mathcentre

A Calculus Refresher

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Contents

Foreword

The material in this refresher course has been designed to enable you to cope better with your university mathematics programme. When your programme starts you will find that the ability to differentiate and integrate confidently will be invaluable. We think that this is so important that we are making this course available for you to work through either before you come to university, or during the early stages of your programme.

Preliminary work

You are advised to work through the companion booklet *An Algebra Refresher* before embarking upon this calculus revision course.

How to use this booklet

You are advised to work through each section in this booklet in order. You may need to revise some topics by looking at an AS-level or A-level textbook which contains information about differentiation and integration.

You should attempt a range of questions from each section, and check your answers with those at the back of the booklet. The more questions that you attempt, the more familiar you will become with these vital topics. We have left sufficient space in the booklet so that you can do any necessary working within it. So, treat this as a work-book.

If you get questions wrong you should revise the material and try again until you are getting the majority of questions correct.

If you cannot sort out your difficulties, do not worry about this. Your university will make provision to help you with your problems. This may take the form of special revision lectures, self-study revision material or a drop-in mathematics support centre.

Level

This material has been prepared for students who have completed an A-level course in mathematics

Reminders

Use this page to note topics and questions which you found difficult.

Seek help with these from your tutor or from other university support services as soon as possible.

Tables

The following tables of common derivatives and integrals are provided for revision purposes. It will be a great advantage to know these derivatives and integrals because they are required so frequently in mathematics courses.

Table of derivatives

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln kx$	$\frac{1}{x}$
e^{kx}	ke^{kx}
a^x	$a^x \ln a$
$\sin kx$	$k \cos kx$
$\cos kx$	$-k \sin kx$
$\tan kx$	$k \sec^2 kx$

Table of integrals

$f(x)$	$\int f(x)\,\mathrm{d}x$
$x^n \quad (n \neq -1)$	$\frac{x^{n+1}}{n+1} + c$
$x^{-1} = \frac{1}{x}$	$\ln\lvert x\rvert + c$
$e^{kx} \quad (k \neq 0)$	$\frac{e^{kx}}{k} + c$
$\sin kx \quad (k \neq 0)$	$-\frac{\cos kx}{k} + c$
$\cos kx \quad (k \neq 0)$	$\frac{\sin kx}{k} + c$
$\sec^2 kx \quad (k \neq 0)$	$\frac{\tan kx}{k} + c$

1. Derivatives of basic functions

Try to find all the derivatives in this section without referring to a table of derivatives. The derivatives of these functions occur so frequently that you should try to memorise the appropriate rules. If you are really stuck, consult the table on page 4.

1. Differentiate each of the following with respect to x.

(a) x (b) x^6 (c) 6 (d) $\sqrt{x}$ (e) x^{-1} (f) $x^{1/7}$

(g) $\dfrac{1}{x^3}$ (h) x^{79} (i) $x^{1.3}$ (j) $\dfrac{1}{\sqrt[3]{x}}$ (k) $x^{-5/3}$ (l) $\dfrac{1}{x^{0.71}}$

2. Differentiate each of the following with respect to θ.

(a) $\cos\theta$ (b) $\cos 4\theta$ (c) $\sin\theta$ (d) $\sin\dfrac{2\theta}{3}$ (e) $\tan\theta$ (f) $\tan\pi\theta$

(g) $\sin(-8\theta)$ (h) $\tan\dfrac{\theta}{4}$ (i) $\cos 3\pi\theta$ (j) $\cos\left(-\dfrac{5\theta}{2}\right)$ (k) $\sin 0.7\theta$

3. Find the following derivatives.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(e^x)$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}y}(e^{2y})$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}t}(e^{-7t})$ (d) $\dfrac{\mathrm{d}}{\mathrm{d}x}(e^{-x/3})$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}z}(e^{2z/\pi})$ (f) $\dfrac{\mathrm{d}}{\mathrm{d}x}(e^{-1.4x})$ (g) $\dfrac{\mathrm{d}}{\mathrm{d}x}(3^x)$

4. Find the following derivatives.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\ln x)$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}z}(\ln 5z)$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\ln \dfrac{2x}{3}\right)$

2. Linearity in differentiation

The **linearity rules** enable us to differentiate sums and differences of functions, and constant multiples of functions. Specifically

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x) \pm g(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(f(x)) \pm \frac{\mathrm{d}}{\mathrm{d}x}(g(x)), \qquad \frac{\mathrm{d}}{\mathrm{d}x}(kf(x)) = k\frac{\mathrm{d}}{\mathrm{d}x}(f(x)).$$

1. Differentiate each of the following with respect to x.

(a) $3x + 2$ (b) $2x - x^2$ (c) $-\cos x - \sin x$ (d) $3x^{-3} + 4\sin 4x$

(e) $2e^x + e^{-2x}$ (f) $\dfrac{1}{x} - 4 - 3\ln x$ (g) $4x^5 - 3\tan 8x - 2e^{5x}$

2. Find the following derivatives.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(5t^{1/5} + \dfrac{t^8}{8}\right)$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(2\cos\dfrac{\theta}{4} - 3e^{-\theta/4}\right)$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{3e^{3x/5}}{5}\right)$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{2}{9}\tan\dfrac{3x}{2} - \dfrac{3}{4}\cos 8x\right)$ (e) $\dfrac{\mathrm{d}}{\mathrm{d}z}\left(\dfrac{1}{4}z^{4/3} - \dfrac{1}{3}e^{-4z/3}\right)$

In Questions 3-5 you don't need the product rule, quotient rule or chain rule to differentiate any of these if you do the algebra first!

3. Expand the powers or roots and hence find the following derivatives.

(a) $\frac{\mathrm{d}}{\mathrm{d}y}\left(\sqrt{2y}\right)$ (b) $\frac{\mathrm{d}}{\mathrm{d}x}\left((2x)^3 - \frac{1}{(2x)^3}\right)$ (c) $\frac{\mathrm{d}}{\mathrm{d}y}\left(\left(\frac{1}{2}e^y\right)^4\right)$ (d) $\frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt[3]{5e^{-2t}}\right)$

4. Simplify or expand each of the following expressions, and then differentiate with respect to x.

(a) $\frac{x - x^2}{x^3}$ (b) $x(\sqrt{x} - x^2)$ (c) $\left(2x - \frac{2}{x}\right)\left(\frac{3}{x^2} + x\right)$

(d) $(e^{2x} - 1)(3 - e^{3x})$ (e) $\frac{1 - e^{-2x}}{e^{-4x}}$

5. Use the laws of logarithms to find the following derivatives.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln x^{9/2}\right)$ (b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(\frac{1}{\sqrt{6x}}\right)\right)$ (c) $\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln\left(\frac{t^3}{e^{3t}}\right)\right)$ (d) $\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln\left(te^{-2t}\right)^{1/3}\right)$

3. Higher derivatives

1. Find the following second derivatives.

(a) $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}(x^5)$ (b) $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}(\cos 3x)$ (c) $\dfrac{\mathrm{d}^2}{\mathrm{d}z^2}(e^{2z} - e^{-2z})$

(d) $\dfrac{\mathrm{d}^2}{\mathrm{d}y^2}(8 - 13y)$ (e) $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\dfrac{1}{x} - 3x - 3x^3\right)$ (f) $\dfrac{\mathrm{d}^2}{\mathrm{d}t^2}(\ln 2t - \sqrt{6t})$

