NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1 EXAMINATION 2000 – 2001

MA1102 CALCULUS

October/November 2000 – Time Allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SIX (6)** questions and comprises **FIVE (5)** printed pages.

2. Answer **ALL** questions in **Section A** The marks for questions in Section A are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Answer not more than **TWO (2)** questions from Section B. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

*Answer **ALL** questions in this section.*

Question 1 [**20** marks]

Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} 2x^3 + 1, & x < -1 \\ x^6 \sin\left(\dfrac{\pi}{2x^3}\right), & -1 \le x \le 1 \text{ and } x \ne 0 \\ x^2 + 1, & x > 1 \\ 0, & x = 0 \end{cases}.$$

(a) Find the *range* of the function f.

(b) Determine if f is surjective.

(c) Determine all x in $\mathbf{R}$ at which the function f is *continuous.*

(d) Is the function f *differentiable* at $x = -1$? Justify your answer.

(e) Is the function f *differentiable* at $x = 0$? Is it also *twice differentiable* at $x = 0$? Justify your answer.

Question 2 [**20** marks]

Evaluate, if it exists, each of the following limits.

(a) $\displaystyle\lim_{x \to -\infty} \frac{x^2 - 7|x^3| + 10000}{51x^3 + x + 100}.$

(b) $\displaystyle\lim_{x \to 8} \frac{\sqrt{14 + \sqrt[3]{x}} - 4}{x - 8}.$

(c) $\displaystyle\lim_{x \to 0} \frac{1 - \cos(x) - 2\sin^2(x)}{x}.$

(d) $\displaystyle\lim_{x \to \infty} \left(\sqrt{1 + \frac{1}{2}x + x^2} - \sqrt{1 - \frac{1}{2}x + x^2} \right).$

(e) $\displaystyle\lim_{x \to \infty} (\ln(\ln(x)))^{\frac{1}{x}}.$

Question 3 [**20** marks]

(a) Evaluate $\int \frac{1}{(x^2+x+1)(x^2+x+2)}dx$.

(b) Determine the derivative of $\sin^{-1}(a\,x)$, where a is a non zero constant in the interval (-1, 1).

Use this or otherwise, evaluate $\int \sin^{-1}(29x)dx$.

(c) Evaluate $\int_0^2 [x^2]dx$, where $[t]$ denotes the greatest integer less than or equal to t.

(d) Find an antiderivative of $g(x)$, which is defined by

$$g(x) = \begin{cases} 5x^4+1, & x \ge 1 \\ x^2+5, & x < 1 \end{cases}.$$

SECTION B

Answer not more than ***TWO (2)*** *questions from this section. Each question in this section carries 20 marks.*

Question 4 [**20** marks]

(a) Find the critical points of the function g, defined by

$$g(x) = \begin{cases} x^2-2x+1, & 0 \le x \le 3 \\ 157-x^3+18x^2-96x, & 3 < x \le 9 \end{cases},$$

in the interval (0, 9). Determine the absolute maximum and the absolute minimum values of the function in the interval [0, 9].

(b) Show that

$$\frac{1}{65}\ln(2) \le \int_{\ln(2)}^{2\ln(2)} \frac{1}{e^{3x}+1}\,dx \le \frac{1}{9}\ln(2).$$

(c) Suppose f is a continuous function defined on the closed interval $[0, 1]$ such that $f(0) = f(1)$. Prove that there exists a point c in $\left[\frac{1}{5}, 1\right]$ such that $f(c) = f\left(\frac{1}{4}(c-\frac{1}{5})\right)$. Hence or otherwise deduce that

there exists a point c in $\left[\frac{1}{5}, 1\right]$ such that

$$\sin(\pi c) = \sin\left(\frac{\pi c}{4} - \frac{\pi}{20}\right).$$

Question 5 [**20** marks]

Let the function f be defined on $\mathbf{R}$ by

$$f(x) = \begin{cases} \dfrac{2x}{x+1}, & x \ge 1 \\ 2x^3 - 3x + 2, & x < 1 \end{cases}.$$

(a) Determine if the function f is *continuous* at $x = 1$.

