NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 2 EXAMINATION 1998-99

MA1102 CALCULUS

April 1999 – Time Allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SIX (6)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions in **Section A**. The marks for questions in Section A are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Answer not more than **TWO (2)** questions from Section B. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

*Answer **ALL** questions in this section.*

Question 1 [**20** marks]

Let the function f be defined on **R** by

$$f(x)=\begin{cases} x^2-\pi^2, & x<-\pi \\ \sin(x)\,, & -\pi\le x<\pi \\ x^2-1\,, & x\ge\pi \end{cases} \quad .\pi$$

(a) Find the *range* of the function f .

(b) Find the values of x (if any) where

(i) $f(x) = 1$,

(ii) $f(x) = 0$.

(c) Determine all x in **R** at which the function f is *continuous*.

(d) Is the function f differentiable at x=–π ? Justify your answer.

(e) Compute $\int_{-\pi}^{2\pi} f(x)dx$.

Question 2 [**20** marks]

Evaluate, if it exists, each of the following limits.

(a) $\lim\limits_{x\to-\infty}\dfrac{6x^6+3x+1}{8x^6-4x^2+3x+5}$.

(b) $\lim\limits_{x\to 0}\dfrac{5x^2}{\sqrt{6x^2+64}-8}$.

(c) $\lim\limits_{x\to\infty}\dfrac{(\ln(x))^3}{x^2}$.

(d) $\lim\limits_{x\to\infty}\dfrac{\sin(e^x+x+1)}{x}$.

(e) $\lim\limits_{x\to 0}(e^{(x^2)}+2x)^{(1/x)}$.

Question 3 [**20** marks]

Evaluate the following integrals.

(a) $\int \frac{1}{x(x^2+1)}dx.$

(b) $\int \tan^{-1}(2x)dx.$

(c) $\int_0^3 (|x-1|+|x-2|)dx.$

(d) $\int_0^2 \frac{x+3}{x^2+3x+2}dx.$

SECTION B

*Answer not more than **TWO (2)** questions from this section. Each question in this section carries 20 marks.*

Question 4 [**20** marks]

(a) Determine the *absolute extrema* of the function f on [0, 2] defined by

$$f(x) = \begin{cases} x^2+1, & 0 \le x < 1 \\ 2(x-2)^2, & 1 \le x \le 2 \end{cases}.$$

(b) State the *Mean Value Theorem* for the function $g(x) = \ln(x)$ on the interval [1, 2]. Hence or otherwise, prove that

$$\frac{1}{3} < \ln(3) - \ln(2) < \frac{1}{2}.$$

(c) Determine whether or not the function $h(x) = \frac{|x-3|^{6/5}}{x}$ is differentiable at $x = 3$.

Question 5 [**20** marks]

Let the function f be defined on **R** by

$$f(x)=\begin{cases} 2x^3-9x+10, & x\le 1 \\ 3-\dfrac{3}{x}+\dfrac{3}{x^2}, & x>1 \end{cases}.$$

(a) Find the intervals on which f is (i) *increasing*, (ii) *decreasing*.

(b) Find the *horizontal asymptotes* of the graph of f .

(c) Find the intervals on which the graph of f is *concave upward* or *concave downward*.

(d) Find the *relative extrema* of f .

(e) Find the *points of inflection* of the graph of f.

(f) Sketch the graph of f.

Question 6 [**20** marks]

(a) Differentiate each of the following functions.

(i) $g(x)=\int_{-2x}^{x^2}\dfrac{1}{1+\cos^2(2t)+t^4}dt.$

(ii) $h(x)=(1+x^2)^x.$

(b) Let the function f be defined on **R** by

$$f(x)=\int_2^x\sqrt{1+\sin(t)+\sin^2(t)}\,dt.$$

(i) Without integrating, show that the function f is injective.