(g) $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left(x^{3/2} - \dfrac{1}{x^{3/2}}\right)$ (h) $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}(e^x + e^{-x} + \sin x + \cos x)$ (i) $\dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\dfrac{1}{2}\sin 2t - \dfrac{1}{4}\ln 4t\right)$

4. The product rule for differentiation

The rule for differentiating the product of two functions $f(x)$ and $g(x)$ is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x).$$

1. Differentiate each of the following with respect to x.

(a) $x\,\sin x$ (b) $x^3\cos 2x$ (c) $x^{-1/3}\,e^{-3x}$

(d) $\sqrt{x}\,\ln 4x$ (e) $(x^2 - x)\sin 6x$ (f) $\frac{1}{x}\left(\tan\frac{x}{3} - \cos\frac{x}{3}\right)$

2. Find the following derivatives.

(a) $\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta\,\cos\theta)$ (b) $\frac{\mathrm{d}}{\mathrm{d}t}(\sin 2t\,\tan 5t)$ (c) $\frac{\mathrm{d}}{\mathrm{d}z}(\sin z\,\ln 4z)$ (d) $\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x/2}\,\cos\frac{x}{2}\right)$

(e) $\frac{\mathrm{d}}{\mathrm{d}x}(e^{6x}\,\ln 6x)$ (f) $\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta\cos 3\theta)$ (g) $\frac{\mathrm{d}}{\mathrm{d}t}(\ln t\,\ln 2t)$

5. The quotient rule for differentiation

The rule for differentiating the quotient of two functions $f(x)$ and $g(x)$ is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

1. Differentiate each of the following with respect to x.

(a) $\dfrac{x}{1-x^2}$ (b) $\dfrac{x^4}{1-x}$ (c) $\dfrac{2-x}{1+2x}$ (d) $\dfrac{3x^2-2x^3}{2x^3+3}$ (e) $\dfrac{1+\sqrt{x}}{\sqrt{x}-x}$

2. Find the following derivatives.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\sin x}{x}\right)$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\ln x}{x^{4/3}}\right)$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\dfrac{\theta^2}{\tan 2\theta}\right)$ (d) $\dfrac{\mathrm{d}}{\mathrm{d}z}\left(\dfrac{e^z}{\sqrt{z}}\right)$ (e) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x^2}{\ln 2x}\right)$

3. Find the following derivatives.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\sin 2t}{\sin 5t}\right)$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{e^{-2x}}{\tan x}\right)$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\ln x}{\cos 3x}\right)$ (d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\ln 3x}{\ln 4x}\right)$

6. The chain rule for differentiation

The chain rule is used to differentiate a "function of a function":

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(g(x))\right) = f'(g(x)).g'(x).$$

1. Differentiate each of the following with respect to x.

(a) $(4+3x)^2$ (b) $(1-x^4)^3$ (c) $\dfrac{1}{(1-2x)^2}$ (d) $\sqrt{1+x^2}$

(e) $\left(x-\dfrac{1}{x}\right)^{-1/3}$ (f) $(2x^2-3x+5)^{5/2}$ (g) $\sqrt{x-2\sqrt{x}}$ (h) $\dfrac{1}{\sqrt{4x^2-x^4}}$

2. Find the following derivatives. (Remember the notation for powers of trigonometric functions: "$\sin^2 x$" means $(\sin x)^2$, etc.)

(a) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\sin^2\theta)$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta^2)$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\sin(\sin\theta))$ (d) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan(3-4x))$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}z}(\cos^5 5z)$ (f) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{\cos^3 x}\right)$ (g) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\sin(2-t-3t^2))$

3. Find the following derivatives. (The notation "$\exp x$" is used rather than "e^x" where it is clearer.)

(a) $\dfrac{\mathrm{d}}{\mathrm{d}y}\left(\exp(-y^2)\right)$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\exp(\cos 3x))$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\cos(e^{3x}))$ (d) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\ln(\sin 4x))$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin(\ln 4x))$ (f) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\ln(e^x-e^{-x}))$ (g) $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{e^{3t}-3\cos 3t}\right)$

7. Differentiation of functions defined implicitly

1. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of y when x and y are related by the following equations. You will need the formula $\frac{\mathrm{d}y}{\mathrm{d}x} = 1/\frac{\mathrm{d}x}{\mathrm{d}y}$.

(a) $x = y - y^3$ (b) $x = y^2 + \dfrac{1}{y}$ (c) $x = e^y + e^{2y}$ (d) $x = \ln(y - e^{-y})$

2. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and/or y when x and y are related by the following equations.

(a) $\cos 2x = \tan y$ (b) $x + y^2 = y - x^2$ (c) $y - \sin y = \cos x$

(d) $e^x - x = e^{2y} + 2y$ (e) $x + e^y = \ln x + \ln y$ (f) $y = (x - y)^3$

8. Differentiation of functions defined parametrically

If x and y are both functions of a parameter t, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$$

1. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t when x and y are related by the following pairs of parametric equations.

(a) $x = \sin t, \quad y = \cos t$ (b) $x = t - \dfrac{1}{t}, \quad y = 1 - t^2$

(c) $x = e^{2t} + t, \quad y = e^t + t^2$ (d) $x = \ln t + t, \quad y = t - \ln t$

2. Find $\frac{\mathrm{d}x}{\mathrm{d}y}$ in terms of t when x and y are related by the following pairs of parametric equations.

(a) $x = 3t + t^3, \quad y = 2t^2 + t^4$ (b) $x = \cos 2t, \quad y = \tan 2t$

9. Miscellaneous differentiation exercises

1. Find the following derivatives, each of which requires one of the techniques covered in previous sections. You have to decide which technique is required for each derivative!

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^3 \tan 4x)$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\tan^3 4t)$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\exp(3\tan 4x))$ (d) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\dfrac{3\theta}{\tan 4\theta}\right)$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\exp(x - e^x))$ (f) $\dfrac{\mathrm{d}}{\mathrm{d}y}\left(\dfrac{y^4 + y^{-4}}{y + y^{-1}}\right)$ (g) $\dfrac{\mathrm{d}}{\mathrm{d}x}(2^x x^2)$ (h) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{\ln x - x}\right)$

(i) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(5^{-3x}\right)$ (j) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\ln(\ln t))$ (k) $\dfrac{\mathrm{d}}{\mathrm{d}z}\left(\ln\left(\dfrac{1-z}{1+z}\right)^2\right)$

2. Find the following derivatives, which require both the product and quotient rules.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x\cos x}{1 - \cos x}\right)$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}z}\left(\dfrac{e^z}{z\ln z}\right)$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\dfrac{\sin 3\theta \cos 2\theta}{\tan 4\theta}\right)$

3. Find the following derivatives, which require the chain rule as well as either the product rule or the quotient rule.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}(e^{-t}\ln(e^t+1))$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\sin^2 3\theta\,\cos^4 3\theta)$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\left(\dfrac{1-x^2}{1+x^2}\right)^{3/2}\right)$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\exp(\theta\cos\theta))$ (e) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left((x\ln x)^3\right)$ (f) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\exp\left(\dfrac{1-x}{1+x}\right)\right)$