(b) Find the intervals on which f is (i) *increasing*, and (ii) *decreasing*.

(c) Find the *relative extrema* of f.

(d) Find the intervals on which the graph of f is *concave upward* or *concave downward*.

(e) Find the *points of inflection* of the graph of f.

(f) Sketch the graph of f.

Question 6 [**20** marks]

(a) Differentiate the following functions.

(i) $h(x) = (\tan(x))^{\left(\frac{1}{x^3}\right)}$, $x > 0$.

(ii) $j(x)=\int_{\sin(x)}^{\cos(x^2)} \frac{1}{2+t^2+\cos(t)}dt.$

(b) Suppose that f is a differentiable function with the property that

(1) $f(x+y)=f(x)+f(y)+7xy$ and

(2) $\lim_{h\to 0}\frac{f(h)}{h}=4.$

Find $f(0)$ and $f'(x)$.

(c) Find the following limit $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^3}{n^4}+\frac{\sqrt{i}}{n\sqrt{n}}\right).$

END OF PAPER

Answer To MA1102 Calculus

SECTION A (Compulsory)

1. The function f is defined by $f(x) = \begin{cases} 2x^3+1, & x<-1 \\ x^6\sin(\frac{\pi}{2x^3}), & -1\le x\le 1 \text{ and } x\ne 0 \\ x^2+1, & x>1 \\ 0, & x=0 \end{cases}$.

(a) For $x<-1$, $f(x)=2x^3+1<-1$. Also, for $x<-1$, $2x^3+1<-1 \Leftrightarrow x<-1$. Thus f maps $(-\infty,-1)$ onto $(-\infty,-1)$. (Because for any $y<-1$, we can take $x=\sqrt[3]{\frac{y-1}{2}}$ (<-1) so that $f(x)=y$) Also, for $-1\le x\le 1$, $-1\le f(x)\le 1$. This is seen as follows.
For $-1\le x\le 1$ and $x\ne 0$, $|f(x)| = \left|x^6 sin(\frac{\pi}{2x^3})\right| \le |x|^6 \le 1$. Now $f(0)=0$. Thus $-1\le f(x)\le 1$. Therefore, $f(-1) = -1$ is the absolute minimum of f on $[-1, 1]$ and $f(1) = 1$ is the absolute maximum of f on $[-1, 1]$. Assuming that f is continuous on $[-1, 1]$ (as we shall show in part (d) below), by the *Intermediate Value Theorem*, f maps the interval $[-1,1]$ onto $[-1,1]$.

One alternative answer for deducing this: Examine the behaviour of the sine function. Note that $x^6\sin(\frac{\pi}{2x^3})$ is continuous on the interval $[1/\sqrt[3]{2}, 1]$. The interval $[1/\sqrt[3]{2}, 1]$ is mappped in a one-one way onto $[\pi/2, \pi]$ under the function $\pi/(2x^3)$. Now the derivative of $x^6\sin(\frac{\pi}{2x^3})$ is $6x^5\sin(\frac{\pi}{2x^3}) - x^2\frac{3\pi}{2}\cos(\frac{\pi}{2x^3})$ for x in $[1/\sqrt[3]{2}, 1]$ and is positive on $(1/\sqrt[3]{2}, 1)$ since $6x^5\sin(\frac{\pi}{2x^3})$ and $-x^2\frac{3\pi}{2}\cos(\frac{\pi}{2x^3})$ are positive there. Thus f is increasing on $[1/\sqrt[3]{2}, 1]$ and the image of $[1/\sqrt[3]{2}, 1]$ under the function $x^6\sin(\frac{\pi}{2x^3})$ and so under f is $[1/4\sin(\pi), \sin(\pi/2)] = [0, 1]$. Similarly we can deduce that $x^6\sin(\frac{\pi}{2x^3})$ and (therefore) f maps $[-1, -1/\sqrt[3]{2}]$ onto $[-1,0]$. Since for $-1\le x\le 1$ and $x\ne 0$, $|f(x)| = \left|x^6 sin(\frac{\pi}{2x^3})\right| \le |x|^6 \le 1$ and $f(0)=0$, the above argument says that f maps the interval $[-1,1]$ onto $[-1,1]$.] (All the while we are assuming the continuity of $x^6\sin(\frac{\pi}{2x^3})$ on the respective intervals.)

