Indices or Powers

A knowledge of powers, or indices as they are often called, is essential for an understanding of most algebraic processes. In this section of text you will learn about powers and rules for manipulating them through a number of worked examples.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- simplify expressions involving indices
- use the rules of indices to simplify expressions involving indices
- use negative and fractional indices.

Contents

1. Introduction

In the section we will be looking at **indices** or **powers**. Either name can be used, and both names mean the same thing.

Basically, they are a shorthand way of writing multiplications of the *same* number.
So, suppose we have

$$4 \times 4 \times 4$$

We write this as '4 to the power 3':

$$4^3$$

So

$$4 \times 4 \times 4 = 4^3$$

The number 3 is called the **power** or **index**. Note that the plural of index is indices.

Key Point

An index, or power, is used to show that a quantity is repeatedly multiplied by itself.

This can be done with letters as well as numbers. So, we might have:

$$a \times a \times a \times a \times a$$

Since there are five a's multiplied together we write this as 'a to the power 5'.

$$a^5$$

So

$$a \times a \times a \times a \times a = a^5.$$

What if we had $2x^2$ raised to the power 4 ? This means four factors of $2x^2$ multiplied together, that is,

$$2x^2 \times 2x^2 \times 2x^2 \times 2x^2$$

This can be written

$$2 \times 2 \times 2 \times 2 \times x^2 \times x^2 \times x^2 \times x^2$$

which we will see shortly can be written as $16x^8$.

Use of a power or index is simply a form of notation, that is, a way of writing something down. When mathematicians have a way of writing things down they like to use their notation in other ways. For example, what might we mean by

$$a^{-2} \quad \text{or} \quad a^{\frac{1}{2}} \quad \text{or} \quad a^0 \qquad ?$$

To proceed further we need **rules** to operate with so we can find out what these notations actually mean.

Exercises

1. Evaluate each of the following.

   a) 3^5 b) 7^3 c) 2^9

   d) 5^3 e) 4^4 f) 8^3

2. The first rule

Suppose we have a^3 and we want to multiply it by a^2. That is

$$a^3 \times a^2 \quad = \quad a \times a \times a \quad \times \quad a \times a$$

Altogether there are five a's multiplied together. Clearly, this is the same as a^5. This suggests our first rule.

The first rule tells us that if we are multiplying expressions such as these then we add the indices together. So, if we have

$$a^m \times a^n$$

we add the indices to get

$$a^m \times a^n = a^{m+n}$$

Key Point

$$a^m \times a^n = a^{m+n}$$

3. The second rule

Suppose we had a^4 and we want to raise it all to the power 3. That is

$$(a^4)^3$$

This means

$$a^4 \times a^4 \times a^4$$

Now our first rule tells us that we should add the indices together. So that is

$$a^{12}$$

But note also that 12 is 4×3. This suggests that if we have a^m all raised to the power n the result is obtained by multiplying the two powers to get $a^{m\times n}$, or simply a^{mn}.

Key Point

$$(a^m)^n = a^{mn}$$

4. The third rule

Consider dividing a^7 by a^3.

$$a^7 \div a^3 = \frac{a^7}{a^3} = \frac{a \times a \times a \times a \times a \times a \times a}{a \times a \times a}$$

We can now begin dividing out the common factors of a. Three of the a's at the top and the three a's at the bottom can be divided out, so we are now left with

$$\frac{a^4}{1} \qquad \text{that is} \qquad a^4$$

The same answer is obtained by subtracting the indices, that is, $7 - 3 = 4$. This suggests our third rule, that $a^m \div a^n = a^{m-n}$.

Key Point

$$a^m \div a^n = a^{m-n}$$

5. What can we do with these rules ? The fourth rule

Let's have a look at a^3 divided by a^3. We know the answer to this. We are dividing a quantity by itself, so the answer has got to be 1.

$$a^3 \div a^3 = 1$$

Let's do this using our rules; rule 3 will help us do this. Rule 3 tells us that to divide the two quantities we subtract the indices:

$$a^3 \div a^3 = a^{3-3} = a^0$$

We appear to have obtained a different answer. We have done the same calculation in two different ways. We have done it correctly in two different ways. So the answers we get, even if they look different, must be the same. So, what we have is $a^0 = 1$.

