

# Quadratic Equations

This unit is about the solution of quadratic equations. These take the form  $ax^2 + bx + c = 0$ . We will look at four methods: solution by factorisation, solution by completing the square, solution using a formula, and solution using graphs

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- solve quadratic equations by factorisation
- solve quadratic equations by completing the square
- solve quadratic equations using a formula
- solve quadratic equations by drawing graphs

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# 1. Introduction

This unit is about how to solve quadratic equations. A **quadratic equation** is one which must contain a term involving  $x^2$ , e.g.  $3x^2$ ,  $-5x^2$  or just  $x^2$  on its own. It may also contain terms involving  $x$ , e.g.  $5x$  or  $-7x$ , or  $0.5x$ . It can also have constant terms - these are just numbers:  $6$ ,  $-7$ ,  $\frac{1}{2}$ .

It cannot have terms involving higher powers of  $x$ , like  $x^3$ . It cannot have terms like  $\frac{1}{x}$  in it.

In general a quadratic equation will take the form

$$ax^2 + bx + c = 0$$

$a$  can be any number excluding zero.  $b$  and  $c$  can be any numbers including zero. If  $b$  or  $c$  is zero then these terms will not appear.



## Key Point

A quadratic equation takes the form

$$ax^2 + bx + c = 0$$

where  $a$ ,  $b$  and  $c$  are numbers. The number  $a$  cannot be zero.

In this unit we will look at how to solve quadratic equations using four methods:

- solution by factorisation
- solution by completing the square
- solution using a formula
- solution using graphs

Factorisation and use of the formula are particularly important.

## 2. Solving quadratic equations by factorisation

In this section we will assume that you already know how to factorise a quadratic expression. If this is not the case you can study other material in this series where factorisation is explained.

### Example

Suppose we wish to solve  $3x^2 = 27$ .

We begin by writing this in the standard form of a quadratic equation by subtracting 27 from each side to give  $3x^2 - 27 = 0$ .

We now look for common factors. By observation there is a common factor of 3 in both terms. This factor is extracted and written outside a pair of brackets. The contents of the brackets are adjusted accordingly:

$$3x^2 - 27 = 3(x^2 - 9) = 0$$

Notice here the difference of two squares which can be factorised as

$$3(x^2 - 9) = 3(x - 3)(x + 3) = 0$$

If two quantities are multiplied together and the result is zero then either or both of the quantities must be zero. So either

$$x - 3 = 0 \quad \text{or} \quad x + 3 = 0$$

so that

$$x = 3 \quad \text{or} \quad x = -3$$

These are the two solutions of the equation.

### Example

Suppose we wish to solve  $5x^2 + 3x = 0$ .

We look to see if we can spot any common factors. There is a common factor of  $x$  in both terms. This is extracted and written in front of a pair of brackets:

$$x(5x + 3) = 0$$

Then either  $x = 0$  or  $5x + 3 = 0$  from which  $x = -\frac{3}{5}$ . These are the two solutions.

In this example there is no constant term. A common error that students make is to cancel the common factor of  $x$  in the original equation:

$$5x^{\cancel{2}} + 3x = 0 \quad \text{so that} \quad 5x + 3 = 0 \quad \text{giving} \quad x = -\frac{3}{5}$$

But if we do this we lose the solution  $x = 0$ . In general, when solving quadratic equations we are looking for two solutions.

### Example

Suppose we wish to solve  $x^2 - 5x + 6 = 0$ .

We factorise the quadratic by looking for two numbers which multiply together to give 6, and add to give  $-5$ . Now

$$-3 \times -2 = 6 \quad -3 + -2 = -5$$

so the two numbers are  $-3$  and  $-2$ . We use these two numbers to write  $-5x$  as  $-3x - 2x$  and proceed to factorise as follows:

$$\begin{aligned} x^2 - 5x + 6 &= 0 \\ x^2 - 3x - 2x + 6 &= 0 \\ x(x - 3) - 2(x - 3) &= 0 \\ (x - 3)(x - 2) &= 0 \end{aligned}$$

from which

$$x - 3 = 0 \quad \text{or} \quad x - 2 = 0$$

so that

$$x = 3 \quad \text{or} \quad x = 2$$

These are the two solutions.