(g) $\dfrac{\mathrm{d}}{\mathrm{d}y}\left(\dfrac{1}{y^2\sqrt{y^2-1}}\right)$

4. Find the following derivatives, which require use of the chain rule more than once.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sqrt{1-\cos^3 x})$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\exp\left((x-x^2)^{1/4}\right)\right)$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\ln\left(\tan\dfrac{1}{\theta}\right)\right)$

5. Find the following second derivatives.

(a) $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}(\sqrt{1+x^2})$ (b) $\dfrac{\mathrm{d}^2}{\mathrm{d}z^2}(\exp(z^2))$ (c) $\dfrac{\mathrm{d}^2}{\mathrm{d}\theta^2}(\sin^3\theta)$ (d) $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\dfrac{1}{(1-x^4)^4}\right)$

6. Remembering that $\operatorname{cosec} x = \dfrac{1}{\sin x}$, $\sec x = \dfrac{1}{\cos x}$ and $\cot x = \dfrac{\cos x}{\sin x}$, find the following derivatives.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} 2x)$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\sec^2\theta)$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}z}(\sqrt{1+\cot z})$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\operatorname{cosec}^2\theta\cot^3\theta)$ (e) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\ln(\sec x+\tan x))$ (f) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\tan(\sec\theta))$

10. Integrals of basic functions

Try to find all the integrals in this section without referring to a table of integrals. The integrals of these functions occur so frequently that you should try to memorise the appropriate rules. If you are really stuck, consult the Tables on page 4.

1. Integrate each of the following with respect to x.

(a) x^4 (b) x^7 (c) $x^{1/2}$ (d) $x^{1/3}$ (e) $\sqrt{x}$ (f) $x^{-1/2}$ (g) $\sqrt[4]{x}$

(h) $\dfrac{1}{x^3}$ (i) $x^{0.2}$ (j) $\dfrac{1}{x^{0.3}}$ (k) $\dfrac{1}{\sqrt{x}}$ (l) $\dfrac{1}{\sqrt{x^3}}$ (m) x^{-2} (n) $x^{4/3}$

2. Integrate each of the following with respect to x.

(a) $\cos 5x$ (b) $\sin 2x$ (c) $\sin \frac{1}{2}x$ (d) $\cos \dfrac{x}{2}$ (e) $\dfrac{1}{x}$ (f) e^{2x}

(g) e^{-2x} (h) $\mathrm{e}^{x/3}$ (i) $\mathrm{e}^{0.5x}$ (j) $\dfrac{1}{\mathrm{e}^x}$ (k) $\dfrac{1}{\mathrm{e}^{2x}}$ (l) $\cos(-7x)$

11. Linearity in integration

The **linearity rules** enable us to integrate sums (and differences) of functions, and constant multiples of functions. Specifically

$$\int (f(x) \pm g(x))\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\mathrm{d}x, \qquad \int k\, f(x)\mathrm{d}x = k \int f(x)\mathrm{d}x$$

1. Integrate each of the following with respect to x.

(a) $7x^4$ (b) $-4x^7$ (c) $x^{1/2} + x^{1/3}$ (d) $17x^{1/3}$ (e) $\sqrt{x} - \dfrac{1}{\sqrt{x}}$ (f) $x^2 + \dfrac{1}{x}$

(g) $x^3 + \dfrac{1}{x^2}$ (h) $\dfrac{1}{7x^3}$ (i) 11 (j) $\dfrac{11}{x^{0.3}}$ (k) $2x - \dfrac{2}{x}$ (l) $7x - 11$

2. Integrate each of the following with respect to x.

(a) $3x + \cos 4x$ (b) $4 + \sin 3x$ (c) $\dfrac{x}{2} + \sin \dfrac{x}{2}$ (d) $4\mathrm{e}^x + \cos \dfrac{x}{2}$

(e) $\mathrm{e}^{-2x} + \mathrm{e}^{2x}$ (f) $3 \sin 2x + 2 \sin 3x$ (g) $\dfrac{1}{kx}$, k constant (h) $-1 - \dfrac{4}{x}$

(i) $1 + x + x^2$ (j) $\dfrac{1}{3x} - 7$ (k) $\frac{1}{2} \cos \frac{1}{2} x$ (l) $\frac{1}{2} x^2 - 3x^{-1/2}$

3. Simplify each of the following expressions first and then integrate them with respect to x.

(a) $6x(x+1)$ (b) $(x+1)(x-2)$ (c) $\dfrac{x^3+2x^2}{\sqrt{x}}$ (d) $(\sqrt{x}+2)(\sqrt{x}-3)$

(e) $e^{2x}(e^x - e^{-x})$ (f) $\dfrac{e^{3x}-e^{2x}}{e^x}$ (g) $\dfrac{x+4}{x}$ (h) $\dfrac{x^2+3x+2}{x+2}$

12. Evaluating definite integrals

1. Evaluate each of the following definite integrals.

(a) $\int_0^1 7x^4 \mathrm{d}x$ (b) $\int_{-2}^3 -4t^7 \mathrm{d}t$ (c) $\int_1^2 (x^{1/2} + x^{1/3}) \mathrm{d}x$ (d) $\int_{-2}^{-1} 17t^{1/3} \mathrm{d}t$

(e) $\int_1^3 (2s + 8s^3) \mathrm{d}s$ (f) $\int_1^5 \frac{1}{x^2} \mathrm{d}x$ (g) $\int_0^3 (t^2 + 2t) \mathrm{d}t$ (h) $\int_0^{\pi/4} \cos 2x \, \mathrm{d}x$

(i) $\int_0^{1/2} \mathrm{e}^{3x} \mathrm{d}x$ (j) $\int_2^4 \frac{1}{\sqrt{\mathrm{e}^x}} \mathrm{d}x$ (k) $\int_0^{\pi/4} (2\lambda + \sin\lambda) \mathrm{d}\lambda$ (l) $\int_0^1 (\mathrm{e}^x + \mathrm{e}^{-x}) \mathrm{d}x$

2. For the function $f(x) = x^2 + 3x - 2$ verify that

$$\int_0^2 f(x) \mathrm{d}x + \int_2^3 f(x) \mathrm{d}x = \int_0^3 f(x) \mathrm{d}x$$

3. For the function $f(x) = 4x^2 - 7x$ verify that $\int_{-1}^1 f(x) \mathrm{d}x = -\int_1^{-1} f(x) \mathrm{d}x$.

13. Integration by parts

Integration by parts is a technique which can often be used to integrate products of functions. If u and v are both functions of x then

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

When dealing with definite integrals the relevant formula is

$$\int_a^b u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = [uv]_a^b - \int_a^b v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

1. Integrate each of the following with respect to x.

(a) $x\mathrm{e}^x$ (b) $5x\cos x$ (c) $x\sin x$ (d) $x\cos 2x$ (e) $x\ln x$ (f) $2x\sin\dfrac{x}{2}$

2. Evaluate the following definite integrals.

(a) $\displaystyle\int_0^{\pi/2} x\cos x\,\mathrm{d}x$ (b) $\displaystyle\int_1^2 4x\mathrm{e}^x\mathrm{d}x$ (c) $\displaystyle\int_{-1}^1 5t\mathrm{e}^{-2t}\mathrm{d}t$ (d) $\displaystyle\int_1^3 x\ln x\,\mathrm{d}x$

3. In the following exercises it may be necessary to apply the integration by parts formula more than once. Integrate each of the following with respect to x.