(ii) Determine $(f^{-1})'(0).$

(c) Find the area of the region bounded by the graphs of the functions $f(x)=x\sqrt{2x+3}$ and $g(x)=x^2.$

END OF PAPER

Answer To MA1102 Calculus

SECTION A (Compulsory)

1. The function f is defined by $f(x)=\begin{cases} x^2-\pi^2, & x<-\pi \\ \sin(x), & -\pi\le x<\pi \\ x^2-1, & x\ge\pi \end{cases}$.

(a) For $x<-\pi$, $f(x)=x^2-\pi^2>0$. Also, for $x<-\pi$, $x^2-\pi^2>0 \Leftrightarrow x<-\pi$. Thus f maps $(-\infty,-\pi)$ onto $(0,+\infty)$. (Because for any $y>0$, we can take $x=-\sqrt{\pi^2+y}$.) Also, for $-\pi\le x<\pi$, $-1\le f(x)\le 1$, assuming that the sine function maps the interval $[-\pi,\pi)$ onto the interval $[-1, 1]$. Thus f maps $[-\pi,\pi)$ onto the interval $[-1, 1]$. Finally for $x\ge\pi$, $f(x)=x^2-1\ge\pi^2-1$. And for any $y\ge\pi^2-1$, we can take $x=\sqrt{y+1}\ge\pi$. Hence f maps $[\pi,\infty)$ onto $[\pi^2-1,\infty)$.

Hence the range of f is $(0,\infty)\cup[-1,1]\cup[\pi^2-1,\infty)=[-1,\infty)$.

(b) (i) From part (a) 1 is in the image of $(-\infty,-\pi)$ under f. Thus, to find the preimage we need to solve the equation $x^2-\pi^2=1$ for $x<-\pi$. Solving this gives $x=-\sqrt{\pi^2+1}$. Also 1 is in the image of $[-\pi,\pi)$ under f. Solving $\sin(x)=1$ for x in the interval $[-\pi,\pi)$ gives $x=\frac{\pi}{2}$. Thus the desired values of x are $\{-\sqrt{\pi^2+1},\frac{\pi}{2}\}$.

(ii) From part (a) 0 is in the image of $[-\pi,\pi)$ under f. Thus, solving $\sin(x)=0$ for x in the interval $(-\pi,\pi)$ gives x = 0 or $-\pi$.

(c) When $x<-\pi$, $f(x)=x^2-\pi^2$, which is a polynomial function, therefore f is continuous on $(-\infty,-\pi)$, since any polynomial function is continuous on the reals and so is continuous on any open interval. When $-\pi<x<\pi$, $f(x)=\sin(x)$ and since $\sin(x)$ is a continuous function, f is continuous on this interval. Finally when $x>\pi$, $f(x)$ is a polynomial function and so it is continuous for $x>\pi$. Thus it remains to check if f is continuous at $x=-\pi$ or π. Consider the left limit at $x=-\pi$,

$\lim_{x\to(-\pi)^-} f(x)=\lim_{x\to(-\pi)^-} x^2-\pi^2=0$ and the right limit at $x=-\pi$

$\lim_{x\to(-\pi)^+} f(x)=\lim_{x\to(-\pi)^+}\sin(x)=\sin(-\pi)=0.$

Thus the limit at $x=-\pi$ is equal to 0 and is the same as $f(-\pi)=\sin(-\pi)=0$. Therefore f is continuous at $x=-\pi$. Now consider the left limit of f at $x=\pi$,

$\lim_{x\to\pi^-} f(x)=\lim_{x\to\pi^-}\sin(x)=0$ and the right limit at $x=\pi$,

$\lim_{x\to\pi^+} f(x)=\lim_{x\to\pi^+} x^2-1=\pi^2-1>0$.

Thus the left and the right limits of f at $x=\pi$ are not the same and so f is not continuous at $x=\pi$.

Hence f is continuous at x for all $x\ne\pi$.

(d) f is differentiable at $x=-\pi$. This is seen as follows.

$$\lim_{h\to0^-}\frac{f(-\pi+h)-f(-\pi)}{h}=\lim_{h\to0^-}\frac{(-\pi+h)^2-\pi^2-0}{h}=-2\pi \quad\text{and}$$

$$\lim_{h\to0^+}\frac{f(-\pi+h)-f(-\pi)}{h}=\lim_{h\to0^+}\frac{\sin(-\pi+h)}{h}=\lim_{h\to0^+}\frac{\cos(-\pi+h)}{1}=-1.$$

Thus f is not differentiable at $x=-\pi$.