### Example

Suppose we wish to solve the equation  $2x^2 + 3x - 2 = 0$ .

To factorise this we seek two numbers which multiply to give  $-4$  (the coefficient of  $x^2$  multiplied by the constant term) and which add together to give  $3$ .

$$4 \times -1 = -4 \quad 4 + -1 = 3$$

so the two numbers are  $4$  and  $-1$ . We use these two numbers to write  $3x$  as  $4x - x$  and then factorise as follows:

$$\begin{aligned} 2x^2 + 3x - 2 &= 0 \\ 2x^2 + 4x - x - 2 &= 0 \\ 2x(x + 2) - (x + 2) &= 0 \\ (x + 2)(2x - 1) &= 0 \end{aligned}$$

from which

$$x + 2 = 0 \quad \text{or} \quad 2x - 1 = 0$$

so that

$$x = -2 \quad \text{or} \quad x = \frac{1}{2}$$

These are the two solutions.

### Example

Suppose we wish to solve  $4x^2 + 9 = 12x$ .

First of all we write this in the standard form:

$$4x^2 - 12x + 9 = 0$$

We should look to see if there is a common factor - but there is not. To factorise we seek two numbers which multiply to give  $36$  (the coefficient of  $x^2$  multiplied by the constant term) and add to give  $-12$ . Now, by inspection,

$$-6 \times -6 = 36 \quad -6 + -6 = -12$$

so the two numbers are  $-6$  and  $-6$ . We use these two numbers to write  $-12x$  as  $-6x - 6x$  and proceed to factorise as follows:

$$\begin{aligned} 4x^2 - 12x + 9 &= 0 \\ 4x^2 - 6x - 6x + 9 &= 0 \\ 2x(2x - 3) - 3(2x - 3) &= 0 \\ (2x - 3)(2x - 3) &= 0 \end{aligned}$$

from which

$$2x - 3 = 0 \quad \text{or} \quad 2x - 3 = 0$$

so that

$$x = \frac{3}{2} \quad \text{or} \quad x = \frac{3}{2}$$

These are the two solutions, but we have obtained the same answer twice. So we can have quadratic equations for which the solution is repeated.

### Example

Suppose we wish to solve  $x^2 - 3x - 2 = 0$ .

We are looking for two numbers which multiply to give  $-2$  and add together to give  $-3$ . Never mind how hard you try you will not find any such two numbers. So this equation will not factorise. We need another approach. This is the topic of the next section.

### Exercise 1

Use factorisation to solve the following quadratic equations

- a)  $x^2 - 3x + 2 = 0$     b)  $5x^2 = 20$     c)  $x^2 - 5 = 4x$     d)  $2x^2 = 10x$   
 e)  $x^2 + 19x + 60 = 0$     f)  $2x^2 + x - 6 = 0$     g)  $2x^2 - x - 6 = 0$     h)  $4x^2 = 11x - 6$

## 3. Solving quadratic equations by completing the square

### Example

Suppose we wish to solve  $x^2 - 3x - 2 = 0$ .

In order to complete the square we look at the first two terms, and try to write them in the form  $(\quad)^2$ . Clearly we need an  $x$  in the brackets:

$(x + ?)^2$  because when the term in brackets is squared this will give the term  $x^2$

We also need the number  $-\frac{3}{2}$ , which is half of the coefficient of  $x$  in the quadratic equation,

$\left(x - \frac{3}{2}\right)^2$  because when the term in brackets is squared this will give the term  $-3x$

However, removing the brackets from  $\left(x - \frac{3}{2}\right)^2$  we see there is also a term  $\left(-\frac{3}{2}\right)^2$  which we do not want, and so we subtract this again. So the quadratic equation can be written

$$x^2 - 3x - 2 = \left(x - \frac{3}{2}\right)^2 - \left(-\frac{3}{2}\right)^2 - 2 = 0$$