(There is a distinction between saying f is continuous on a non trivial interval $[a, b]$ and continuity at a point. The function f is continuous on $[a, b]$ means that f is continuous at each point of the open interval (a, b), the right limit at $x = a$, $\lim_{x\to a^+} f(x)$ is equal to $f(a)$ and the left limit at $x = b$, $\lim_{x\to b^-} f(x)$ is equal to $f(b)$. It does not imply that f is continuous at a or at b. In fact it need not be. The left limit at a, $\lim_{x\to a^-} f(x)$ may not exist and when it does, it may not be equal to $f(a)$. The same thing can be said about the right limit at b, it need not exist or equal to $f(b)$. For our function f, f is not continuous at $x = 1$; but it is continuous on $[-1, 1]$.)

Finally for $x>1$, $f(x)=x^2+1>2$. And for any $y>2$, we can take $x=\sqrt{y-1} >1$ so that $f(x)=y$. Hence f maps $(1,\infty)$ onto $(2,\infty)$. Hence the range of f is $(2,\infty)\cup[-1,1]\cup(-\infty,-1)=(-\infty,1]\cup(2,\infty)$.

(b) By part (a) Range(f) = $(-\infty,1]\cup(2,\infty) \ne \mathbf{R}$ = codomain(f), therefore f is not surjective.

(c) When $x < -1$, $f(x) = 2x^3 + 1$, which is a polynomial function, therefore f is continuous on $(-\infty, -1)$, since any polynomial function is continuous on the real numbers and so is continuous on any open interval. When $-1 < x < 1$ and $x \neq 0$, $f(x) = x^6 \sin(\frac{\pi}{2x^3})$ and since $x^6 \sin(\frac{\pi}{2x^3})$ is continuous on (–1, 0) and on (0, 1), f is continuous on the union of these two intervals. Finally when $x > 1$, $f(x)$ is a polynomial function and so it is continuous for $x > 1$. Thus it remains to check if f is continuous at $x = -1$, 0 or 1. Consider the left limit at $x = 1$,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^6 \sin(\frac{\pi}{2x^3}) = 1^6 \sin(\frac{\pi}{2}) = 1 \text{ and the right limit at } x = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 + 1 = 2.$$

Thus $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$ and so the $\lim_{x\to 1} f(x)$ does not exist and f is not continuous at $x =$ 1.

Now consider the left limit of f at $x = -1$,

$$\lim_{x\to(-1)^-} f(x) = \lim_{x\to(-1)^-} 2x^3 + 1 = -1 \text{ and the right limit at } x = -1,$$

$$\lim_{x\to(-1)^+} f(x) = \lim_{x\to(-1)^+} x^6 \sin(\frac{\pi}{2x^3}) = 1\sin(-\tfrac{\pi}{2}) = -1 = f(-1).$$

Thus $\lim_{x\to 1} f(x) = f(-1)$ and so f is continuous at $x = -1$.

Now $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^6 \sin(\frac{\pi}{2x^3}) = 0$ by the Squeeze Theorem and $f(0) = 0$. Therefore, f is continuous at $x = 0$. Hence f is continuous at x for all $x \neq 1$.

(d) f is differentiable at $x = -1$. This is seen as follows.

$$\lim_{x\to -1^+} \frac{f(x) - f(-1)}{x+1} = \lim_{x\to -1^+} \frac{x^6 \sin(\frac{\pi}{2x^3}) + 1}{x+1} = \lim_{x\to -1^+} \frac{6x^5 \sin(\frac{\pi}{2x^3}) - x^2 \frac{3\pi}{2} \cos(\frac{\pi}{2x^3})}{1}$$

by L' Hôpital's Rule

$$= 6.$$

$$\lim_{x\to -1^-} \frac{f(x) - f(-1)}{x+1} = \lim_{x\to -1^-} \frac{2x^3 + 2}{x+1} = \lim_{x\to -1^-} 6x^2 = 6.$$

Therefore $\lim_{x\to -1^+} \frac{f(x) - f(-1)}{x+1} = \lim_{x\to -1^-} \frac{f(x) - f(-1)}{x+1}$. Thus f is differentiable at $x = -1$ and $f'(-1) = 6$.

