SECTION A

Question 1

This tests concept of range of a function, continuity, differentiability and integration over a piece-wise polynomial.

The function f is defined by

$$f(x) = \begin{cases} x^2 - 9, \quad x < -3 \\ 3, \; x = -3 \\ 3x + 9, \; -3 < x < 1 \\ x^3 + 11, \quad x \geq 1 \end{cases}.$$

(a) For $x < -3$,

$$f(x) = x^2 - 9 = (-x)^2 - 9 > 0.$$

Also, for $x < -3$,

$$x^2 - 9 > 0 \Leftrightarrow x < -3.$$

Thus f maps $(-\infty, -3)$ onto $(0, +\infty)$.

(Because for any $y > 0$, we can take

$$x = -\sqrt{9+y}\,.)$$

Also,

$-3 < x < 1$ if and only if $0 < 3x + 9 < 12$.

Thus f maps $(-3, 1)$ onto $(0, 12)$.

Finally for $x \geq 1$, $f(x) = x^3 + 11 \geq 12$.

And for any $y \geq 12$, we can take

$$x = \sqrt[3]{y-11} \geq 1.$$

Hence f maps $[1, \infty)$ onto $[12, \infty)$.

Therefore, the range of f is

$(0, \infty) \cup \{3\} \cup (0, 12) \cup [12, \infty) = (0, \infty)$.

(b) (i) From part (a) 3 is in the image of $(-\infty, -3)$ under f.

Thus, to find the preimage we need to solve the equation $x^2 - 9 = 3$ for $x < -3$. Solving this gives $x = -\sqrt{12} = -2\sqrt{3}$.

Obviously $f(-3) = 3$. Also 3 is in the image of $(-3, 1)$ under f. Solving $3x + 9 = 3$ gives $x = -2$. Thus the desired values of x are

$$\{-2\sqrt{3}, -3, -2\}$$

(ii) From part (a) 0 is not in the range of f. Thus, there is no value of x for which

$$f(x) = 0.$$

(c) When $x < -3$, $f(x) = x^2 - 9$, which is a polynomial function, therefore f is continuous on $(-\infty, -3)$, since any polynomial function is continuous on the real numbers and so is continuous on any interval.

Similarly, f is continuous on the interval

$(-3, 1)$ and on the interval $(1, \infty)$. Thus it

remains to check if f is continuous at $x = -$

3 or 1. Consider the left limit of f at $x = -3$,

$$\lim_{x \to (-3)^-} f(x) = \lim_{x \to (-3)^-} x^2 - 9 = 0$$

and the right limit at $x = -3$

$$\lim_{x \to (-3)^+} f(x) = \lim_{x \to (-3)^+} 3x + 9 = 0.$$

Thus the limit at $x = -3$ is equal to 0 but not

equal to $f(-3) = 3$. Therefore f is

not continuous at $x = -3$.

Now consider the left limit of f at $x = 1$,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 3x + 9 = 12$$

and the right limit at $x = 1$,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^3 + 11 = 1 + 11 = 12 = f(1) .$$

Thus the left and the right limits of f at $x = 1$ are the same and is equal to the value of the function f at $x = 1$ and so f is continuous at $x = 1$.

Hence f is continuous at x for all $x \neq -3$.

(d) f is differentiable at $x = 1$. This is seen as follows.

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0^-} \frac{3(1+h) + 9 - 12}{h} = 3 \quad \text{and}$$

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1+h)^3 + 11 - 12}{h}$$

$$= \lim_{h \to 0^+} \frac{h^3 + 3h^2 + 3h}{h} = 3.$$

Thus f is differentiable at $x = 1$ and $f'(1) = 3$.

(e) $\int_{-1}^{2} f(x)dx = \int_{-1}^{1} f(x)dx + \int_{1}^{2} f(x)dx$

$$= \int_{-1}^{1}(3x+9)dx + \int_{1}^{2}(x^3+11)dx$$

$$= [3x^2+9x]_{-1}^{1} + \left[\frac{x^4}{4} + 11x\right]_{1}^{2}$$

$$= 18 + 11 + 4 - \frac{1}{4} = 32\tfrac{3}{4}.$$

Question 2.