(a) $x^2\mathrm{e}^x$ (b) $5x^2\cos x$ (c) $x(\ln x)^2$ (d) $x^3\mathrm{e}^{-x}$ (e) $x^2\sin\dfrac{x}{2}$

4. Evaluate the following definite integrals.

(a) $\displaystyle\int_0^{\pi/2} x^2 \cos x \, \mathrm{d}x$ (b) $\displaystyle\int_0^1 7x^2 \mathrm{e}^x \mathrm{d}x$ (c) $\displaystyle\int_{-1}^1 t^2 \mathrm{e}^{-2t} \mathrm{d}t$

5. By writing $\ln x$ as $1 \times \ln x$ find $\displaystyle\int \ln x \, \mathrm{d}x$.

6. Let I stand for the integral $\displaystyle\int \frac{\ln t}{t} \, \mathrm{d}t$. Using integration by parts show that

$$I = (\ln t)^2 - I \qquad \text{(plus a constant of integration)}$$

Hence deduce that $I = \frac{1}{2}(\ln t)^2 + c$.

7. Let I stand for the integral $\displaystyle\int \mathrm{e}^t \sin t \, \mathrm{d}t$. Using integration by parts twice show that

$$I = \mathrm{e}^t \sin t - \mathrm{e}^t \cos t - I \qquad \text{(plus a constant of integration)}$$

Hence deduce that $\displaystyle\int \mathrm{e}^t \sin t \, \mathrm{d}t = \frac{\mathrm{e}^t(\sin t - \cos t)}{2} + c$

14. Integration by substitution

1. Find each of the following integrals using the given substitution.

(a) $\int \cos(x-3)\mathrm{d}x, \quad u = x-3$

(b) $\int \sin(2x+4)\mathrm{d}x, \quad u = 2x+4$

(c) $\int \cos(\omega t+\phi)\mathrm{d}t, \quad u = \omega t+\phi$

(d) $\int \mathrm{e}^{9x-7}\mathrm{d}x, \quad u = 9x-7$

(e) $\int x(3x^2+8)^3\mathrm{d}x, \quad u = 3x^2+8$

(f) $\int x\sqrt{4x-3}\,\mathrm{d}x, \quad u = 4x-3$

(g) $\int \frac{1}{(x-2)^4}\mathrm{d}x, \quad u = x-2$

(h) $\int \frac{1}{(3-t)^5}\,\mathrm{d}t, \quad u = 3-t$

(i) $\int x\,e^{-x^2}\mathrm{d}x, \quad u = -x^2$

(j) $\int \sin x\,\cos^3 x\,\mathrm{d}x, \quad u = \cos x$

(k) $\int \frac{t}{\sqrt{1+t^2}}\,\mathrm{d}t, \quad u = 1+t^2$

(l) $\int \frac{x}{2x+1}\mathrm{d}x, \quad u = 2x+1$

2. Find each of the following integrals using the given substitution.

(a) $\displaystyle\int \frac{x}{\sqrt{x-3}}\mathrm{d}x, \qquad z=\sqrt{x-3}$ (b) $\displaystyle\int (x-5)^4(x+3)^2\mathrm{d}x, \qquad u=x-5$

3. Evaluate each of the following definite integrals using the given substitution.

(a) $\displaystyle\int_0^{\pi/4} \cos(x-\pi)\mathrm{d}x, \quad z=x-\pi$ (b) $\displaystyle\int_8^9 (x-8)^5(x+1)^2\mathrm{d}x, \quad u=x-8$

(c) $\displaystyle\int_2^3 t\sqrt{t-2}\,\mathrm{d}t,$ by letting $u=t-2$, and also by letting $u=\sqrt{t-2}$

(d) $\displaystyle\int_1^2 t(t^2+5)^3\mathrm{d}t, \quad u=t^2+5$ (e) $\displaystyle\int_0^{\pi/4} \tan x\,\sec^2 x\,\mathrm{d}x, \quad u=\tan x$

(f) $\displaystyle\int_{\pi^2/4}^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\mathrm{d}x, \quad u=\sqrt{x}$ (g) $\displaystyle\int_1^2 \frac{e^{\sqrt{t}}}{\sqrt{t}}\mathrm{d}t, \quad u=\sqrt{t}$

4. By means of the substitution $u = 4x^2 - 7x + 2$ show that

$$\int \frac{8x-7}{4x^2-7x+2}\mathrm{d}x = \ln|4x^2-7x+2| + c$$

5. The result of the previous exercise is a particular case of a more general rule with which you should become familiar: when the integrand takes the form

$$\frac{\text{derivative of denominator}}{\text{denominator}}$$

the integral is the logarithm of the denominator. Use this rule to find the following integrals, checking each example by making an appropriate substitution.

(a) $\int \frac{1}{x+1}\,\mathrm{d}x$ (b) $\int \frac{1}{x-3}\,\mathrm{d}x$ (c) $\int \frac{3}{3x+4}\,\mathrm{d}x$ (d) $\int \frac{2}{2x+1}\,\mathrm{d}x$

6. Use the technique of Question 5 together with a linearity rule to find the following integrals. For example, to find $\int \frac{x}{x^2-7}\,\mathrm{d}x$ we note that the numerator can be made equal to the derivative of the denominator as follows:

$$\int \frac{x}{x^2-7}\mathrm{d}x = \frac{1}{2}\int \frac{2x}{x^2-7}\mathrm{d}x = \frac{1}{2}\ln|x^2-7| + c.$$

(a) $\int \frac{x}{x^2+1}\mathrm{d}x$ (b) $\int \frac{\sin 3\theta}{1+\cos 3\theta}\mathrm{d}\theta$ (c) $\int \frac{3e^{2x}}{1+e^{2x}}\,\mathrm{d}x$

15. Integration using partial fractions

1. By expressing the integrand as the sum of its partial fractions, find the following integrals.

(a) $\displaystyle\int \frac{2x+1}{x^2+x}\,\mathrm{d}x$ (b) $\displaystyle\int \frac{3x+1}{x^2+x}\,\mathrm{d}x$ (c) $\displaystyle\int \frac{5x+6}{x^2+3x+2}\,\mathrm{d}x$

(d) $\displaystyle\int \frac{x+1}{(1-x)(x-2)}\,\mathrm{d}x$ (e) $\displaystyle\int \frac{9x+25}{x^2+10x+9}\,\mathrm{d}x$ (f) $\displaystyle\int \frac{5x-11}{x^2+10x+9}\,\mathrm{d}x$

(g) $\displaystyle\int \frac{4x}{4-x^2}\,\mathrm{d}x$ (h) $\displaystyle\int \frac{15x+51}{x^2+7x+10}\,\mathrm{d}x$ (i) $\displaystyle\int \frac{\mathrm{d}s}{s^2-1}$

2. By expressing the integrand as the sum of its partial fractions, find the following definite integrals.

(a) $\displaystyle\int_1^2 \frac{2-8x}{x^2+2x}\,\mathrm{d}x$ (b) $\displaystyle\int_0^2 \frac{5x+7}{(x+1)(x+2)}\,\mathrm{d}x$ (c) $\displaystyle\int_{-1}^0 \frac{7x-11}{x^2-3x+2}\,\mathrm{d}x$