(e) $\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^6 \sin(\frac{\pi}{2x^3})}{x} = \lim_{x\to 0} x^5 \sin(\frac{\pi}{2x^3}) = 0$ by the *Squeeze Theorem* since $-|x|^5 \leq x^5 \sin(\frac{\pi}{2x^3}) \leq |x|^5$ for $x \neq 0$ and $\lim_{x\to 0} |x|^5 = 0$. Therefore f is differentiable at $x = 0$ and $f'(0) = 0$.

Now for x in (-1, 1)-{0}, $f'(x) = 6x^5 \sin(\frac{\pi}{2x^3}) - x^2 \frac{3\pi}{2} \cos(\frac{\pi}{2x^3})$

Thus

$$\lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} \frac{6x^5 \sin(\frac{\pi}{2x^3}) - x^2 \frac{3\pi}{2} \cos(\frac{\pi}{2x^3})}{x}$$

$$= \lim_{x\to 0} 6x^4 \sin(\frac{\pi}{2x^3}) - x\frac{3\pi}{2} \cos(\frac{\pi}{2x^3}) = 0$$

since $\lim_{x\to 0} 6x^4 \sin(\frac{\pi}{2x^3}) = 0$ and $\lim_{x\to 0} x\frac{3\pi}{2} \cos(\frac{\pi}{2x^3}) = 0$ by the *Squeeze Theorem*.

Therefore f is twice differentiable at $x = 0$ and $f''(0) = 0$.

2. (a) $\lim_{x\to-\infty} \frac{x^2 - 7|x|^3 + 10000}{51x^3 + x + 100} = \lim_{x\to-\infty} \frac{x^2 + 7x^3 + 10000}{51x^3 + x + 100} = \frac{7}{51}$.

(b) $\lim_{x\to 8} \frac{\sqrt{14+\sqrt[3]{x}} - 4}{x-8} = \lim_{x\to 8} \frac{\frac{1}{2}(14+\sqrt[3]{x})^{-\frac{1}{2}} \frac{1}{3} x^{-\frac{2}{3}}}{1}$ by L' Hôpital's Rule

$$= \frac{1}{6\cdot 4\cdot 4} = \frac{1}{96}$$

(c) $\lim_{x\to 0} \frac{1-\cos(x) - 2\sin^2(x)}{x} = \lim_{x\to 0} \frac{\sin(x) - 4\sin(x)\cos(x)}{1} = 0$
by L' Hôpital's rule.

(d) $\lim_{x\to\infty} \left(\sqrt{1+\frac{1}{2}x+x^2} - \sqrt{1-\frac{1}{2}x+x^2}\right) = \lim_{x\to\infty} \frac{(1+\frac{1}{2}x+x^2) - (1-\frac{1}{2}x+x^2)}{\left(\sqrt{1+\frac{1}{2}x+x^2} + \sqrt{1-\frac{1}{2}x+x^2}\right)}$

$$= \lim_{x\to\infty} \frac{x}{\left(\sqrt{1+\frac{1}{2}x+x^2} + \sqrt{1-\frac{1}{2}x+x^2}\right)} = \lim_{x\to\infty} \frac{1}{\left(\sqrt{\frac{1}{x^2}+\frac{1}{2x}+1} + \sqrt{\frac{1}{x^2}-\frac{1}{2x}+1}\right)} = \frac{1}{2}$$

(e) $\lim_{x\to\infty} \frac{\ln(\ln(\ln(x)))}{x} = \lim_{x\to\infty} \frac{1}{\ln(\ln(x))} \frac{1}{x\ln(x)} = 0$ by L' Hôpital's rule and the last equality is a consequence of the fact that $\lim_{x\to\infty} x\ln(x)\ln(\ln(x)) = \infty$.