Simplifying

$$\begin{aligned} \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - 2 &= 0 \\ \left(x - \frac{3}{2}\right)^2 - \frac{17}{4} &= 0 \\ \left(x - \frac{3}{2}\right)^2 &= \frac{17}{4} \\ x - \frac{3}{2} &= \frac{\sqrt{17}}{2} \quad \text{or} \quad -\frac{\sqrt{17}}{2} \\ x &= \frac{3}{2} + \frac{\sqrt{17}}{2} \quad \text{or} \quad x = \frac{3}{2} - \frac{\sqrt{17}}{2} \end{aligned}$$

We can write these solutions as

$$x = \frac{3 + \sqrt{17}}{2} \quad \text{or} \quad \frac{3 - \sqrt{17}}{2}$$

Again we have two answers. These are exact answers. Approximate values can be obtained using a calculator.

## Exercise 2

a) Show that  $x^2 + 2x = (x + 1)^2 - 1$ .

Hence, use completing the square to solve  $x^2 + 2x - 3 = 0$ .

b) Show that  $x^2 - 6x = (x - 3)^2 - 9$ .

Hence use completing the square to solve  $x^2 - 6x = 5$ .

c) Use completing the square to solve  $x^2 - 5x + 1 = 0$ .

d) Use completing the square to solve  $x^2 + 8x + 4 = 0$ .

## 4. Solving quadratic equations using a formula

Consider the general quadratic equation  $ax^2 + bx + c = 0$ .

There is a formula for solving this:  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ . It is so important that you should learn it.



### Key Point

Formula for solving  $ax^2 + bx + c = 0$ :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We will illustrate the use of this formula in the following example.

### Example

Suppose we wish to solve  $x^2 - 3x - 2 = 0$ .

Comparing this with the general form  $ax^2 + bx + c = 0$  we see that  $a = 1$ ,  $b = -3$  and  $c = -2$ . These values are substituted into the formula.

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\&= \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 1 \times (-2)}}{2 \times 1} \\&= \frac{3 \pm \sqrt{9 + 8}}{2} \\&= \frac{3 \pm \sqrt{17}}{2}\end{aligned}$$

These solutions are exact.

### Example

Suppose we wish to solve  $3x^2 = 5x - 1$ .

First we write this in the standard form as  $3x^2 - 5x + 1 = 0$  in order to identify the values of  $a$ ,  $b$  and  $c$ .

We see that  $a = 3$ ,  $b = -5$  and  $c = 1$ . These values are substituted into the formula.

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\&= \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 3 \times 1}}{2 \times 3} \\&= \frac{5 \pm \sqrt{25 - 12}}{6} \\&= \frac{5 \pm \sqrt{13}}{6}\end{aligned}$$

Again there are two exact solutions. Approximate values could be obtained using a calculator.

### Exercise 3

Use the quadratic formula to solve the following quadratic equations.

- a)  $x^2 - 3x + 2 = 0$    b)  $4x^2 - 11x + 6 = 0$    c)  $x^2 - 5x - 2 = 0$    d)  $3x^2 + 12x + 2 = 0$   
e)  $2x^2 = 3x + 1$    f)  $x^2 + 3 = 2x$    g)  $x^2 + 4x = 10$    h)  $25x^2 = 40x - 16$

## 5. Solving quadratic equations by using graphs

In this section we will see how graphs can be used to solve quadratic equations. If the coefficient of  $x^2$  in the quadratic expression  $ax^2 + bx + c$  is positive then a graph of  $y = ax^2 + bx + c$  will take the form shown in Figure 1(a). If the coefficient of  $x^2$  is negative the graph will take the form shown in Figure 1(b).