Key Point

$$a^0 = 1$$

This means that any number raised to the power zero is 1. So

$$2^0 = 1 \qquad (1,000,000)^0 = 1 \qquad \left(\frac{1}{2}\right)^0 = 1 \qquad (-6)^0 = 1$$

However, note that zero itself is an exception to this rule. 0^0 cannot be evaluated. Any number, apart from zero, when raised to the power zero is equal to 1.

6. The fifth rule

Let's have a look now at doing a division again.
Consider a^3 divided by a^7.

$$a^3 \div a^7 = \frac{a^3}{a^7} = \frac{a \times a \times a}{a \times a \times a \times a \times a \times a \times a}$$

Again, we can now begin dividing out the common factors of a. The 3 a's at the top and three of the a's at the bottom can be divided out, so we are now left with

$$a^3 \div a^7 = \frac{1}{a \times a \times a \times a} = \frac{1}{a^4}$$

Now let's use our third rule and do the same calculation by subtracting the indices.

$$a^3 \div a^7 = a^{3-7} = a^{-4}$$

We have done the same calculation in two different ways. We have done it correctly in two different ways. So the answers we get, even if they look different, must be the same. So

$$\frac{1}{a^4} = a^{-4}$$

So a negative sign in the index can be thought of as meaning '1 over'.

Key Point

$$a^{-1} = \frac{1}{a} \qquad \text{and more generally} \qquad a^{-m} = \frac{1}{a^m}$$

Now let's develop this further in the following examples.

In the next two examples we start with an expression which has a negative index, and rewrite it so that it has a positive index, using the rule $a^{-m} = \dfrac{1}{a^m}$.

Examples

$$2^{-2} = \frac{1}{2^2} = \frac{1}{4} \qquad 5^{-1} = \frac{1}{5^1} = \frac{1}{5}$$

We can reverse the process in order to rewrite quantities so that they have a negative index.

Examples

$$\frac{1}{a} = \frac{1}{a^1} = a^{-1} \qquad\qquad \frac{1}{7^2} = 7^{-2}$$

One you should try to remember is $\dfrac{1}{a} = a^{-1}$ as you will probably use it the most.

But now what about an example like $\dfrac{1}{7^{-2}}$. Using the Example above, we see that this means $\dfrac{1}{1/7^2}$. Here we are dividing by a fraction, and to divide by a fraction we need to invert and multiply so:

$$\frac{1}{7^{-2}} = \frac{1}{1/7^2} = 1 \div \frac{1}{7^2} = 1 \times \frac{7^2}{1} = 7^2$$

This illustrates another way of writing the previous keypoint:

Key Point

$$\frac{1}{a^{-m}} = a^m$$

Exercises

2. Evaluate each of the following leaving your answer as a proper fraction.

a) 2^{-9} b) 3^{-5} c) 4^{-4}

d) 5^{-3} e) 7^{-3} f) 8^{-3}

7. The sixth rule

So far we have dealt with integer powers both positive and negative. What would we do if we had a fraction for a power, like $a^{\frac{1}{2}}$. To see how to deal with fractional powers consider the following:

Suppose we have two identical numbers multiplying together to give another number, as in, for example

$$7 \times 7 = 49$$

Then we know that 7 is a square root of 49. That is, if

$$7^2 = 49 \qquad \text{then } 7 = \sqrt{49}$$

Now suppose we found that

$$a^p \times a^p = a$$

That is, when we multiplied a^p by itself we got the result a. This means that a^p must be a square root of a.

However, look at this another way: noting that $a = a^1$, and also that, from the first rule, $a^p \times a^p = a^{2p}$ we see that if $a^p \times a^p = a$ then

$$a^{2p} = a^1$$

from which

$$2p = 1$$

and so

$$p = \frac{1}{2}$$

This shows that $a^{1/2}$ must be the square root of a. That is

$$a^{\frac{1}{2}} = \sqrt{a}$$

Key Point

the power 1/2 denotes a square root: $a^{\frac{1}{2}} = \sqrt{a}$

Similarly

$$a^{\frac{1}{3}} = \sqrt[3]{a} \qquad \text{this is the cube root of } a$$

and

$$a^{\frac{1}{4}} = \sqrt[4]{a} \qquad \text{this is the fourth root of } a$$

More generally,

Key Point

$$a^{\frac{1}{q}} = \sqrt[q]{a}$$

Work through the following examples:

Example
What do we mean by $16^{1/4}$?