Therefore, $\lim_{x\to\infty} (\ln(\ln(x)))^{\frac{1}{x}} = e^{\lim_{x\to\infty} \frac{\ln(\ln(\ln(x)))}{x}} = e^0 = 1$.

3. (a) $\int \frac{dx}{(x^2+x+1)(x^2+x+2)} = \int (\frac{1}{x^2+x+1} - \frac{1}{x^2+x+2})dx$

$= \int (\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}} - \frac{1}{(x+\frac{1}{2})^2+\frac{7}{4}})dx = \frac{2}{\sqrt{3}}\tan^{-1}\left(.\frac{2x+1}{\sqrt{3}}\right) - \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{2x+1}{\sqrt{7}}\right) + C$

(b) Use the formula $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$.

Therefore $\frac{d}{dx}\sin^{-1}(ax) = a\frac{d}{dy}\sin^{-1}(y)|_{y=ax} = a\frac{1}{\cos(\sin^{-1}(y))} = \frac{a}{\sqrt{1-a^2x^2}}$

$\int \sin^{-1}(29x)dx = x\sin^{-1}(29x) - \int x\frac{29}{\sqrt{1-29^2x^2}}dx = x\sin^{-1}(29x) + \frac{1}{58}\int \frac{-2\cdot 29^2 x}{\sqrt{1-29^2x^2}}dx$

$= x\sin^{-1}(29x) + \frac{1}{29}\sqrt{1-29^2x^2} + C.$

(c) $\int_0^2 [x^2]dx = \int_0^1 [x^2]dx + \int_1^{\sqrt{2}} [x^2]dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2]dx + \int_{\sqrt{3}}^2 [x^2]dx$

$= \int_0^1 0dx + \int_1^{\sqrt{2}} 1dx + \int_{\sqrt{2}}^{\sqrt{3}} 2dx + \int_{\sqrt{3}}^2 3dx$

$= 0 + (\sqrt{2}-1) + 2(\sqrt{3}-\sqrt{2}) + 3(2-\sqrt{3}) = 5-\sqrt{2}-\sqrt{3}$

(d) Note $g(x)$ is defined by $g(x) = \begin{cases} 5x^4+1, & x \ge 1 \\ x^2+5, & x<1 \end{cases}$. We claim that g is a continuous function. For $x<1$, $g(x)$ is given by the polynomial function x^2+5, which we know is continuous and so g is continuous at x for all $x<1$. Similarly for $x>1$, $g(x)$ is given by the polynomial function $5x^4+1$, and so g is continuous at x for all $x>1$. Now $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} x^2+5 = 6$, $\lim_{x\to 1^+} g(x) = \lim_{x\to 1^-} 5x^4+1 = 6$, and $g(1)=6$ and so $\lim_{x\to 1} g(x) = g(1)$ and g is continuous at $x=1$. Hence g is a continuous function. So we can use the Fundamental Theorem of Calculus to obtain the antiderivative.

For $x \le 1$, $\int_0^x g(t)dt = \int_0^x (t^2+5)dt = \left[\frac{t^3}{3}+5t\right]_0^x = \frac{x^3}{3}+5x$ for $x \le 1$

and for $x>1$, $\int_0^x g(t)dt = \int_0^1 g(t)dt + \int_1^x g(t)dt$

$= \left[\frac{t^3}{3}+5t\right]_0^1 + \int_1^x (5t^4+1)dt = 5\frac{1}{3} + [t^5+t]_1^x = x^5+x+\frac{10}{3}.$

Therefore, any antiderivative of g is given by $h(x)+C$ where C is a constant and

$h(x) = \begin{cases} \frac{x^3}{3}+5x, & x \le 1 \\ x^5+x+\frac{10}{3}, & x>1 \end{cases}$.

Question 4.

(a) Note that g is continuous on the closed interval [0, 9]. For $x < 3$ $g(x)$ is given by the polynomial function $x^2 - 2x+1$ and so g is continuous for $x < 3$. Likewise for $x > 3$ $g(x)$ is given by the polynomial function $157 - x^3 + 18x^2 - 96x$ and so $g(x)$ is continuous for $x > 3$. Now the left limit of g at $x = 3$, $\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-} x^2 - 2x + 1 = 4 = g(3)$ and the right limit $\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+} 157 - x^3 + 18x^2 - 96x = 4$. Hence $\lim_{x\to 3} g(x) = g(3)$. Thus g is continuous at $x = 3$. Hence g is continuous on [0, 9]. Therefore the *Extreme value Theorem* says that g has an absolute maximum value and an absolute minimum value on [0, 9].