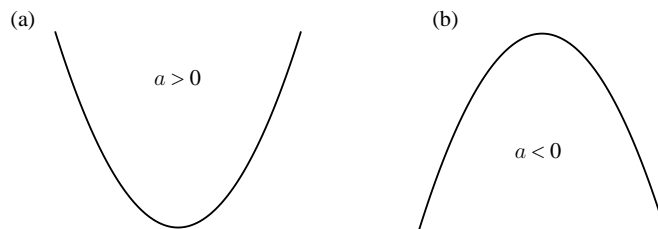


Figure 1. Graphs of  $y = ax^2 + bx + c$  have these general shapes

We will now add  $x$  and  $y$  axes. Figure 2 shows what can happen when we plot a graph of  $y = ax^2 + bx + c$  for the case in which  $a$  is positive.

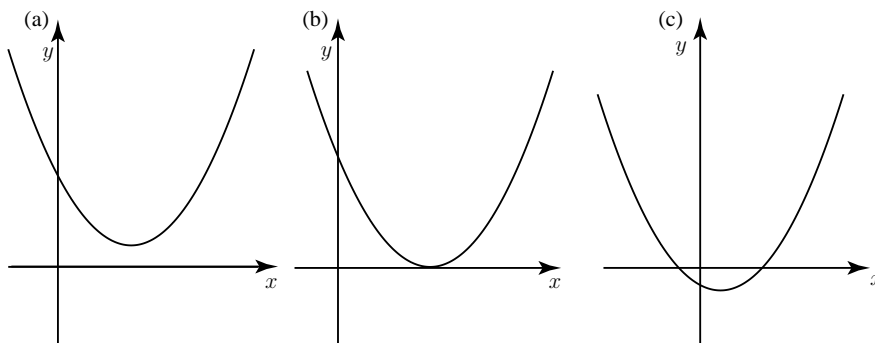


Figure 2. Graphs of  $y = ax^2 + bx + c$  when  $a$  is positive

The horizontal line, the  $x$  axis, corresponds to points on the graph where  $y = 0$ . So points where the graph touches or crosses this axis correspond to solutions of  $ax^2 + bx + c = 0$ .

In Figure 2, the graph in (a) never cuts or touches the horizontal axis and so this corresponds to a quadratic equation  $ax^2 + bx + c = 0$  having no real roots.

The graph in (b) just touches the horizontal axis corresponding to the case in which the quadratic equation has two equal roots, also called ‘repeated roots’.

The graph in (c) cuts the horizontal axis twice, corresponding to the case in which the quadratic equation has two different roots.

What we have done in Figure 2 for the the case in which  $a$  is positive we can do for the case in which  $a$  is negative. This case is shown in Figure 3.

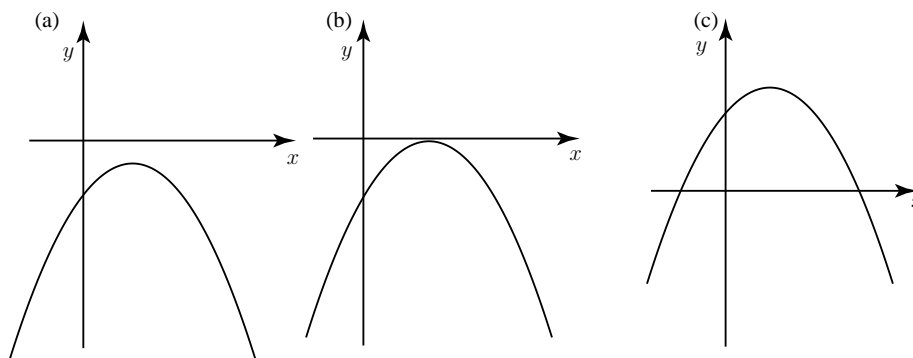


Figure 3. Graphs of  $y = ax^2 + bx + c$  when  $a$  is negative

Referring to Figure 3: in case (a) there are no real roots. In case (b) there will be repeated roots. Case (c) corresponds to there being two real roots.

### Example

Suppose we wish to solve  $x^2 - 3x - 2 = 0$ .

We consider  $y = x^2 - 3x - 2$  and produce a table of values so that we can plot a graph.