(e) $\int_{-\pi}^{2\pi} f(x)dx=\int_{-\pi}^{\pi} f(x)dx+\int_{\pi}^{2\pi} f(x)dx=\int_{-\pi}^{\pi}\sin(x)dx+\int_{\pi}^{2\pi}(x^2-1)dx$

$=[-\cos(x)]_{-\pi}^{\pi}+\left[\frac{x^3}{3}-x\right]_{\pi}^{2\pi}=0+\frac{7\pi^3}{3}-\pi=\frac{7\pi^3}{3}-\pi.$

2. (a) $\lim_{x\to\infty} \dfrac{6x^6+3x+1}{8x^6-4x^2+3x+5} = \lim_{x\to\infty} \dfrac{6+\frac{3}{x^5}+\frac{1}{x^6}}{8-\frac{4}{x^4}+\frac{3}{x^5}+\frac{5}{x^6}} = \dfrac{6+0+0}{8-0+0+0} = \dfrac{3}{4}.$

(b) $\lim_{x\to 0} \dfrac{5x^2}{\sqrt{6x^2+64}-8} = \lim_{x\to 0} \dfrac{5x^2(\sqrt{6x^2+64}+8)}{(\sqrt{6x^2+64}-8)(\sqrt{6x^2+64}+8)} = \lim_{x\to 0} \dfrac{5x^2(\sqrt{6x^2+64}+8)}{6x^2+64-64}$

$= \lim_{x\to 0} \dfrac{5(\sqrt{6x^2+64}+8)}{6} = \frac{40}{3}.$

(c) $\lim_{x\to\infty} \dfrac{(\ln(x))^3}{x^2} = \lim_{x\to\infty} \dfrac{3(\ln(x))^2}{2x^2} = \lim_{x\to\infty} \dfrac{6\ln(x)}{4x^2} = \lim_{x\to\infty} \dfrac{6}{8x^2} = 0$ by L' Hôpital's rule.

(d) $\lim_{x\to\infty} \dfrac{\sin(e^x+x+1)}{x}.$

For $x > 0$, $\left|\dfrac{\sin(e^x+x+1)}{x}\right| \le \left|\dfrac{1}{x}\right|$. Thus, for $x > 0$, $-\left|\dfrac{1}{x}\right| \le \dfrac{\sin(e^x+x+1)}{x} \le \left|\dfrac{1}{x}\right|.$

Since $\lim_{x\to\infty}\left|\dfrac{1}{x}\right| = 0$, by the Squeeze Theorem, $\lim_{x\to\infty} \dfrac{\sin(e^x+x+1)}{x} = 0.$

(e) Let $y = (e^{(x^2)}+2x)^{(1/x)}$. Then $\ln(y) = \frac{1}{x}\ln(e^{(x^2)}+2x)$.

$\lim_{x\to 0} \ln(y) = \lim_{x\to 0} \dfrac{\ln(e^{(x^2)}+2x)}{x} = \lim_{x\to 0} \dfrac{2xe^{(x^2)}+2}{e^{(x^2)}+2x} = \dfrac{0+2}{1+0} = 2.$

Therefore, $\lim_{x\to 0} y = \lim_{x\to 0} e^{\ln(y)} = e^{\lim_{x\to 0}\ln(y)} = e^2.$

3. (a) $\displaystyle\int \frac{dx}{x(x^2+1)} = \int\left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx = \ln(|x|) - \frac{1}{2}\ln(x^2+1) + C.$

(b) $\displaystyle\int \tan^{-1}(2x)dx = x\tan^{-1}(2x) - \int x\frac{2}{1+4x^2}dx = x\tan^{-1}(2x) - \frac{1}{4}\int \frac{8x}{1+4x^2}dx$

$= x\tan^{-1}(2x) - \frac{1}{4}\ln(1+4x^2) + C$.