3. By expressing the integrand as the sum of its partial fractions, find the following integrals.

(a) $\displaystyle\int \frac{x}{x^2 - 2x + 1}\,\mathrm{d}x$ (b) $\displaystyle\int \frac{4x + 6}{(x + 1)^2}\,\mathrm{d}x$ (c) $\displaystyle\int \frac{7x - 23}{x^2 - 6x + 9}\,\mathrm{d}x$

4. By expressing the integrand as the sum of its partial fractions, find the following definite integrals.

(a) $\displaystyle\int_0^1 \frac{x + 8}{x^2 + 6x + 9}\,\mathrm{d}x$ (b) $\displaystyle\int_{-1}^1 \frac{2x + 19}{x^2 + 18x + 81}\,\mathrm{d}x$ (c) $\displaystyle\int_0^2 -\frac{8x}{x^2 + 2x + 1}\,\mathrm{d}x$

5. Find $\int \frac{2x^2+6x+5}{(x+2)(x+1)^2} \mathrm{d}x$.

6. Find $\int \frac{5x^2+x-34}{(x-2)(x-3)(x+4)} \, \mathrm{d}x$.

7. Find $\int \frac{x^2-11x-4}{(2x+1)(x+1)(3-x)} \, \mathrm{d}x$.

8. In this example note that the degree of the numerator is greater than the degree of the denominator. Find $\int \frac{x^3}{(x+1)(x+2)}\,\mathrm{d}x$.

9. Show that $\frac{2x^3+1}{(x+1)(x+2)^2}$ can be written in the form

$$2 - \frac{1}{x+1} + \frac{15}{(x+2)^2} - \frac{9}{x+2}$$

Hence find $\int \frac{2x^3+1}{(x+1)(x+2)^2}\,\mathrm{d}x$.

10. Find $\int \frac{4x^3+10x+4}{2x^2+x}\,\mathrm{d}x$ by expressing the integrand in partial fractions.

16. Integration using trigonometrical identities

Trigonometrical identities can often be used to write an integrand in an alternative form which can then be integrated. Some identities which are particularly useful for integration are given in the table below.

Table of trigonometric identities

$$\sin A \,\sin B = \tfrac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$
$$\cos A \,\cos B = \tfrac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$$
$$\sin A \,\cos B = \tfrac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$$
$$\sin A \,\cos A = \tfrac{1}{2}\sin 2A$$
$$\cos^2 A = \tfrac{1}{2}(1+\cos 2A)$$
$$\sin^2 A = \tfrac{1}{2}(1-\cos 2A)$$
$$\sin^2 A + \cos^2 A = 1$$
$$\tan^2 A = \sec^2 A - 1$$

1. In preparation for what follows find each of the following integrals.

(a) $\int \sin 3x\,\mathrm{d}x$ (b) $\int \cos 8x\,\mathrm{d}x$ (c) $\int \sin 7t\,\mathrm{d}t$ (d) $\int \cos 6x\,\mathrm{d}x$

2. Use the identity $\sin^2 A = \frac{1}{2}(1-\cos 2A)$ to find $\int \sin^2 x\,\mathrm{d}x$.

3. Use the identity $\sin A\,\cos A = \frac{1}{2}\sin 2A$ to find $\int \sin t\cos t\,\mathrm{d}t$.

4. Find (a) $\int 2\sin 7t\cos 3t\,\mathrm{d}t$ (b) $\int 8\cos 9x\cos 4x\,\mathrm{d}x$ (c) $\int \sin t\sin 7t\,\mathrm{d}t$

5. Find $\int \tan^2 t\,\mathrm{d}t$.
(Hint: use one of the identities and note that the derivative of $\tan t$ is $\sec^2 t$).

6. Find $\int \cos^4 t\,\mathrm{d}t$. (Hint: square an identity for $\cos^2 A$).

7. (a) Use the substitution $u = \cos x$ to show that

$$\int \sin x\cos^n x\,\mathrm{d}x = -\frac{1}{n+1}\cos^{n+1} x + c$$

(b) In this question you are required to find the integral $\int \sin^5 t\,\mathrm{d}t$. Start by writing the integrand as $\sin^4 t\,\sin t$. Take the identity $\sin^2 t = 1 - \cos^2 t$ and square it to produce an identity for $\sin^4 t$. Finally use the result in part (a) to find the required integral, $\int \sin^5 t\,\mathrm{d}t$.

17. Miscellaneous integration exercises

To find the integrals in this section you will need to select an appropriate technique from any of the earlier techniques.

1. Find $\int (9x-2)^5 \mathrm{d}x$.

2. Find $\int \frac{1}{\sqrt{t}+1} \, \mathrm{d}t$.

3. Find $\int t^4 \ln t \, \mathrm{d}t$.

4. Find $\int (5\sqrt{t} - 3t^3 + 2) \, \mathrm{d}t$.

5. Find $\int (\cos 3t + 3\sin t)\, \mathrm{d}t$.

6. By taking logarithms to base e show that a^x can be written as $\mathrm{e}^{x \ln a}$. Hence find $\int a^x \, \mathrm{d}x$ where a is a constant.

7. Find $\int x\mathrm{e}^{3x+1}\, \mathrm{d}x$.

8. Find $\int \frac{2x}{(x+2)(x-2)}\, \mathrm{d}x$.

9. Find $\int \tan\theta \sec^2\theta \, \mathrm{d}\theta$.

10. Find $\int_0^{\pi/2} \sin^3 x \, \mathrm{d}x$.

11. Using the substitution $x = \sin\theta$ find $\int \sqrt{1 - x^2} \, \mathrm{d}x$.

12. Find $\int \mathrm{e}^x \sin 2x \, \mathrm{d}x$.

13. Find $\displaystyle\int \frac{x-9}{x(x-1)(x+3)}\,\mathrm{d}x$.

14. Find $\displaystyle\int \frac{x^3+2x^2-10x-9}{(x-3)(x+3)}\,\mathrm{d}x$.

15. Let I_n stand for the integral $\displaystyle\int x^n \mathrm{e}^{2x}\mathrm{d}x$. Using integration by parts show that $I_n = \dfrac{x^n\mathrm{e}^{2x}}{2} - \dfrac{n}{2}I_{n-1}$. This result is known as a **reduction formula**. Use it repeatedly to find I_4, that is $\displaystyle\int x^4\mathrm{e}^{2x}\mathrm{d}x$.