$x$	-2	-1	0	1	2	3	4	5
$x^2$	4	1	0	1	4	9	16	25
$-3x$	6	3	0	-3	-6	-9	-12	-15
$-2$	-2	-2	-2	-2	-2	-2	-2	-2
$x^2 - 3x - 2$	8	2	-2	-4	-4	-2	2	8

From this table of values a graph can be plotted, or sketched as shown in Figure 4. From the graph we observe that solutions of the equation  $x^2 - 3x - 2 = 0$  lie between  $-1$  and  $0$ , and between  $3$  and  $4$ .

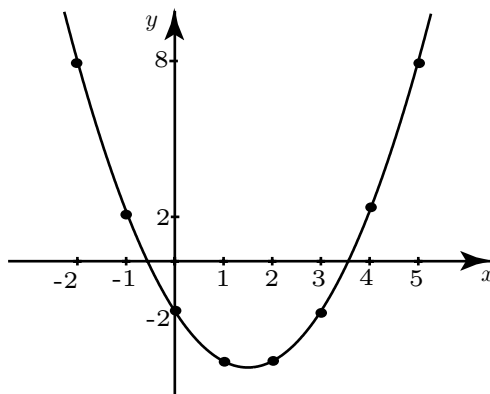


Figure 4. Graph of  $y = x^2 - 3x - 2$



**Example**

We can use the same graph to solve other equations. For example to solve  $x^2 - 3x - 2 = 6$  we can simply locate points where the graph crosses the line  $y = 6$  as shown in Figure 5.

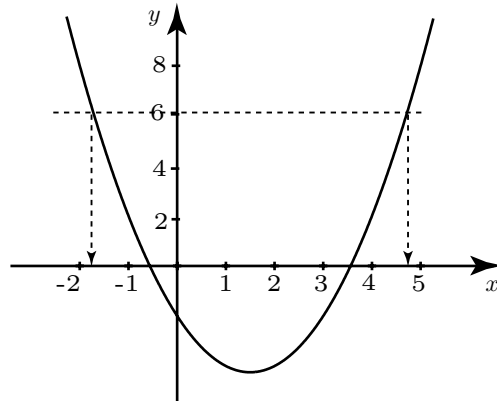


Figure 5. Using the graph of  $y = x^2 - 3x - 2$  to solve  $x^2 - 3x - 2 = 6$

**Example**

We can use the same graph to solve  $x^2 - 3x - 5 = 0$  by rewriting the equation as  $x^2 - 3x - 2 - 3 = 0$  and then as  $x^2 - 3x - 2 = 3$ . We can then locate points where the graph crosses the line  $y = 3$  in order to solve the equation.

**Exercise 4**

By plotting the graph  $y = x^2 - 5x + 2$ , solve the equation  $x^2 - 5x + 2 = 0$ , giving your answers to 1 decimal place.

Use your graph to solve the equations  $x^2 - 5x + 2 = 4$ ,  $x^2 - 5x - 1 = 0$ ,  $x^2 - 5x + 2 = 2x$ .

**Answers****Exercise 1**

a) 1, 2   b) 2, -2   c) 5, -1   d) 0, 5   e) -4, -15   f)  $-2, \frac{3}{2}$    g)  $2, -\frac{3}{2}$    h)  $2, \frac{3}{4}$

**Exercise 2**

a) 1, -3   b)  $3 \pm \sqrt{14}$    c)  $\frac{5 \pm \sqrt{21}}{2}$    d)  $-4 \pm \sqrt{12}$

**Exercise 3**

a) 1, 2   b)  $2, \frac{3}{4}$    c)  $\frac{5 + \sqrt{33}}{2}$    d)  $\frac{-12 \pm \sqrt{120}}{6}$    e)  $\frac{3 \pm \sqrt{17}}{4}$    f) No real roots  
g)  $-2 \pm \sqrt{14}$    h)  $\frac{4}{5}$  repeated.

**Exercise 4**

a) 4.6, 0.4   b) 5.4, -0.4   c) 5.2, -0.2   d) 6.7, 0.3