(a) $$\lim_{x\to\infty} \frac{7x^5+41x+5}{14x^5-103x^2+7} = \lim_{x\to\infty} \frac{7+\frac{41}{x^4}+\frac{5}{x^5}}{14-\frac{103}{x^3}+\frac{7}{x^5}}$$

$$= \frac{7+0+0}{14-0+0} = \frac{1}{2}.$$

(b) $$\lim_{x\to 0} \frac{\sqrt{17x+49}-7}{4x}$$

$$= \lim_{x\to 0} \frac{(\sqrt{17x+49}-7)(\sqrt{17x+49}+7)}{4x(\sqrt{17x+49}+7)}$$

$$= \lim_{x\to 0} \frac{17x+49-49}{4x(\sqrt{17x+49}+7)}$$

$$= \lim_{x\to 0} \frac{17}{4(\sqrt{17x+49}+7)} = \frac{17}{56}.$$

(c) $$\lim_{x\to\infty} \frac{x^3}{7^x} = \lim_{x\to\infty} \frac{3x^2}{\ln(7)7^x} = \lim_{x\to\infty} \frac{6x}{(\ln(7))^2 7^x}$$

$$= \lim_{x \to \infty} \frac{6}{(\ln(7))^3 7^x} = 0 \text{ by L' Hopital's rule.}$$

(d) $\displaystyle \lim_{x \to \infty} \frac{\cos(e^x + 1)}{x}$.

For $x > 0$, $\displaystyle \left| \frac{\cos(e^x + 1)}{x} \right| \le \left| \frac{1}{x} \right|$.

Thus, for $x > 0$, $\displaystyle -\left| \frac{1}{x} \right| \le \frac{\cos(e^x + 1)}{x} \le \left| \frac{1}{x} \right|$.

Since $\displaystyle \lim_{x \to \infty} \left| \frac{1}{x} \right| = 0$, by the Squeeze Theorem,

$$\lim_{x \to \infty} \frac{\cos(e^x + 1)}{x} = 0.$$

(e) For $0 < x < \pi/2, 0 < \sin(x) < 1$ and so

$[\sin(x)] = 0$.

Also for $-\pi/2 < x < 0, -1 < \sin(x) < 0$

so that $[\sin(x)] = -1$. Hence,

$$\lim_{x \to 0^+} [\sin(x)] = 0 \text{ and}$$

$$\lim_{x \to 0^-} [\sin(x)] = -1.$$

Thus the limit $\lim_{x\to 0}[\sin(x)]$ does not exist.

Question 3

(a) $\int \frac{dx}{(x+5)(x+6)} = \int (\frac{1}{x+5} - \frac{1}{x+6})dx$

$$= \ln(\frac{|x+5|}{|x+6|}) + C.$$

(b) $\int x^5\sqrt{x^6+8}\,dx = \frac{1}{6}\int y^{\frac{1}{2}}\frac{dy}{dx}dx = \frac{1}{6}\int y^{\frac{1}{2}}dy$

$$= \frac{1}{9}\cdot\ y^{\frac{3}{2}} + C = \frac{1}{9}(x^6+8)^{\frac{3}{2}} + C,$$

where $y = (x^6+8)$.

(c) $\int_0^2 \sqrt{3+|x-1|}\,dx$

$$= \int_0^1 \sqrt{3+|x-1|}\,dx + \int_1^2 \sqrt{3+|x-1|}\,dx$$

$$= \int_0^1 \sqrt{4-x}\,dx + \int_1^2 \sqrt{2+x}\,dx$$

$$= -\frac{2}{3}\left[(4-x)^{\frac{3}{2}}\right]_0^1 + \frac{2}{3}\left[(2+x)^{\frac{3}{2}}\right]_1^2$$

$$= \frac{2}{3}\{(-3^{\frac{3}{2}} + 4^{\frac{3}{2}}) + (4^{\frac{3}{2}} - 3^{\frac{3}{2}})\}$$

$$= \frac{32}{3} - 4\sqrt{3}.$$

(d) $\int_0^{\ln(4)} e^x(5+e^x)^2 dx = \left[\frac{(5+e^x)^3}{3}\right]_0^{\ln(4)}$

$$= \tfrac{1}{3}(9^3 - 6^3) = 81 + 36 + 54 = 171.$$

Question 4.

(a) $g(x) = \dfrac{x^3 + 3x^2 + 2x + 1}{x + 1}$ on [0, 1]. The function g is continuous on [0, 1] and

$g'(x) = \dfrac{2x^3 + 6x^2 + 6x + 1}{(x + 1)^2} \neq 0$ on the open interval (0, 1).

Thus there are no critical points in (0, 1).

$$g(0) = 1, \quad g(1) = 7/2.$$

Thus the absolute minimum value of g on [0, 1] is 1 and the absolute maximum value of g on [0,1] is 7/2.

(b) (i) For $h(x) = x^4 - 4x^3 + 2x^2 - 12x + 2$, $h(0)$ =2 and $h(1)$= 1–4 +2–12 +2 = –11 < 0.

Since h is a polynomial function on [0, 1], h is continuous on [0, 1]. Therefore, by the *Intermediate Value Theorem*, there is a point k in (0, 1) such that $h(k) = 0$. It follows that there is a point k in (0, 3) such that $h(k) = 0$.