16. Find $\displaystyle\int_0^1 \frac{1}{(4-t^2)^{3/2}}\mathrm{d}t$, by letting $t = 2\sin\theta$.

Answers

Section 1. Derivatives of basic functions

1. (a) 1 (b) $6x^5$ (c) 0 (d) $\frac{1}{2\sqrt{x}}$
   (e) $-x^{-2}$ (f) $\frac{1}{7}x^{-6/7}$ (g) $-\frac{3}{x^4}$ (h) $79x^{78}$
   (i) $1.3x^{0.3}$ (j) $-\frac{1}{3}x^{-4/3}$ (k) $-\frac{5}{3}x^{-8/3}$ (l) $-\frac{0.71}{x^{1.71}}$

2. (a) $-\sin\theta$ (b) $-4\sin 4\theta$ (c) $\cos\theta$ (d) $\frac{2}{3}\cos\frac{2\theta}{3}$
   (e) $\sec^2\theta$ (f) $\pi\sec^2\pi\theta$ (g) $-8\cos(-8\theta) = -8\cos 8\theta$
   (h) $\frac{1}{4}\sec^2\frac{\theta}{4}$ (i) $-3\pi\sin 3\pi\theta$ (j) $\frac{5}{2}\sin\left(-\frac{5\theta}{2}\right) = -\frac{5}{2}\sin\frac{5\theta}{2}$
   (k) $0.7\cos 0.7\theta$

3. (a) e^x (b) $2e^{2y}$ (c) $-7e^{-7t}$ (d) $-\frac{1}{3}e^{-x/3}$ (e) $\frac{2}{\pi}e^{2z/\pi}$ (f) $-1.4e^{-1.4x}$ (g) $3^x\ln 3$

4. (a) $\frac{1}{x}$ (b) $\frac{1}{z}$ (c) $\frac{1}{x}$

Section 2. Linearity in differentiation

1. a) 3 b) $2-2x$ c) $\sin x - \cos x$ d) $-9x^{-4} + 16\cos 4x$
   e) $2e^x - 2e^{-2x}$ f) $-\frac{1}{x^2} - \frac{3}{x}$ g) $20x^4 - 24\sec^2 8x - 10e^{5x}$

2. (a) $t^{-4/5} + t^7$ (b) $-\frac{1}{2}\sin\frac{\theta}{4} + \frac{3}{4}e^{-\theta/4}$ (c) $\frac{9e^{3x/5}}{25}$ (d) $\frac{1}{3}\sec^2\frac{3x}{2} + 6\sin 8x$
   (e) $\frac{1}{3}z^{1/3} + \frac{4}{9}e^{-4z/3}$

3. (a) $\frac{1}{\sqrt{2y}}$ (b) $24x^2 + \frac{3}{8x^4}$ (c) $\frac{1}{4}e^{4y}$ (d) $-\frac{2}{3}\sqrt[3]{5e^{-2t}}$

4. (a) $-\frac{2}{x^3} + \frac{1}{x^2}$ (b) $\frac{3}{2}\sqrt{x} - 3x^2$ (c) $-\frac{6}{x^2} + \frac{18}{x^4} + 4x$ (d) $6e^{2x} - 5e^{5x} + 3e^{3x}$
   (e) $4e^{4x} - 2e^{2x}$

5. (a) $\frac{9}{2x}$ (b) $-\frac{1}{2x}$ (c) $\frac{3}{t} - 3$ (d) $\frac{1}{3t} - \frac{2}{3}$

Section 3. Higher derivatives

1. (a) $20x^3$ (b) $-9\cos 3x$ (c) $4e^{2z} - 4e^{-2z}$ (d) 0
   (e) $\frac{2}{x^3} - 18x$ (f) $-\frac{1}{t^2} + \frac{\sqrt{6}}{4}t^{-3/2}$ (g) $\frac{3}{4}x^{-1/2} - \frac{15}{4}x^{-7/2}$ (h) $e^x + e^{-x} - \sin x - \cos x$
   (i) $-2\sin 2t + \frac{1}{4t^2}$

Section 4. The product rule for differentiation

1. (a) $\sin x + x\cos x$ (b) $3x^2\cos 2x - 2x^3\sin 2x$
(c) $-\frac{1}{3}x^{-4/3}e^{-3x} - 3x^{-1/3}e^{-3x}$ (d) $\frac{1}{2\sqrt{x}}\ln 4x + \frac{1}{\sqrt{x}}$
(e) $(2x-1)\sin 6x + 6(x^2-x)\cos 6x$ (f) $-\frac{1}{x^2}\left(\tan\frac{x}{3} - \cos\frac{x}{3}\right) + \frac{1}{3x}\left(\sec^2\frac{x}{3} + \sin\frac{x}{3}\right)$

2. (a) $\cos^2\theta - \sin^2\theta$ (b) $2\cos 2t\tan 5t + 5\sin 2t\sec^2 5t$
(c) $\cos z\ln 4z + \frac{\sin z}{z}$ (d) $-\frac{1}{2}e^{-x/2}\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)$
(e) $6e^{6x}\ln 6x + \frac{e^{6x}}{x}$ (f) $-\sin\theta\cos 3\theta - 3\cos\theta\sin 3\theta$
(g) $\frac{1}{t}(\ln 2t + \ln t)$

Section 5. The quotient rule for differentiation

1. (a) $\frac{1+x^2}{(1-x^2)^2}$ (b) $\frac{4x^3-3x^4}{(1-x)^2}$ (c) $-\frac{5}{(1+2x)^2}$
(d) $\frac{18x-18x^2-6x^4}{(2x^3+3)^2}$ (e) $\left(\frac{1}{2}\sqrt{x} + 1 - \frac{1}{2\sqrt{x}}\right)/(\sqrt{x}-x)^2$

2. (a) $\frac{x\cos x - \sin x}{x^2}$ (b) $(1 - \frac{4}{3}\ln x)x^{-7/3}$ (c) $\frac{2\theta\tan 2\theta - 2\theta^2\sec^2 2\theta}{\tan^2 2\theta}$
(d) $\frac{e^z}{z}(\sqrt{z} - \frac{1}{2\sqrt{z}})$ (e) $\frac{2x\ln 2x - x}{(\ln 2x)^2}$

3. (a) $\frac{2\cos 2t\sin 5t - 5\cos 5t\sin 2t}{\sin^2 5t}$ (b) $\frac{-2e^{-2x}\tan x - e^{-2x}\sec^2 x}{\tan^2 x}$
(c) $\frac{(\cos 3x)/x + 3\ln x\sin 3x}{\cos^2 3x}$, (d) $\frac{\ln(4/3)}{x(\ln 4x)^2}$

Section 6. The chain rule for differentiation

1. (a) $6(4+3x)$ (b) $-12x^3(1-x^4)^2$ (c) $\frac{4}{(1-2x)^3}$
(d) $\frac{x}{\sqrt{1+x^2}}$ (e) $-\frac{1}{3}\left(1+\frac{1}{x^2}\right)\left(x-\frac{1}{x}\right)^{-4/3}$ (f) $\frac{5}{2}(4x-3)(2x^2-3x+5)^{3/2}$
(g) $\left(1-\frac{1}{\sqrt{x}}\right)\frac{1}{2\sqrt{x-2\sqrt{x}}}$ (h) $-\frac{4x-2x^3}{(4x^2-x^4)^{3/2}}$

2. (a) $2\sin\theta\cos\theta$ (b) $2\theta\cos\theta^2$ (c) $\cos\theta\cos(\sin\theta)$
(d) $-4\sec^2(3-4x)$ (e) $-25\cos^4 5z\sin 5z$ (f) $\frac{3\sin x}{\cos^4 x}$
(g) $(-1-6t)\cos(2-t-3t^2)$

3. (a) $-2y\exp(-y^2)$ (b) $-3\sin 3x\exp(\cos 3x)$ (c) $-3e^{3x}\sin(e^{3x})$ (d) $\frac{4\cos 4x}{\sin 4x} = 4\cot 4x$
(e) $\frac{1}{x}\cos(\ln 4x)$ (f) $\frac{e^x+e^{-x}}{e^x-e^{-x}}$ (g) $\frac{3e^{3t}+9\sin 3t}{2\sqrt{e^{3t}-3\cos 3t}}$