(c) $\int_0^3(|x-1|+|x-2|)dx = \int_0^1(-(x-1)-(x-2))dx + \int_1^2((x-1)-(x-2))dx$

$+\int_2^3((x-1)+(x-2))dx$

$= -\int_0^1(2x-3)dx + \int_1^2 1dx + \int_2^3(2x-3)dx$

$= -[x^2-3x]_0^1 + 1 + [x^2-3x]_2^3$

$= -1+1+3^2-2^2$

$= 5.$

(d) $\displaystyle\int_0^2 \frac{x+3}{x^2+3x+2}dx = \int_0^2\left(\frac{2}{x+1} - \frac{1}{x+2}\right)dx = [2\ln|x+1| - \ln|x+2|]_0^2$

$= 2\ln(3) - 2\ln(2) + \ln(2) = 2\ln(3) - \ln(2).$

SECTION B

Question 4.

(a) $f(x)=\begin{cases} x^2+1, & 0\le x<1 \\ 2(x-2)^2, & 1\le x\le 2\end{cases}$ on [0, 2]. The function f is continuous on [0, 1) and on the interval (1, 2] since it is a polynomial function on each of theses intervals. Now $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2+1 = 2$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2(x-2)^2 = 2 = f(1)$. Therefore, f is continuous at $x = 1$. Thus f is continuous on [0, 2].

$$f'(x)=\begin{cases} 2x, & 0<x<1 \\ 4(x-2), & 1<x<2\end{cases}$$

Clearly $f'(x)\ne 0$ for $0<x<2$ and $x\ne 1$. Therefore, there are no critical points in (0, 1) and in (1, 2). Now $f(0) = 1$, $f(1) = 2$ and $f(2) = 0$.

Thus the absolute minimum value of g on [0, 2] is 0 and the absolute maximum value of g on [0,2] is 2.

(b) For $g(x) = \ln(x)$, applying the Mean Value Theorem on the interval [1, 2] gives a point c in (1, 2) such that $\frac{1}{c} = g'(c) = \frac{g(2)-g(1)}{2-1} = \frac{\ln(2)-\ln(1)}{1} = \ln(2)$.

Analogously, applying the Mean Value Theorem on the interval [2, 3] gives a point d in the interval (2, 3) such that $\frac{1}{d} = g'(d) = \frac{g(3)-g(2)}{3-2} = \ln(3)-\ln(2)$. Since $2<d<3$, $.\frac{1}{3} < \frac{1}{d} < \frac{1}{2}$, therefore $\frac{1}{3} < \ln(3)-\ln(2) < \frac{1}{2}$.

(c) Note that $h(x) = \frac{|x-3|^{6/5}}{x}$. $h(3) = 0$. Thus

$$\lim_{x\to 3}\frac{h(x)-h(3)}{x-3} = \lim_{x\to 3}\frac{\frac{|x-3|^{6/5}}{x}-0}{x-3} = \lim_{x\to 3}\frac{\frac{(x-3)^{6/5}}{x}}{x-3} = \lim_{x\to 3}\frac{(x-3)^{1/5}}{x} = 0.$$

Therefore, h is differentiable at $x = 3$ and $h'(3) = 0$.

5. Since $f(x)=\begin{cases} 2x^3-9x+10, & x\le 1 \\ 3-\frac{3}{x}+\frac{3}{x^2}, & x>1\end{cases}$, we note that f is continuous on **R** because f is a polynomial function on $(-\infty, 1)$ and a rational function on $(1,\infty)$ and that

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = 3 = f(1).$$

Then $f'(x)=\begin{cases} 6x^2-9, & x<1 \\ \frac{3}{x^2}-\frac{6}{x^3}, & x>1\end{cases} = \begin{cases} 6(x^2-\frac{3}{2}), & x<1 \\ \frac{3(x-2)}{x^3}, & x>1\end{cases}$ ------------------------- (1)

