Linear equations in one variable

In this unit we give examples of simple linear equations and show you how these can be solved. In any equation there is an unknown quantity, x say, that we are trying to find. In a linear equation this unknown quantity will appear only as a multiple of x, and not as a function of x such as x^2, x^3, $\sqrt{x}$, $\sin x$ and so on. Linear equations occur so frequently in the solution of other problems that a thorough understanding of them is essential.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that all this becomes second nature. To help you to achieve this, the unit includes a substantial number of such exercises.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- recognise simple linear equations
- solve simple linear equations
- check that your solutions are correct by substitution

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1. Introduction

In this unit we are going to be looking at simple equations in one variable, and the equations will be linear - that means there'll be no x^2 terms and no x^3's, just x's and numbers. For example, we will see how to solve the equation $3x + 15 = x + 25$.

2. Solving equations by collecting terms

Suppose we wish to solve the equation

$$3x + 15 = x + 25$$

The important thing to remember about any equation is that the equals sign represents a balance. What an equals sign says is that what's on the left-hand side is exactly the same as what's on the right-hand side. So, if we do anything to one side of the equation we have to do it to the other side. If we don't, the balance is disturbed. Therefore, whatever operation we perform on either side of the equation, so long as it's done in **exactly** the same way on each side the balance will be preserved.

Our first step in solving any equation is to attempt to gather all the x's together and to gather all the numbers together.

From

$$3x + 15 = x + 25$$

we can subtract x from each side, because this will remove it entirely from the right, to give

$$2x + 15 = 25$$

We can subtract 15 from each side to give

$$2x = 10$$

and finally, by dividing each side by 2 we obtain

$$x = 5$$

So the solution of the equation is $x = 5$. This solution should be checked by substitution into the original equation in order to check that both sides are the same. If we do this, the left is $3(5) + 15 = 30$. The right is $5 + 25 = 30$. So the left equals the right and we have checked that the solution is correct.

Example

Solve the equation $2x + 3 = 6 - (2x - 3)$.

Solution

From $2x + 3 = 6 - (2x - 3)$ we first remove the brackets on the right to give

$$2x + 3 = 6 - 2x + 3$$

so that

$$2x + 3 = 9 - 2x$$

We are now in the same position as we were in during the first Example. We need to get the x's together by adding $2x$ to each side.

$$4x + 3 = 9$$

Now take 3 away from each side:

$$4x = 6$$

so that

$$\begin{aligned} x &= \frac{6}{4} \qquad \text{(by dividing both sides by 4)} \\ &= \frac{3}{2} \\ &= 1\frac{1}{2} \end{aligned}$$

When solving simple equations we should always check the solution by taking our answer and substituting it in the original equation to check that the left- and right- hand sides are the same.

Substituting $x = 1\frac{1}{2}$ in the left-hand side gives:

$$2 \times \left(1\frac{1}{2}\right) + 3 = 3 + 3 = 6$$

Substituting $x = 1\frac{1}{2}$ in the right-hand side gives:

$$6 - \left(2 \times \left(1\frac{1}{2}\right) - 3\right) = 6 - 0 = 6$$

So again, the left- and right- hand sides are equal - we've got that balance, so we know that we've got the right answer.

Exercises

1. Solve the following equations.

a)	$x + 5 = 9$	b)	$12 - x = 7$	c)	$5x = 3$
d)	$4x + 10 = 2$	e)	$5 - 3x = -4$	f)	$2 + 14x = 30$
g)	$9 + 5x = 3x + 13$	h)	$4 - 3x = 8 + x$	i)	$5 + 3(x - 1) = 5x - 6$

3. Solving equations by removing brackets & collecting terms

Example

Solve the equation

$$8(x - 3) - (6 - 2x) = 2(x + 2) - 5(5 - x)$$

We begin by multiplying out the brackets, taking care, in particular, with any minus signs.

$$8x - 24 - 6 + 2x \quad = \quad 2x + 4 - 25 + 5x$$

Each side can be tidied up by collecting the x terms and the numbers together.

$$10x - 30 = 7x - 21$$

Now take $7x$ from each side, and then add 30 to each side:

$$\begin{aligned} 3x - 30 &= -21 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

And again you should take the solution ($x = 3$), substitute it back into the original equation to check that we have got the correct answer. On the left:

$$8(x-3) - (6-2x) = 8(3-3) - (6-2(3)) = 0 - 0 = 0.$$

On the right:

$$2(x+2) - 5(5-x) = 2(3+2) - 5(5-3) = 10 - 10 = 0.$$

So both sides equal zero. The equation balances and so $x = 3$ is the solution.