For this we need to know what number when multiplied together four times gives 16. The answer is 2. So $16^{1/4} = 2$.

Example

What do we mean by $81^{1/2}$? For this we need to know what number when multiplied by itself gives 81. The answer is 9. So $81^{\frac{1}{2}} = \sqrt{81} = 9$.

Example

What about $243^{\frac{1}{5}}$? What number when multiplied together five times gives us 243 ? If we are familiar with times-tables we might spot that $243 = 3 \times 81$, and also that $81 = 9 \times 9$. So

$$243^{1/5} = (3 \times 81)^{1/5} = (3 \times 9 \times 9)^{1/5} = (3 \times 3 \times 3 \times 3 \times 3)^{1/5}$$

So 3 multiplied by itself five times equals 243. Hence

$$243^{1/5} = 3$$

Notice in doing this how important it is to be able to recognise what factors numbers are made up of. For example, it is important to be able to recognise that:

$$16 = 2^4, \qquad 16 = 4^2, \qquad 81 = 9^2, \qquad 81 = 3^4 \qquad \text{and so on.}$$

You will find calculations much easier if you can recognise in numbers their composition as powers of simple numbers such as 2, 3, 4 and 5. Once you have got these firmly fixed in your mind, this sort of calculation becomes straightforward.

Exercises

3. Evaluate each of the following.

a)	$125^{1/3}$	b)	$243^{1/5}$	c)	$256^{1/4}$
d)	$512^{1/9}$	e)	$343^{1/3}$	f)	$512^{1/3}$

8. A final result

What happens if we take $a^{\frac{3}{4}}$?
We can write this as follows:

$$a^{\frac{3}{4}} = (a^{\frac{1}{4}})^3 \qquad \text{using the 2nd rule } (a^m)^n = a^{mn}$$

Example
What do we mean by $16^{\frac{3}{4}}$?

$$\begin{aligned} 16^{\frac{3}{4}} &= (16^{\frac{1}{4}})^3 \\ &= (2)^3 \\ &= 8 \end{aligned}$$

We can also think of this calculation performed in a slightly different way. Note that instead of writing $(a^m)^n = a^{mn}$ we could write $(a^n)^m = a^{mn}$ because mn is the same as nm.

Example

What do we mean by $8^{\frac{2}{3}}$? One way of calculating this is to write

$$\begin{aligned} 8^{\frac{2}{3}} &= (8^{\frac{1}{3}})^2 \\ &= (2)^2 \\ &= 4 \end{aligned}$$

Alternatively,

$$\begin{aligned} 8^{\frac{2}{3}} &= (8^2)^{\frac{1}{3}} \\ &= (64)^{\frac{1}{3}} \\ &= 4 \end{aligned}$$

Additional note

Doing this calculation the first way is usually easier as it requires recognising powers of smaller numbers. For example, it is straightforward to evaluate $27^{5/3}$ as

$$27^{5/3} = (27^{1/3})^5 = 3^5 = 243$$

because, at least with practice, you will know that the cube root of 27 is 3. Whereas, evaluation in the following way

$$27^{5/3} = (27^5)^{1/3} = 14348907^{1/3}$$

would require knowledge of the cube root of 14348907.

Writing these results down algebraically we have the following important point:

Key Point

$$a^{\frac{p}{q}} = (a^p)^{\frac{1}{q}} = \sqrt[q]{a^p}$$

$$a^{\frac{p}{q}} = (a^{\frac{1}{q}})^p = (\sqrt[q]{a})^p$$

Both results are exactly the same.

Exercises

4. Evaluate each of the following.

a)	$343^{2/3}$	b)	$512^{2/3}$	c)	$256^{3/4}$
d)	$125^{4/3}$	e)	$512^{7/9}$	f)	$243^{6/5}$

5. Evaluate each of the following.

a)	$512^{-7/9}$	b)	$243^{-6/5}$	c)	$256^{-3/4}$
d)	$125^{-4/3}$	e)	$343^{-2/3}$	f)	$512^{-2/3}$

9. Further examples

The remainder of this unit provides examples illustrating the use of the rules of indices.

