - F'(-x)(-1)$ by the *Chain Rule*

$$= \frac{\sin(x)\cos(x)}{2+\sin(\sin^2(x))} - \frac{x}{2+\sin(x^2)}.$$

(b) $f(x) = \int_0^x e^{-t^2}dt$

(i) By the Fundamental Theorem of Calculus,

$$f'(x) = e^{-x^2} > 0 \text{ ------------------ (1)}$$

for all x in **R**.

Therefore, f is an increasing function on **R** and hence is an injective function

Consequently, f has an inverse function f^{-1} with domain $f(\mathbf{R})$.

(ii) By (1) $f'(x) \neq 0$. Therefore,

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \text{ --------------- (2)}$$

Now since $f(0) = \int_0^0 e^{-t^2}dt = 0$ and f is injective, $f^{-1}(0) = 0$ and so by (2) and (1)

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(0)} = \frac{1}{e^0} = 1 .$$

(iii) Consider the function $g(x) = e^{x^2} - (1+x^2)$. Then g is differentiable on **R** and $g'(x) = 2x\,e^{x^2} - 2x = 2x(e^{x^2}-1)$. Since $x^2 > 0$ implies that $e^{x^2} > e^0 = 1$,

$e^{x^2} - 1 > 0$ for $x \neq 0$.

Therefore, for $x > 0$, $g'(x) = 2x(e^{x^2}-1) > 0$. Hence g is increasing on the interval $[0, \infty)$ since g is continuous at 0 as well. Thus for any $x > 0$,

$g(x) > g(0) = e^0 - (1+0) = 0$.

That means for $x \geq 0$, $e^{x^2} \geq (1+x^2)$.

Alternative solution:

$$e^{x^2} \geq (1+x^2) \Leftrightarrow x^2 \geq \ln(1+x^2) = \int_1^{1+x^2} \frac{1}{t}dt \text{ ------------ (3)}$$

Now for $t \geq 1$, $\dfrac{1}{t} \leq 1$ so that $\int_1^{1+x^2} \dfrac{1}{t}dt \leq \int_1^{1+x^2} 1dt = x^2$. Thus $e^{x^2} \geq (1+x^2)$.

Therefore, $\frac{1}{e^{x^2}} \le \frac{1}{1+x^2}$. Thus for any $x \ge 0$, $\int_0^x \frac{1}{e^{t^2}}dt \le \int_0^x \frac{1}{1+t^2}dt = \tan^{-1}(x)$.
Hence $f(1) = \int_0^1 \frac{1}{e^{t^2}}dt \le \tan^{-1}(1) = \frac{\pi}{4}$.

(iv) Since f is an increasing function (by (i)), for $x \le 1$, $f(x) \le f(1) = \pi/4 < 1$.

Now for $x > 1$, $f(x) = \int_0^1 e^{-t^2}dt + \int_1^x e^{-t^2}dt = f(1) + \int_1^x e^{-t^2}dt \le \frac{\pi}{4} + \int_1^x e^{-t^2}dt$ ---- (4)
Now for $t \ge 1, e^{-t^2} \le t\,e^{-t^2}$ and so for $x > 1$,
$\int_1^x e^{-t^2}dt \le \int_1^x te^{-t^2}dt = \frac{1}{2}\int_1^x e^{-t^2}2tdt = \frac{1}{2}\int_1^x e^{-t^2}\frac{du}{dt}dt$, where $u = t^2$
$\le \frac{1}{2}\int_1^{x^2} e^{-u}du = \frac{1}{2}[-e^{-u}]_1^{x^2} = \frac{1}{2}(\frac{1}{e} - \frac{1}{e^{x^2}}) < \frac{1}{2e}$.
Therefore, by (4) for any $x > 1$,

$$f(x) \le \frac{\pi}{4} + \frac{1}{2e} < 0.8 + 0.19 < 1.$$

Hence for *any x*, $f(x) < 1$.