Section 7. Differentiation of functions defined implicitly

1. (a) $\frac{1}{1-3y^2}$ (b) $\frac{y^2}{2y^3-1}$ (c) $\frac{1}{e^y+2e^{2y}}$ (d) $\frac{y-e^{-y}}{1+e^{-y}}$

2. (a) $\frac{-2\sin 2x}{\sec^2 y}$ (b) $\frac{-2x-1}{2y-1}$ (c) $\frac{-\sin x}{1-\cos y}$ (d) $\frac{e^x-1}{2e^{2y}+2}$ (e) $\left(\frac{1}{x}-1\right)/\left(e^y-\frac{1}{y}\right)$
   (f) $\frac{3(x-y)^2}{1+3(x-y)^2}$

Section 8. Differentiation of functions defined parametrically

1. (a) $-\tan t$ (b) $\frac{-2t^3}{t^2+1}$ (c) $\frac{e^t+2t}{2e^{2t}+1}$ (d) $\frac{t-1}{1+t}$

2. (a) $\frac{3}{4t}$ (b) $-\frac{\sin 2t}{\sec^2 2t}$

Section 9. Miscellaneous differentiation exercises

1. (a) $3x^2\tan 4x + 4x^3\sec^2 4x$ (b) $12\tan^2 4t\sec^2 4t$ (c) $12\exp(3\tan 4x)\sec^2 4x$
   (d) $\frac{3\tan 4\theta-12\theta\sec^2 4\theta}{\tan^2 4\theta}$ (e) $(1-e^x)\exp(x-e^x)$ (f) $\frac{3y^4+5y^2-5y^{-4}-3y^{-6}}{(y+y^{-1})^2}$
   (g) $(2^x\ln 2)x^2+2^{x+1}x$ (h) $\frac{x-1}{x(\ln x-x)^2}$ (i) $(-3\ln 5)5^{-3x}$
   (j) $\frac{1}{t\ln t}$ (k) $\frac{-4}{1-z^2}$

2. (a) $\frac{\cos x-\cos^2 x-x\sin x}{(1-\cos x)^2}$ (b) $\frac{e^z(z\ln z-\ln z-1)}{(z\ln z)^2}$
   (c) $\frac{(3\cos 3\theta\cos 2\theta-2\sin 3\theta\sin 2\theta)\tan 4\theta-4\sec^2 4\theta\sin 3\theta\cos 2\theta}{\tan^2 4\theta}$

3. (a) $-e^{-t}\ln(e^t+1)+\frac{1}{e^t+1}$ (b) $6\sin 3\theta\cos^5 3\theta - 12\sin^3 3\theta\cos^3 3\theta$
   (c) $\frac{-6x(1-x^2)^{1/2}}{(1+x^2)^{5/2}}$ (d) $(\cos\theta-\theta\sin\theta)\exp(\theta\cos\theta)$
   (e) $3(x\ln x)^2(1+\ln x)$ (f) $\frac{-2}{(1+x)^2}\exp\left(\frac{1-x}{1+x}\right)$
   (g) $\frac{2-3y^2}{y^3(y^2-1)^{3/2}}$

4. (a) $\frac{3\cos^2 x\sin x}{2\sqrt{1-\cos^3 x}}$ (b) $\frac{1}{4}(x-x^2)^{-3/4}(1-2x)\exp\left((x-x^2)^{1/4}\right)$ (c) $-\frac{\sec^2(1/\theta)}{\theta^2\tan(1/\theta)}$

5. (a) $\frac{1}{(1+x^2)^{3/2}}$ (b) $(2+4z^2)\exp(z^2)$ (c) $6\sin\theta\cos^2\theta-3\sin^3\theta$
   (d) $\frac{320x^6}{(1-x^4)^6}+\frac{48x^2}{(1-x^4)^5}$

6. (a) $-2\cot 2x\,\mathrm{cosec}\,2x$ (b) $2\sec^2\theta\tan\theta$ (c) $-\frac{\mathrm{cosec}^2 z}{2\sqrt{1+\cot z}}$
   (d) $-2\mathrm{cosec}^2\theta\cot^4\theta-3\mathrm{cosec}^4\theta\cot^2\theta$ (e) $\sec x$ (f) $\sec^2(\sec\theta)\sec\theta\tan\theta$

Section 10. Integrals of basic functions

1. (a) $\frac{x^5}{5}+c$, (b) $\frac{x^8}{8}+c$, (c) $\frac{2x^{3/2}}{3}+c$ (d) $\frac{3x^{4/3}}{4}+c$, (e) same as (c), (f) $2x^{1/2}+c$, (g) $\frac{4x^{5/4}}{5}+c$, (h) $-\frac{1}{2x^2}+c$, (i) $\frac{x^{1.2}}{1.2}+c$, (j) $\frac{x^{0.7}}{0.7}+c$ (k) same as (f) (l) $-2x^{-1/2}+c$ (m) $-\frac{1}{x}+c$ (n) $\frac{3x^{7/3}}{7}+c$

2. (a) $\frac{1}{5}\sin 5x+c$, (b) $-\frac{1}{2}\cos 2x+c$ (c) $-2\cos\frac{1}{2}x+c$ (d) $2\sin\frac{x}{2}+c$ (e) $\ln|x|+c$ (f) $\frac{1}{2}\mathrm{e}^{2x}+c$ (g) $-\frac{1}{2}\mathrm{e}^{-2x}+c$ (h) $3\mathrm{e}^{x/3}+c$ (i) $2\mathrm{e}^{0.5x}+c$ (j) $-\mathrm{e}^{-x}+c$ (k) same as (g) (l) $-\frac{1}{7}\sin(-7x)+c=\frac{1}{7}\sin 7x+c$

Section 11. Linearity in integration

1. (a) $\frac{7x^5}{5}+c$, (b) $-\frac{x^8}{2}+c$, (c) $\frac{2x^{3/2}}{3}+\frac{3x^{4/3}}{4}+c$ (d) $\frac{51x^{4/3}}{4}+c$, (e) $\frac{2x^{3/2}}{3}-2x^{1/2}+c$ (f) $\frac{x^3}{3}+\ln|x|+c$ (g) $\frac{x^4}{4}-\frac{1}{x}+c$ (h) $-\frac{1}{14x^2}+c$ (i) $11x+c$ (j) $\frac{11x^{0.7}}{0.7}+c$ (k) $x^2-2\ln|x|+c$ (l) $\frac{7x^2}{2}-11x+c$

2. (a) $\frac{3x^2}{2}+\frac{1}{4}\sin 4x+c$ (b) $4x-\frac{1}{3}\cos 3x+c$ (c) $\frac{x^2}{4}-2\cos\frac{x}{2}+c$ (d) $4\mathrm{e}^x+2\sin\frac{x}{2}+c$ (e) $-\frac{\mathrm{e}^{-2x}}{2}+\frac{\mathrm{e}^{2x}}{2}+c$ (f) $-\frac{3\cos 2x}{2}-\frac{2\cos 3x}{3}+c$ (g) $\frac{1}{k}\ln|x|+c$ (h) $-x-4\ln|x|+c$ (i) $x+\frac{x^2}{2}+\frac{x^3}{3}+c$ (j) $\frac{1}{3}\ln|x|-7x+c$ (k) $\sin\frac{1}{2}x+c$ (l) $\frac{x^3}{6}-6x^{1/2}+c$