Now $$g'(x) = \begin{cases} 2x-2, 0<x<3 \\ -3x^2+36x-96, 3<x<9 \end{cases}.$$
$$= \begin{cases} 2x-2, 0<x<3 \\ -3(x^2-12x+32), 3<x<9 \end{cases} = \begin{cases} 2(x-1), 0<x<3 \\ -3(x-4)(x-8), 3<x<9 \end{cases} \quad \text{--- (1)}$$

Note that g is not differentiable at $x = 3$. This is seen as follows:

$\lim_{x\to 3^-} \frac{g(x)-g(3)}{x-3} = \lim_{x\to 3^-} \frac{g'(x)}{1} = \lim_{x\to 3^-} 2(x-1) = 4$ by L' Hôpital's Rule, and

$\lim_{x\to 3^+} \frac{g(x)-g(3)}{x-3} = \lim_{x\to 3^+} \frac{g'(x)}{1} = \lim_{x\to 3^+} -3(x-4)(x-8) = -15$ and so g is not differentiable at $x = 3$. From (1) $g'(x) = 0$ in (0, 9) if and only if $x = 1$, 4 or 8. Hence the critical points are 1,3, 4 and 8. Now $g(0) = 1$, $g(1) = 0$, $g(3) = 4$ and $g(4) = -3$, $g(8) = 29$ and $g(9) = 22$. Therefore the absolute maximum value of g is 29 and the absolute minimum value of g is -3.

(b) By the *Mean Value Theorem for Integral*,

$$\frac{\int_{\ln(2)}^{2\ln(2)} \frac{1}{1+e^{3x}}dx}{2\ln(2)-\ln(2)} = \frac{1}{1+e^{3c}} \quad \text{--------------------------- (2)}$$

for some c in the interval [ln(2), 2ln(2)].

Since $1 + e^{3x}$ is an increasing function, $\frac{1}{1+e^{3x}}$ is a decreasing function. Therefore,

$\frac{1}{1+e^{3(2\ln(2))}} \le \frac{1}{1+e^{3c}} \le \frac{1}{1+e^{3\ln(2)}}$ and since $e^{3(2\ln(2))} = 2^6 = 64$ and $e^{3\ln(2)} = 2^3 = 8$,

$\frac{1}{1+64} \le \frac{1}{1+e^{3c}} \le \frac{1}{1+8}$. Thus from (2) $\frac{1}{1+64} \le \frac{\int_{\ln(2)}^{2\ln(2)} \frac{1}{1+e^{3x}}dx}{\ln(2)} \le \frac{1}{1+8}$ and so

$\frac{\ln(2)}{65} \le \int_{\ln(2)}^{2\ln(2)} \frac{1}{1+e^{3x}}dx \le \frac{\ln(2)}{9}$.

(c) Let $g(x) = f(x) - f(\frac{1}{4}(x-\frac{1}{5}))$. Then g is a continuous function on [1/5, 1] since f is continuous on [0, 1] and $\frac{1}{4}(x-\frac{1}{5})$ is a continuous function and because we know that the difference and composite of two continuous functions are also continuous functions. Now $g(\frac{1}{5}) = f(\frac{1}{5}) - f(0) = f(\frac{1}{5}) - f(1)$ since $f(0) = f(1)$. Also $g(1) = f(1) - f(\frac{1}{5}) = -g(\frac{1}{5})$. Therefore either g(1) = g(1/5) = 0 or they have opposite signs. So if $g(1) \neq 0$ (and so $g(1/5) \neq 0$), by the *Intermediate Value Theorem*, there exists a point c in [1/5, 1] such that $g(c) = 0$. In any case, we have a point c in [1/5, 1] such that $g(c) = 0$. That is, $f(c) = f(\frac{1}{4}(c-\frac{1}{5}))$. Take $f(x) = \sin(\pi x)$. Then we have a point c in [1/5,1] such that

$$\sin(\pi c) = \sin(\frac{1}{4}\pi(c-\frac{1}{5})) = \sin(\frac{\pi c}{4} - \frac{\pi}{20}).$$

5. $f(x) = \begin{cases} 2 - \dfrac{2}{x+1}, x \geq 1 \\ 2x^3 - 3x + 2, x < 1 \end{cases}$.