$f''(x)=\begin{cases} 12x, & x<1 \\ -\frac{6}{x^3}+\frac{18}{x^4}, & x>1\end{cases} = \begin{cases} 12x, & x<1 \\ \frac{6(3-x)}{x^4}, & x>1\end{cases}$ -------------------------------- (2)

(a) From (1)

For $x < -\sqrt{\frac{3}{2}}$, $x^2 > \frac{3}{2}$ and $x^2-\frac{3}{2} > 0$ so that $f'(x) > 0$. Since f is continuous at $x = -\sqrt{\frac{3}{2}}$, f is increasing on the interval $(-\infty, -\sqrt{\frac{3}{2}}\,]$.

For $-\sqrt{\frac{3}{2}} < x < 0$, $\frac{3}{2} > x^2 > 0$ so that $f'(x) = 6(x^2 - \frac{3}{2}) < 0$. Hence f is decreasing on $[-\sqrt{\frac{3}{2}}, 0]$ since f is continuous at $x = 0$ and at $x = -\sqrt{\frac{3}{2}}$.

For $0 < x < 1 < \sqrt{\frac{3}{2}}$, $0 < x^2 < \frac{3}{2}$ so that $f'(x) < 0$ on (0, 1). Thus we have that f is decreasing on $[0, 1]$ since f is continuous at $x = 1$. Hence f is decreasing on $[-\sqrt{\frac{3}{2}}, 1]$. For $1 < x < 2$,, $x - 2 < 0$ and so by (1) $f'(x) < 0$ and we conclude that f is decreasing on $[1, 2]$. Therefore, f is decreasing on the interval $[-\sqrt{\frac{3}{2}}, 2]$. Finally for $x > 2$, $f'(x) > 0$and so f is increasing on the interval $[2, \infty)$.

(b) Now $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 3 - \frac{3}{x} + \frac{3}{x^2} = 3$ and so the line $y = 3$ is a horizontal asymptote of the graph of f. Next we check the following limit.
$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 2x^3 - 9x + 10 = \lim_{x\to-\infty} x^3(2 - \frac{9}{x^2} + \frac{10}{x^3}) = -\infty$ since
$\lim_{x\to-\infty} x^3 = -\infty$ and $\lim_{x\to-\infty} (2 - \frac{9}{x^2} + \frac{10}{x^3}) = 2 > 0$. Therefore there is no other horizontal asymptote other than the one we have found.

(c) From (2) when $x < 0$, $f''(x) < 0$. Hence the graph of f is concave downward on the interval $(-\infty, 0)$. Also from (2), when $1 > x > 0$, $f''(x) > 0$. Thus the graph of f is concave upward on the interval (0, 1). Again from (2), for $x1 < x < 3$,
$f''(x) = 6\frac{3-x}{x^4} > 0$and so the graph of f is concave upward on $(1, 3)$. Finally for $x > 3$,
$f''(x) = 6\frac{3-x}{x^4} < 0$and so the graph of f is concave downward on $(3, \infty)$. It remains to check the concavity of f at the point $x = 1$. For a start we need to check if f is differentiable at $x = 1$. Thus we compute the limit,

$$\lim_{x\to1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to1^-} \frac{2x^3 - 9x + 10 - 3}{x - 1} = \lim_{x\to1^-} 2x^2 + 2x - 7 = 2 + 2 - 7 = -3$$

and the limit

$$\lim_{x\to1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to1^+} \frac{3 - \frac{3}{x} + \frac{3}{x^2} - 3}{x - 1} = \lim_{x\to1^+} -3\frac{\frac{x-1}{x^2}}{x - 1} = \lim_{x\to1^+} -3\frac{1}{x^2} = -3.$$