(ii) For x in **R**,

$$h'(x) = 4x^3 - 12x^2 + 4x - 12 = 4(x^2 + 1)(x - 3) .$$

If h has two distinct roots in $(-\infty, 3)$, then since h is differentiable on the whole of $(-\infty, 3)$ by Rolle's theorem, there is a point c in $(-\infty, 3)$ with $h'(c) = 0$. But for any c in $(-\infty, 3)$, $c < 3$ so that $c - 3 < 0$ and so $h'(c) = 4(c^2 + 1)(c - 3) < 0$, i.e. $h'(c) \neq 0$. This contradiction shows that h can have only one root. Thus by (i) h has exactly one such root k in $(-\infty, 3)$.

Alternatively, since $h'(x) < 0$ for $x < 3$, h is decreasing on $(-\infty, 3]$ and so h is injective on $(-\infty, 3]$. Therefore by part (i) there is exactly one root k in $(-\infty, 3)$.

(c) Consider the limit

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x}$$

since $f(0) = 0$.

If this limit exists, then so is the following limit

$$\lim_{x\to 0} \left| \frac{f(x)}{x} \right|.$$

Therefore, if $\lim_{x\to 0} \left| \frac{f(x)}{x} \right|$ does not exist, then

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x} \text{ does not exist.}$$

Now by property 2, for $x \neq 0$,

$$\left|\frac{f(x)}{x}\right| = \frac{|f(x)|}{|x|} \geq \frac{\sqrt{|x|}}{|x|} = \frac{1}{\sqrt{|x|}}.$$

Since $\lim_{x\to 0} \frac{1}{\sqrt{|x|}} = +\infty$, the limit $\lim_{x\to 0} \left|\frac{f(x)}{x}\right|$ dose not exist.

Thus $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x}$ does not

exist. Therefore, f is not differentiable at $x =$ 0.

Question 5

Since $f(x) = \begin{cases} x^3 - 1, & x \le 1 \\ 4x - x^2 - 3, & x > 1 \end{cases}$, we note that f is continuous on **R** because f is a polynomial function on $(-\infty, 1)$ and also on $(1, \infty)$ and that

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 0 = f(1).$$

Then $f'(x) = \begin{cases} 3x^2, & x < 1 \\ 4 - 2x, & x > 1 \end{cases}$ ---------------------------- (1)

$f''(x) = \begin{cases} 6x, & x < 1 \\ -2, & x > 1 \end{cases}$ -------------------- (2)

(a) Deducing from (1):
for $x < 0$, $x^2 > 0$ and $3x^2 > 0$ so that $f'(x) > 0$. Since f is continuous at $x = 0$, f is increasing on the interval $(-\infty, 0]$.

For $0 < x < 1$, $3x^2 > 0$ so that $f'(x) > 0$.
Hence f is increasing on $[0, 1]$ since f is continuous at $x = 0$ and at $x = 1$.
For $1 < x < 2$, $(4 - 2x) > 0$ so that $f'(x) > 0$.
Thus we have that
f is increasing on $[1, 2]$ since f is continuous at $x = 2$.
Therefore, f is increasing on $(-\infty, 2]$.
Finally for $x > 2$, $4 - 2x < 0$ and so by (1) $f'(x) < 0$ and we conclude that f is decreasing on $[2, \infty)$.

(b) From (2) when $x < 0$, $f''(x) < 0$. Hence the graph of f is concave downward on the interval $(-\infty, 0)$. Also from (2), when $1 > x > 0$, $f''(x) > 0$. Thus the graph of f is concave upward on the interval (0, 1). Again from (2), for $x > 1$, $f''(x) = -2 < 0$ and so the graph of f is concave downward on $(1, \infty)$.

(c) Now $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} -(x-2)^2 + 1 = -\infty$.
Thus f has no absolute minimum. From part (a), f is increasing on $(-\infty, 2]$ and decreasing