(a) f is continuous at $x = 1$ if and only if $\lim_{x \to 1} f(x) = f(1)$.
Now $f(1) = 2 - \dfrac{2}{1+1} = 1$, $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 - \dfrac{2}{x+1} = 2 - 1 = 1$ and $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x^3 - 3x + 2 = 2 - 3 + 2 = 1$. Thus $\lim_{x \to 1} f(x) = f(1)$ and so f is continuous at $x = 1$.

(b) $f'(x) = \begin{cases} \dfrac{2}{(x+1)^2}, x > 1 \\ 6x^2 - 3, x < 1 \end{cases} = \begin{cases} \dfrac{2}{(x+1)^2}, x > 1 \\ 6(x + \frac{\sqrt{2}}{2})(x - \frac{\sqrt{2}}{2}), x < 1 \end{cases}$ -------------------- (1)

When $x < -\frac{\sqrt{2}}{2}$, by (1), $f'(x) > 0$ and so since f is continuous at $x = -\frac{\sqrt{2}}{2}$, f is increasing on the interval $(-\infty, -\frac{\sqrt{2}}{2}]$. Also from (1) when $-\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2} (< 1)$, so that $(x + \frac{\sqrt{2}}{2}) > 0$ and $(x - \frac{\sqrt{2}}{2}) < 0$ and $f'(x) < 0$. Hence again since f is continuous at $x = -\frac{\sqrt{2}}{2}$ and at $x = \frac{\sqrt{2}}{2}$, f is decreasing on the closed interval $[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$. Now when $\frac{\sqrt{2}}{2} < x < 1$, $(x + \frac{\sqrt{2}}{2}) > 0$ and $(x - \frac{\sqrt{2}}{2}) > 0$ and so by (1) $f'(x) > 0$. Thus f is increasing on the interval $[\frac{\sqrt{2}}{2}, 1]$ since f is also continuous at $x = 1$ by part (a). Clearly, by (1), for $x > 1$, $f'(x) > 0$. Thus f is increasing on the interval $[1, \infty)$. Hence f is increasing on the interval $[\frac{\sqrt{2}}{2}, \infty)$.

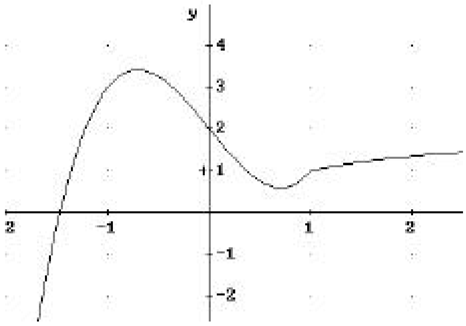
(c) From part (b), $f(-\frac{\sqrt{2}}{2}) = -2\frac{2\sqrt{2}}{8} + 3\frac{\sqrt{2}}{2} + 2 = 2 + \sqrt{2}$ is a relative maximum and $f(\frac{\sqrt{2}}{2}) = 2\frac{2\sqrt{2}}{8} - 3\frac{\sqrt{2}}{2} + 2 = 2 - \sqrt{2}$ is a relative minimum value.

(d) $f''(x) = \begin{cases} \dfrac{-4}{(x+1)^3}, x > 1 \\ 12x, x < 1 \end{cases}$ -- (2)

From (2) for $x < 0$, $f''(x) = 12x < 0$ and so the graph of f is concave downward on the interval $(-\infty, 0)$. From (2) for $0 < x < 1$ $f''(x) = 12x > 0$ and so the graph of f is concave upward on the interval (0, 1). Finally from (2) again, for $x > 1$, $f''(x) = \dfrac{-4}{(x+1)^3} < 0$ and so the graph of f is concave downward on the interval $(1, \infty)$.