Example

Solve the equation

$$(x+1)(2x+1) = (x+3)(2x+3) - 14$$

We begin by removing the brackets.

$$2x^2 + x + 2x + 1 = 2x^2 + 3x + 6x + 9 - 14$$

So

$$2x^2 + 3x + 1 = 2x^2 + 9x - 5$$

Remember we stated that we are dealing in this unit with linear equations, so there should be no x^2 terms. In fact, they all cancel out:
There is a term $2x^2$ on both sides. We can subtract $2x^2$ from both sides to leave

$$3x + 1 = 9x - 5$$

We can now proceed as in the earlier examples.

$$\begin{aligned} 3x + 1 &= 9x - 5 \\ 1 &= 6x - 5 \\ 6 &= 6x \\ 1 &= x \end{aligned}$$

So the solution is $x = 1$. As before, we can substitute it back into the original equation as a check.
On the left:

$$(x+1)(2x+1) = (1+1)(2+1) = (2)(3) = 6$$

On the right

$$(x+3)(2x+3) - 14 = (1+3)(2+3) - 14 = (4)(5) - 14 = 20 - 14 = 6$$

So both sides equal 6 and the equations balance when $x = 1$. The solution is $x = 1$.

Exercises

2. Solve the following equations.

   a) $5(3-x)-2(4-3x)=11-2(x-1)$ b) $6-4(x+3)=2(x-1)$

   c) $5(1-2x)+2(3-x)=3(x+4)+14$

4. Linear equations with fractional coefficients

Example

Solve the equation

$$\frac{4(x+2)}{5}=7+\frac{5x}{13}$$

Solution

In this Example the fractions are the cause of the difficulty. We want to try to remove them and work with whole numbers. Multiplying both sides by 5 and then by 13 will remove the fractions. This is equivalent to multiplying both sides by the lowest common denominator, which is $5\times 13=65$.

$$\begin{aligned}
\frac{4(x+2)}{5} &= 7+\frac{5x}{13}\\
65\times\frac{4(x+2)}{5} &= 65\left(7+\frac{5x}{13}\right)\\
65\times\frac{4(x+2)}{5} &= 65\times 7 \qquad +65\times\frac{5x}{13}\\
{}^{13}\cancel{65}\times\frac{4(x+2)}{\cancel{5}_1} &= 65\times 7 \qquad +{}^{5}\,\cancel{65}\times\frac{5x}{\cancel{13}_1}\\
52(x+2) &= 455+25x
\end{aligned}$$

This is a much more familiar form, like the earlier examples. Multiply out the brackets, collect together x terms and collect together the numbers.

$$\begin{aligned}
52x+104 &= 455+25x\\
27x &= 351\\
x &= \frac{351}{27}\\
&= 13
\end{aligned}$$

We should go back and check this solution to make sure it is correct. So let's do that.
On the left hand side:

$$\frac{4(x+2)}{5}=\frac{4(13+2)}{5}=\frac{60}{5}=12$$

On the right:

$$7+\frac{5x}{13}=7+\frac{(5)(13)}{13}=7+5=12$$

We see that the left and right sides are equal. So the solution $x=13$ is correct.