Example

Write $2x^{-\frac{1}{4}}$ using a positive index.

$$2x^{-\frac{1}{4}} = 2 \times \frac{1}{x^{\frac{1}{4}}} = \frac{2}{x^{\frac{1}{4}}}$$

Example

Write $4x^{-2}a^3$ using positive indices.

$$4x^{-2}a^3 = 4 \times \frac{1}{x^2} \times a^3 = \frac{4a^3}{x^2}$$

Example

Write $\frac{1}{4a^{-2}}$ using a positive index.

$$\frac{1}{4a^{-2}} = \frac{1}{4} \times \frac{1}{a^{-2}} = \frac{1}{4} \times a^2 = \frac{a^2}{4}$$

Example

Simplify $a^{-\frac{1}{3}} \times 2a^{-\frac{1}{2}}$.

$$\begin{aligned} a^{-\frac{1}{3}} \times 2a^{-\frac{1}{2}} &= 2a^{-\frac{1}{3}} \times a^{-\frac{1}{2}} \\ &= 2a^{-\frac{5}{6}} \qquad \text{adding the indices} \\ &= 2 \times \frac{1}{a^{\frac{5}{6}}} \\ &= \frac{2}{a^{\frac{5}{6}}} \end{aligned}$$

Example

Simplify $\frac{2a^{-2}}{a^{-\frac{3}{2}}}$.

$$\begin{aligned}\frac{2a^{-2}}{a^{-\frac{3}{2}}} &= 2a^{-2} \quad \div \quad a^{-\frac{3}{2}} \\ &= 2a^{-2-(-3/2)} \qquad \text{subtracting the indices} \\ &= 2a^{-\frac{1}{2}} \\ &= \frac{2}{a^{\frac{1}{2}}}\end{aligned}$$

Example

Simplify $\sqrt[3]{a^2} \ \times \ \sqrt[2]{a^3}$.

$$\begin{aligned}\sqrt[3]{a^2} \ \times \ \sqrt[2]{a^3} &= a^{\frac{2}{3}} \times a^{\frac{3}{2}} \\ &= a^{\frac{13}{6}} \qquad \text{by adding the indices}\end{aligned}$$

Example

Simplify $16^{\frac{3}{4}}$.

$$16^{\frac{3}{4}} = (16^{\frac{1}{4}})^3 = 2^3 = 8$$

Example

Simplify $4^{-\frac{5}{2}}$.

$$4^{-\frac{5}{2}} = \frac{1}{4^{\frac{5}{2}}} = \frac{1}{(4^{\frac{1}{2}})^5} = \frac{1}{2^5} = \frac{1}{32}$$

Example

Simplify $125^{\frac{2}{3}}$.

$$125^{\frac{2}{3}} = (125^{\frac{1}{3}})^2 = 5^2 = 25$$

Example

Simplify $8^{-\frac{2}{3}}$.

$$8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{(8^{\frac{1}{3}})^2} = \frac{1}{2^2} = \frac{1}{4}$$

Example

Simplify $\frac{1}{25^{-2}}$.

$$\frac{1}{25^{-2}} = 25^2 = 625$$

Example

Simplify $(243)^{\frac{3}{5}}$.

$$(243)^{\frac{3}{5}} = (243^{\frac{1}{5}})^3 = 3^3 = 27$$

Example

Simplify $\left(\dfrac{81}{16}\right)^{-\frac{3}{4}}$.

$$\begin{aligned}\left(\frac{81}{16}\right)^{-\frac{3}{4}} &= \frac{1}{\left(\frac{81}{16}\right)^{\frac{3}{4}}} \\ &= \left(\frac{16}{81}\right)^{\frac{3}{4}} \\ &= \left(\left(\frac{16}{81}\right)^{\frac{1}{4}}\right)^3 \\ &= \left(\frac{2}{3}\right)^3 \\ &= \frac{8}{27}\end{aligned}$$

Exercises

6. Evaluate each of the following.

   a) $\left(\frac{4}{9}\right)^2$ b) $\left(\frac{5}{7}\right)^3$ c) $\left(\frac{2}{3}\right)^6$

   d) $\left(\frac{8}{5}\right)^3$ e) $\left(\frac{5}{9}\right)^3$ f) $\left(\frac{4}{3}\right)^4$