We see that they are equal and so the derivative of f at $x = 1$ is

$$f'(1) = \lim_{x\to1} \frac{f(x) - f(1)}{x - 1} = -3.$$

Now there are two approaches to proceed. The first is to get some information from the existence of the second derivative and in particular we need to know if it is non zero; the second is to show if there is a small interval containing $x = 1$ such that the graph of f is above or below the tangent line at $x = 1$. We shall try the first approach. We compute:

$$\lim_{x\to1^-} \frac{f'(x) - f'(1)}{x - 1} = \lim_{x\to1^-} \frac{6x^2 - 9 + 3}{x - 1} = \lim_{x\to1^-} 6\frac{x^2 - 1}{x - 1} = \lim_{x\to1^-} 6(x + 1) = 6 \cdot 2 = 12$$ and

$$\lim_{x\to1^+} \frac{f'(x) - f'(1)}{x - 1} = \lim_{x\to1^+} \frac{\frac{3(x-2)}{x^3} + 3}{x - 1} = \lim_{x\to1^+} 3\frac{(x - 2 + x^3)}{x^3(x - 1)} = \lim_{x\to1^+} 3\frac{x^2 + x + 2}{x^3} = 12$$

and so $\lim_{x\to1} \frac{f'(x) - f'(1)}{x - 1} = 12$. Therefore, $f''(1) = 12 > 0$ and so the graph of f is concave upward at $x = 1$. Hence the graph of f is concave upward on the interval (0, 3).

(d) Now from part (b) $\lim_{x\to-\infty} f(x) = -\infty$. Thus f has no absolute minimum. From part (a), f is increasing on $(-\infty, -\sqrt{\frac{3}{2}}\,]$ and decreasing on $[-\sqrt{\frac{3}{2}}, 2]$. Therefore

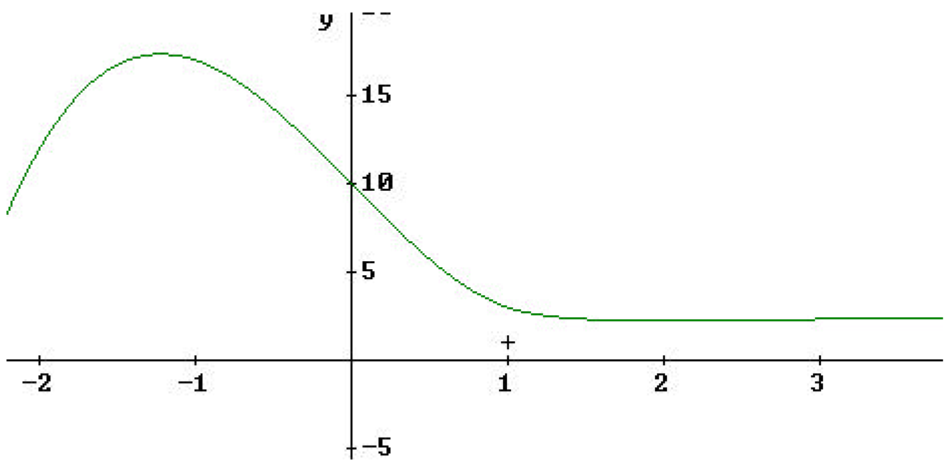
$$f(-\sqrt{\tfrac{3}{2}}) = -2\cdot\frac{3}{2}\cdot\sqrt{\frac{3}{2}} + 9\sqrt{\frac{3}{2}} + 10 = 10 + 3\sqrt{6}$$

is a relative maximum of f. Also from part (a) f is increasing on $[2,\infty)$. Thus $f(2) = 3 - \frac{3}{2} + \frac{3}{4} = 2\frac{1}{4}$ is a relative minimum value of f.

(e) From part (c), (0, $f(0)$)=(0, 10) and (3, $f(3)$)=(3, 2 1/3) are the points of inflection of the graph of f since at x = 0 and at x = 3 respectively, there is a change of concavity before and after the point x.