on $[2, \infty)$. Therefore

$$f(2) = 1$$

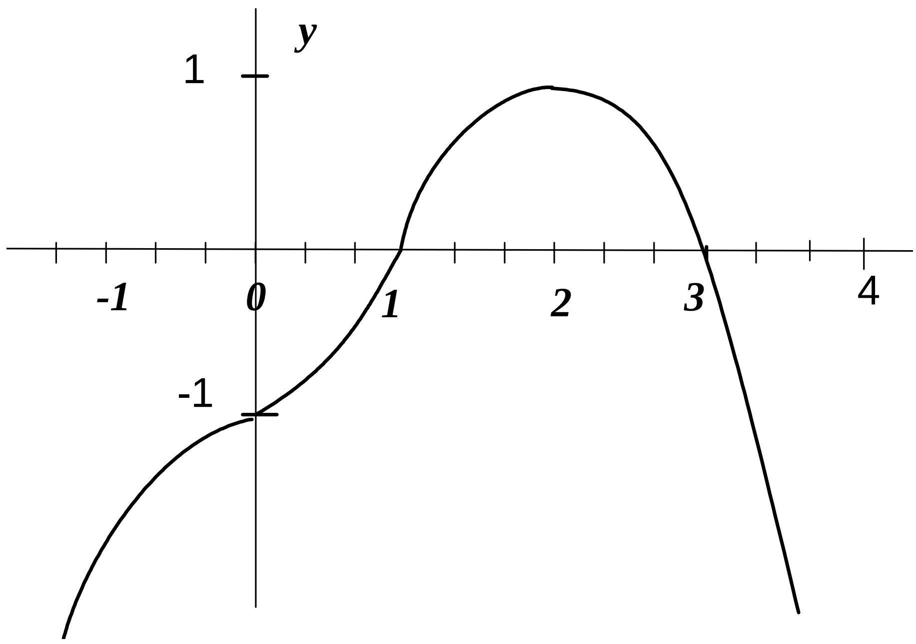
is the absolute maximum of f.

(d) From part (b),

$$(0, f(0)) = (0, -1) \text{ and } (1, f(1)) = (1, 0)$$

are the points of inflection of the graph of f since at $x = 0$ and at $x = 1$ respectively, there is a change of concavity before and after the point x.

[4] (e)



The graph of f (not drawn to scale)

Question 6

(a) (i)

$$g(x) = \int_{-x}^{x^2} \frac{1}{2+t^2+\sin(2t)}dt$$

$$= \int_{0}^{x^2} \frac{1}{2+t^2+\sin(2t)}dt + \int_{-x}^{0} \frac{1}{2+t^2+\sin(2t)}dt$$

$$= \int_{0}^{x^2} \frac{1}{2+t^2+\sin(2t)}dt - \int_{0}^{-x} \frac{1}{2+t^2+\sin(2t)}dt$$

$$= F(x^2) - F(-x)$$

where $F(x) = \int_{0}^{x} \frac{1}{2+t^2+\sin(2t)}dt.$

Therefore,

$$g'(x) = F'(x^2)\cdot 2x - F'(-x)\cdot(-1)$$

by the *Chain Rule*

$$= \frac{2x}{2+x^4+\sin(2x^2)} + \frac{1}{2+x^2-\sin(2x)}$$

by the FT C.

(ii) Since $h(x) = 5^{x^2}(x^3+1)$,

$$\ln(h(x)) = x^2\ln(5) + \ln(x^3+1).$$

Thus, differentiating the above on both sides gives

$$\frac{h'(x)}{h(x)} = \ln(5)(2x) + \frac{3x^2}{x^3+1}.$$

Therefore,

$$h'(x) = 5^{x^2}(x^3+1)\left(\ln(5)(2x) + \frac{3x^2}{x^3+1}\right)$$
$$= 5^{x^2}(2\ln(5)x(x^3+1) + 3x^2).$$

(b) (i)

Since $f(x) = \int_1^x \sqrt{1+3t^2+t^4}\,dt$, by the FTC,

$$f'(x) = \sqrt{1+3x^2+x^4} \geq 1 > 0.$$

Therefore, f is increasing on the whole of **R.** Thus f is injective.

(ii) $(f^{-1})'(0) = \dfrac{1}{f'(f^{-1}(0))}$. So we need to know the value of $f^{-1}(0)$. Now

$$f^{-1}(0) = x \Leftrightarrow f(x) = 0$$
$$\Leftrightarrow \int_1^x \sqrt{1+3t^2+t^4}\,dt = 0.$$

Since

$f(1)=\int_1^1 \sqrt{1+3t^2+t^4}\,dt=0$ and f is injective, $x = 1$.

Therefore,

$$(f^{-1})'(0)=\frac{1}{f'(f^{-1}(0))}$$
$$=\frac{1}{f'(1)}=\frac{1}{\sqrt{1+3+1}}=\frac{1}{\sqrt{5}}.$$

(c)

$$|x|+4>|x-1| \quad \text{---------------} \quad (1)$$

For $x < 0$, the inequality (1) becomes

$$-x+4>-(x-1).$$

That is $4 > 1$ which is true.

So the solution set is $(-\infty, 0)$ for this part.

For $0\le x<1$, the inequality (1) becomes

$$x+4>-(x-1).$$

That is

$2x>-3$ or $x>-3/2$. Hence the solution set is [0, 1).

For $x\ge 1$, the inequality becomes $x + 4 > x - 1$ which is always true.

Therefore the solution set to (1) is **R**.

Alternatively,

$$|x-1|-|x|\le|(x-1)-x|=1<4.$$