7. Evaluate each of the following.

   a) $\left(\frac{4}{9}\right)^{-2}$ b) $\left(\frac{5}{7}\right)^{-3}$ c) $\left(\frac{2}{3}\right)^{-6}$

   d) $\left(\frac{8}{5}\right)^{-3}$ e) $\left(\frac{5}{9}\right)^{-3}$ f) $\left(\frac{4}{3}\right)^{-4}$

8. Evaluate each of the following.

   a) $\left(\frac{32}{243}\right)^{6/5}$ b) $\left(\frac{16}{81}\right)^{3/4}$ c) $\left(\frac{625}{256}\right)^{-1/4}$

   d) $\left(\frac{216}{343}\right)^{1/3}$ e) $\left(\frac{125}{512}\right)^{-2/3}$ f) $\left(\frac{125}{729}\right)^{2/3}$

9. Each of the following expressions can be written as a^n for some value of n. In each case determine the value of n.

   a) $a \times a \times a \times a$ b) $\frac{1}{a \times a \times a}$ c) 1

   d) $\sqrt[3]{a^5}$ e) $a^3 \times a^5$ f) $\frac{a^6}{a^2}$

   g) $(a^4)^2$ h) $\frac{a^2 \times a^5}{(a^3)^3}$ i) $\sqrt{a} \times \frac{1}{a^{-2}}$

   j) $a^{1/2} \times a^2$ k) $\frac{1}{a^{-3}} \times \frac{1}{a^{-2}}$ l) $\frac{1}{(a^{-2})^3}$

10. Simplify each of the following expressions giving your answer in the form Cx^n, where C and n are numbers.

a)	$3x^2 \times 2x^4$	b)	$5x \times 4x^5$	c)	$(2x^3)^4$
d)	$\frac{8x^6}{2x^3}$	e)	$\frac{3}{x^2} \times 4x^5$	f)	$12x^8 \times \frac{1}{3x^2}$
g)	$(5x^3)^{-1}$	h)	$(9x^4)^{1/2}$	i)	$2x^6 \times \frac{1}{4x^{-2}}$
j)	$2x^4 \times \frac{1}{x^5}$	k)	$(2x)^4 \times \frac{1}{x^5}$	l)	$6x^3 \times \frac{1}{(2x)^{-1}}$

Answers

1.

a)	243	b)	343	c)	512
d)	125	e)	256	f)	512

2.

a)	$\frac{1}{512}$	b)	$\frac{1}{243}$	c)	$\frac{1}{256}$
d)	$\frac{1}{125}$	e)	$\frac{1}{343}$	f)	$\frac{1}{512}$

3.

a)	5	b)	3	c)	4
d)	2	e)	7	f)	8

4.

a)	49	b)	64	c)	64
d)	625	e)	128	f)	729

5.

a)	$\frac{1}{128}$	b)	$\frac{1}{729}$	c)	$\frac{1}{64}$
d)	$\frac{1}{625}$	e)	$\frac{1}{49}$	f)	$\frac{1}{64}$

6.

a)	$\frac{16}{81}$	b)	$\frac{125}{343}$	c)	$\frac{64}{729}$
d)	$\frac{512}{125}$	e)	$\frac{125}{729}$	f)	$\frac{256}{81}$

7.

a)	$\frac{81}{16}$	b)	$\frac{343}{125}$	c)	$\frac{729}{64}$
d)	$\frac{125}{512}$	e)	$\frac{729}{125}$	f)	$\frac{81}{256}$

8.

a)	$\frac{64}{729}$	b)	$\frac{8}{27}$	c)	$\frac{4}{5}$
d)	$\frac{6}{7}$	e)	$\frac{64}{25}$	f)	$\frac{25}{81}$

9.

a)	4	b)	-3	c)	0
d)	$\frac{5}{3}$	e)	8	f)	4
g)	8	h)	-2	i)	$\frac{5}{2}$
j)	$\frac{5}{2}$	k)	5	l)	6

10.

a)	$6x^6$	b)	$20x^6$	c)	$16x^{12}$
d)	$4x^3$	e)	$12x^3$	f)	$4x^6$
g)	$\frac{1}{5}x^{-3}$	h)	$3x^2$	i)	$\frac{1}{2}x^8$
j)	$2x^{-1}$	k)	$16x^{-1}$	l)	$12x^4$