Example

Solve $\dfrac{x+5}{6} - \dfrac{x+1}{9} = \dfrac{x+3}{4}$

Solution

In this example there are no brackets. Does that make any difference? The thing you have to remember is that a division line also acts as a bracket. For example, $\dfrac{x+5}{6}$ means that all of $x+5$ is divided by 6. So it is helpful to put brackets around these terms.

$$\frac{(x+5)}{6} - \frac{(x+1)}{9} = \frac{(x+3)}{4}$$

Now we need a common denominator; we need a number into which all of the individual denominators (6, 9 and 4) will divide exactly. The lowest number into which they all divide is 36. So let's multiply throughout by 36.

$$\frac{36(x+5)}{6} - \frac{36(x+1)}{9} = \frac{36(x+3)}{4}$$

Notice that we've made it quite clear by using the brackets what the 36 is multiplying. Each term can be simplifed by dividing top and bottom by the common factors.

$$\frac{{}^{6}\cancel{36}(x+5)}{\cancel{6}_1} - \frac{{}^{4}\cancel{36}(x+1)}{\cancel{9}_1} = \frac{{}^{9}\cancel{36}(x+3)}{\cancel{4}_1}$$

$$\frac{6(x+5)}{1} - \frac{4(x+1)}{1} = \frac{9(x+3)}{1}$$

from which

$$6x + 30 - 4x - 4 = 9x + 27$$

Simplifying the left and right hand sides separately

$$2x + 26 = 9x + 27$$

Then take $2x$ away from each side to give

$$26 = 7x + 27$$

Take 27 away from each side

$$-1 = 7x$$

and finally

$$x = -\frac{1}{7}$$

So, the solution is $x = -\dfrac{1}{7}$. This should be checked by substitution into the original equation as follows:

Substitution of $x = -\frac{1}{7}$ into the left-hand side of the original equation we find:

$$\frac{-\frac{1}{7}+5}{6} - \frac{-\frac{1}{7}+1}{9}$$

which simplifies as follows:

$$\frac{\frac{-1+35}{7}}{6} - \frac{\frac{-1+7}{7}}{9}$$

and further to

$$\frac{34}{42} - \frac{6}{63} = \frac{17}{21} - \frac{2}{21} = \frac{15}{21} = \frac{5}{7}$$

Substitution of $x = -\frac{1}{7}$ into the right-hand side of the original equation we find

$$\frac{-\frac{1}{7}+3}{4} = \frac{\frac{-1+21}{7}}{4} = \frac{20}{28} = \frac{5}{7}$$

We see that, with $x = -\frac{1}{7}$ both sides are equal and so the solution is correct.

Example

Solve $\dfrac{4-5x}{6} - \dfrac{1-2x}{3} = \dfrac{13}{42}$.

Solution

First of all remember that the division line means divide **all** of $4 - 5x$ by 6. So let's put in brackets to be absolutely clear:

$$\frac{(4-5x)}{6} - \frac{(1-2x)}{3} = \frac{13}{42}$$

Now we need a common denominator for the denominators 6, 3 and 42. Note that both 6 and 3 will divide into 42 so choose 42 as the common denominator. Multiply everything by 42. So we have

$$\frac{42(4-5x)}{6} - \frac{42(1-2x)}{3} = 42 \times \frac{13}{42}$$

Simplifying each term

$$\frac{{}^{7}\cancel{42}(4-5x)}{\cancel{6}_1} - \frac{{}^{14}\cancel{42}(1-2x)}{\cancel{3}_1} = {}^{1}\cancel{42} \times \frac{13}{\cancel{42}_1}$$

Now multiply out the brackets and simplify the left-hand side.

$$28 - 35x - 14 + 28x = 13$$

From which

$$\begin{aligned} 14 - 7x &= 13 \\ -7x &= -1 \\ x &= \frac{-1}{-7} \\ &= \frac{1}{7} \end{aligned}$$

So, the solution is $x = \dfrac{1}{7}$. Again this should be checked.

Substitution of $x = \dfrac{1}{7}$ into the left-hand side of the original equation gives

$$\frac{4 - \frac{5}{7}}{6} - \frac{1 - \frac{2}{7}}{3}$$

Simplifying we find this equals

$$\frac{28-5}{42} - \frac{7-2}{21} = \frac{23}{42} - \frac{5}{21} = \frac{23}{42} - \frac{10}{42} = \frac{13}{42}$$

and since this is the same as the right-hand side of the original equation the solution is correct.