(f)

The graph of f (not drawn to scale)



6. (a) (i)

$$g(x) = \int_{-2x}^{x^2} \frac{1}{1+\cos^2(2t)+t^4}\,dt = \int_0^{x^2} \frac{1}{1+\cos^2(2t)+t^4}\,dt + \int_{-2x}^{0} \frac{1}{1+\cos^2(2t)+t^4}\,dt$$

$$= \int_0^{x^2} \frac{1}{1+\cos^2(2t)+t^4}\,dt - \int_0^{-2x} \frac{1}{1+\cos^2(2t)+t^4}\,dt.$$

$$= F(x^2) - F(-2x) \quad \text{where } F(x) = \int_0^{x} \frac{1}{1+\cos^2(2t)+t^4}\,dt.$$

Therefore,

$g'(x) = F'(x^2)\cdot 2x - F'(-2x)\cdot(-2)$ by the *Chain Rule*

$$= \frac{2x}{1+\cos^2(2x^2)+x^8} + \frac{2}{1+\cos^2(-4x)+16x^4} \text{ by the FTC.}$$

(ii)

Since $h(x) = (1+x^2)^x$, $\ln(h(x)) = x\ln(1+x^2)$. Thus, differentiating the above on both sides gives

$$\frac{h'(x)}{h(x)} = \ln(1+x^2) + x\frac{2x}{1+x^2}. \text{ Therefore,}$$

$$h'(x) = (1+x^2)^x\left(\ln(1+x^2) + \frac{2x^2}{1+x^2}\right).$$

(b) (i)

Since $f(x) = \int_2^x \sqrt{1+\sin(t)+\sin^2(t)}\, dt$, by the FTC,

$f'(x) = \sqrt{1+\sin(x)+\sin^2(x)} = \sqrt{(1+\frac{1}{2}\sin(x))^2+\frac{3}{4}\sin^2(x)} > 0$ since $1+\frac{1}{2}\sin(x) \ge \frac{1}{2}$.

Therefore, f is increasing on the whole of **R.** Thus f is injective.

(ii)

$(f^{-1})'(0) = \dfrac{1}{f'(f^{-1}(0))}$. So we need to know the value of $f^{-1}(0)$. Now

$f^{-1}(0) = x \Leftrightarrow f(x) = 0 \Leftrightarrow \int_2^x \sqrt{1+\sin(t)+\sin^2(t)}\, dt = 0$. Since

$f(2) = \int_2^2 \sqrt{1+\sin(t)+\sin^2(t)}\, dt = 0$ and f is injective, $x = 2$.

Therefore, $(f^{-1})'(0) = \dfrac{1}{f'(f^{-1}(0))} = \dfrac{1}{f'(2)} = \dfrac{1}{\sqrt{1+\sin(2)+\sin^2(2)}}$.

(c) $f(x) = g(x)$ if and only if $x\sqrt{2x+3} = x^2 \Leftrightarrow x = 0$ or $x = \sqrt{2x+3}$. Solving

Now $x = \sqrt{2x+3} \Leftrightarrow x^2 = 2x+3$ and $x \ge 0 \Leftrightarrow (x-3)(x+1) = 0$and $x \ge 0 \Leftrightarrow x = 3$.

$\int x\sqrt{2x+3}\, dx = \frac{1}{3}x(2x+3)^{3/2} - \int \frac{1}{3}(2x+3)^{3/2} dx$ by integration by parts

$= \frac{1}{3}x(2x+3)^{3/2} - \frac{1}{15}(2x+3)^{5/2} + C.$

$\int_0^3 x\sqrt{2x+3}\, dx = \left[\frac{1}{3}x(2x+3)^{3/2} - \frac{1}{15}(2x+3)^{5/2}\right]_0^3$

$= 9^{3/2} - \frac{1}{15}9^{5/2} + \frac{1}{15}3^{5/2} = \frac{2}{5}9^{3/2} + \frac{3}{5}\sqrt{3} = 10\frac{4}{5} + \frac{3}{5}\sqrt{3}$ and

$\int_0^3 x^2 dx = \left[\frac{1}{3}x^3\right]_0^3 = 9.$

Therefore, the area of the region bounded by the graphs of the functions f and g is $\int_0^3 (f(x) - g(x))dx = 10\frac{4}{5} + \frac{3}{5}\sqrt{3} - 9 = 1\frac{4}{5} + \frac{3}{5}\sqrt{3}$ square units.