(e) From part (d), there is a change of concavity of the graph of f before and after the points $x = 0$ and $x = 1$. Therefore the points of inflection of the graph of f are $(0, f(0)) = (0, 2)$ and $(1, f(1)) = (1, 1)$.

(f) The graph of f:



6. (a)

(i) $\ln(h(x)) = \frac{1}{x^3}\ln(\tan(x))$. Therefore, $\frac{h'(x)}{h(x)} = -\frac{3}{x^4}\ln(\tan(x)) + \frac{1}{x^3}\frac{\sec^2(x)}{\tan(x)}$. Hence

$$h'(x) = \left(-\frac{3}{x^4}\ln(\tan(x)) + \frac{1}{x^3}\frac{\sec^2(x)}{\tan(x)}\right)(\tan(x))^{\left(\frac{1}{x^3}\right)}$$

$$= \left(-\frac{3}{x^4}\ln(\tan(x)) + \frac{1}{x^3}\frac{1}{\sin(x)\cos(x)}\right)(\tan(x))^{\left(\frac{1}{x^3}\right)}$$

(ii) $j(x) = \int_{\sin(x)}^{\cos(x^2)} \frac{1}{2+t^2+\cos(t)}dt = \int_0^{\cos(x^2)} \frac{1}{2+t^2+\cos(t)}dt - \int_0^{\sin(x)} \frac{1}{2+t^2+\cos(t)}dt$.

Therefore, by the *Fundamental Theorem of Calculus* and the *Chain Rule*,

$$j'(x) = \frac{-2x\sin(x^2)}{2+\cos^2(x^2)+\cos(\cos(x^2))} - \frac{\cos(x)}{2+\sin^2(x)+\cos(\sin(x))}.$$

(b) Note that f satisfies $f(x+y) = f(x) + f(y) + 7xy$ ------------ (1)

and $\lim_{h\to 0}\frac{f(h)}{h} = 4$ ------------- (2)

(i) From (1), $f(0) = f(0+0) = f(0) + f(0) + 0 = 2f(0)$ and so $f(0) = 0$.

(ii) $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{f(x)+f(h)+7xh-f(x)}{h} = \lim_{h\to 0}\frac{f(h)+7xh}{h}$ by (1)

$= \lim_{h\to 0}\frac{f(h)}{h} + 7x = 7x + 4$ by (2).

(c) Write the following as a Riemann sum

$$\sum_{i=1}^{n}\left(\frac{i^3}{n^4} + \frac{\sqrt{i}}{n\sqrt{n}}\right) = \sum_{i=1}^{n}\left(\frac{i^3}{n^3} + \frac{\sqrt{i}}{\sqrt{n}}\right)\frac{1}{n} = \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $x_0 < x_1 < \ldots < x_n$ is a regular partition and $\Delta x_i = x_i - x_{i-1}$.

Therefore, we can take $x_i = \frac{i}{n}$ so that $\Delta x = \frac{1}{n}$, $x_0 = 0$ and $x_n = 1$. Thus by comparing

$$f(x_i)\Delta x \text{ with} \left(\frac{i^3}{n^3} + \frac{\sqrt{i}}{\sqrt{n}}\right)\frac{1}{n} = \left(\left(\frac{i}{n}\right)^3 + \sqrt{\frac{i}{n}}\right)\frac{1}{n}$$

we would want $f(x_i) = \left(\left(\frac{i}{n}\right)^3 + \sqrt{\frac{i}{n}}\right)\frac{1}{n} = x_i^3 + \sqrt{x_i}$. Hence $f(x) = x^3 + \sqrt{x}$.

Therefore $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^3}{n^4} + \frac{\sqrt{i}}{n\sqrt{n}}\right) = \int_0^1 (x^3 + \sqrt{x})dx = \left[\frac{x^4}{4} + \frac{2}{3}x^{\frac{3}{2}}\right]_0^1$

$$= \frac{1}{4} + \frac{2}{3} = \frac{11}{12}.$$