Exercises

3. Solve the equations

a) $5 + \dfrac{x}{3} = 7$ b) $\dfrac{1}{2}x - 1 = 5$ c) $\dfrac{3}{4}x - 2 = \dfrac{1}{3}x + 3$

d) $4 - \dfrac{2}{3}x = \dfrac{x-6}{5}$ e) $\dfrac{x+2}{3} = \dfrac{1-2x}{5}$ f) $\dfrac{5x+1}{2} - \dfrac{x-2}{6} = \dfrac{2x+4}{3}$

5. Another form of a linear equation in one variable

In this final section we have a look at some equations which at first sight appear not to be linear equations. However, with some algebraic manipulation they can be recast in a more familiar form.

Example

Solve $\dfrac{3}{5} = \dfrac{6}{x}$.

Solution

Again, we need a common denominator. We need a quantity that will be divisible by 5 and by x. The obvious choice is $5x$. So let's multiply both sides by $5x$ and simplify.

$$^1\cancel{5}x \times \frac{3}{\cancel{5}_1} = 5\cancel{x}^1 \times \frac{6}{\cancel{x}_1}$$

And so

$$3x = 30$$

Finally x must be equal to 10.

We now look at another way of solving this equation:

$$\frac{3}{5} = \frac{6}{x}$$

If two fractions are equal, they are also equal if we invert them.

$$\frac{5}{3} = \frac{x}{6}$$

This makes it easier still because all we need to do now is multiply by the common denominator and we can see what the common denominator is. It's quite clearly 6. Multiply by the common denominator of 6 and simplify the result.

$$^{2}\cancel{6} \times \frac{5}{\cancel{3}_1} = {}^{1}\cancel{6} \times \frac{x}{\cancel{6}_1}$$

so that

$$10 = x$$

from which $x = 10$ as before.

Example

Solve $\dfrac{5}{3x} = \dfrac{25}{27}$.

Solution

We will tackle this by inverting each fraction.

$$\frac{3x}{5} = \frac{27}{25}$$

There is now a common denominator of 25. Multiply both sides by 25 and simplify to get

$$\begin{aligned} 15x &= 27 \\ x &= \frac{27}{15} \\ &= \frac{9}{5} \end{aligned}$$

The answer can be left like this or written as the mixed number $1\frac{4}{5}$.

Now some of you may not like the idea of flipping over the fractions. So let's tackle this in another way. So again, to solve

$$\frac{5}{3x} = \frac{25}{27}$$

Let's look for a common denominator between $3x$ and 27. So we want something $3x$ will divide into exactly and something that 27 will divide into exactly. Such a quantity is $27x$. So that's going to be our common denominator. Multiply both sides by $27x$.

$$27x \times \frac{5}{3x} = 27x \times \frac{25}{27}$$

So

$$45 = 25x$$

from which

$$x = \frac{45}{25} = \frac{9}{5}$$

Example

Solve $\dfrac{19x}{7} = \dfrac{57}{49}$.

Solution

The common denominator of 7 and 49 is 49. Multiplying both sides by 49 and simplifying:

$$49 \times \frac{19x}{7} = 49 \times \frac{57}{49}$$

So

$$7 \times 19x = 57$$

and dividing each side by 19:

$$7x = 3$$

which means

$$x = \frac{3}{7}$$

The important thing in dealing with these kind of equations and any kind of equations is to remember that the equals sign represents a balance. What it tells you is that what's on the left-hand side is exactly equal to what's on the right-hand side. So whatever you do to one side you **have** to do to the other side and you must follow the rules of arithmetic when you do it.

Exercises

4. Solve the following equations.

a)	$6x + 2 = 29 - 3x$	b)	$\frac{1}{3}x + 4 = \frac{4x - 1}{5}$	c)	$\frac{3x}{4} = \frac{2}{5}$
d)	$\frac{8}{x} = 2$	e)	$\frac{7}{3x} = 2$	f)	$\frac{3}{x + 1} = \frac{6}{5x - 1}$

Answers

1.

a)	4	b)	5	c)	3/5
d)	-2	e)	3	f)	2
g)	2	h)	-1	i)	4

2.

a)	2	b)	$-2/3$	c)	-1

3.

a)	6	b)	12	c)	12
d)	6	e)	$-7/11$	f)	3/10

4.

a)	3	b)	9	c)	8/15
d)	4	e)	7/6	f)	1