3. (a) $2x^3+3x^2+c$ (b) $\frac{x^3}{3}-\frac{x^2}{2}-2x+c$ (c) $\frac{2x^{7/2}}{7}+\frac{4x^{5/2}}{5}+c$ (d) $\frac{x^2}{2}-\frac{2x^{3/2}}{3}-6x+c$ (e) $\frac{e^{3x}}{3}-e^x+c$ (f) $\frac{e^{2x}}{2}-e^x+c$ (g) $x+4\ln|x|+c$ (h) $\frac{x^2}{2}+x+c$

Section 12. Evaluating definite integrals

1. (a) $\frac{7}{5}$ (b) $-\frac{6305}{2}$ (c) $\frac{2}{3}\sqrt{8}+\frac{3}{4}\sqrt[3]{16}-\frac{17}{12}$ (d) $\frac{51}{4}(1-\sqrt[3]{16})$ (e) 168 (f) $\frac{4}{5}$ (g) 18 (h) $\frac{1}{2}$ (i) $\frac{1}{3}(\mathrm{e}^{3/2}-1)$ (j) $2(\mathrm{e}^{-1}-\mathrm{e}^{-2})$ (k) $\frac{\pi^2}{16}-\frac{1}{\sqrt{2}}+1$ (l) $\mathrm{e}^1-\mathrm{e}^{-1}$

2. $\int_0^3 f(x)\,\mathrm{d}x=\frac{33}{2}$, $\int_0^2 f(x)\,\mathrm{d}x=\frac{14}{3}$, $\int_2^3 f(x)\,\mathrm{d}x=\frac{71}{6}$. Then $\frac{14}{3}+\frac{71}{6}=\frac{33}{2}$.

3. $\int_{-1}^1 f(x)\,\mathrm{d}x=\frac{8}{3}$, $\int_1^{-1} f(x)\mathrm{d}x=-\frac{8}{3}$.

Section 13. Integration by parts

1. (a) $xe^x - e^x + c$ (b) $5\cos x + 5x\sin x + c$ (c) $\sin x - x\cos x + c$
(d) $\frac{1}{4}\cos 2x + \frac{1}{2}x\sin 2x + c$ (e) $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + c$ (f) $8\sin\frac{x}{2} - 4x\cos\frac{x}{2} + c$

2. (a) $\frac{\pi}{2} - 1$ (b) $4e^2$
(c) $-\frac{15}{4}e^{-2} - \frac{5}{4}e^2$ (d) $\frac{9}{2}\ln 3 - 2$

3. (a) $e^x(x^2 - 2x + 2) + c$ (b) $5x^2\sin x - 10\sin x + 10x\cos x + c$
(c) $\frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + c$ (d) $-e^{-x}(x^3 + 3x^2 + 6x + 6) + c$
(e) $-2x^2\ \cos(\frac{1}{2}\,x) + 16\cos(\frac{1}{2}\,x) + 8x\ \sin(\frac{1}{2}\,x) + c$

4. (a) $\frac{\pi^2}{4} - 2$ (b) $7e - 14$ (c) $-\frac{5}{4}e^{-2} + \frac{1}{4}e^2$

5. $x\ln x - x + c$.

Section 14. Integration by substitution

1. (a) $\sin(x-3) + c$ (b) $-\frac{1}{2}\cos(2x+4) + c$ (c) $\frac{1}{\omega}\sin(\omega t + \phi) + c$
(d) $\frac{1}{9}e^{9x-7} + c$ (e) $\frac{1}{24}(3x^2+8)^4 + c$ (f) $\frac{(4x-3)^{5/2}}{40} + \frac{(4x-3)^{3/2}}{8} + c$
(g) $-\frac{1}{3(x-2)^3} + c$ (h) $\frac{1}{4(3-t)^4} + c$ (i) $-\frac{1}{2}e^{-x^2} + c$
(j) $-\frac{\cos^4 x}{4} + c$ (k) $(1+t^2)^{1/2} + c$ (l) $\frac{2x+1}{4} - \frac{1}{4}\ln|2x+1| + c$

2. (a) $\dfrac{2(x-3)^{3/2}}{3} + 6\sqrt{x-3} + c$
(b) $\dfrac{(x-5)^7}{7} + \dfrac{8(x-5)^6}{3} + \dfrac{64(x-5)^5}{5} + c$
which can be expanded to $\dfrac{1}{7}\,x^7 - \dfrac{7}{3}\,x^6 + \dfrac{39}{5}\,x^5 + 55\,x^4 - \dfrac{1025}{3}\,x^3 - 375\,x^2 + 5625\,x + c$

3. (a) $-\frac{1}{2}\sqrt{2}$ (b) $\frac{907}{56}$
(c) $\frac{26}{15}$ (d) $\frac{5265}{8}$
(e) $\frac{1}{2}$ (f) 2
(g) $2(e^{\sqrt{2}} - e^1)$

5. (a) $\ln|x+1| + c$ (b) $\ln|x-3| + c$ (c) $\ln|3x+4| + c$ (d) $\ln|2x+1| + c$.

6. (a) $\frac{1}{2}\ln|x^2+1| + c$ (b) $-\frac{1}{3}\ln|1+\cos 3\theta| + c$ (c) $\frac{3}{2}\ln|1+e^{2x}| + c$

Section 15. Integration using partial fractions

1. (a) $\ln|x| + \ln|x+1| + c$ (b) $2\ln|x+1| + \ln|x| + c$
   (c) $4\ln|x+2| + \ln|x+1| + c$ (d) $2\ln|x-1| - 3\ln|x-2| + c$
   (e) $7\ln|x+9| + 2\ln|x+1| + c$ (f) $7\ln|x+9| - 2\ln|x+1| + c$
   (g) $-2\ln|x+2| - 2\ln|x-2| + c$ (h) $8\ln|x+5| + 7\ln|x+2| + c$
   (i) $\frac{1}{2}\ln|s-1| - \frac{1}{2}\ln|s+1| + c$

2. (a) $9\ln 3 - 17\ln 2$ (b) $2\ln 3 + 3\ln 2$
   (c) $-3\ln 3 - \ln 2$

3. (a) $\ln|x-1| - \frac{1}{x-1} + c$ (b) $4\ln|x+1| - \frac{2}{x+1} + c$
   (c) $7\ln|x-3| + \frac{2}{x-3} + c$

4. (a) $-\ln 3 + 2\ln 2 + \frac{5}{12}$ (b) $2\ln 5 - 4\ln 2 + \frac{1}{40}$
   (c) $\frac{16}{3} - 8\ln 3$

5. $\ln|x+2| + \ln|x+1| - \frac{1}{x+1} + c.$

6. $\ln|x+4| + 2\ln|x-3| + 2\ln|x-2| + c.$

7. $\ln|x-3| + \frac{1}{2}\ln|2x+1| - 2\ln|x+1| + c.$

8. $\frac{1}{2}x^2 - 3x - \ln|x+1| + 8\ln|x+2| + c.$

9. $2x - \ln|x+1| - \frac{15}{x+2} - 9\ln|x+2| + c.$

10. $x^2 - x + 4\ln|x| + \frac{3}{2}\ln|2x+1| + c$

Section 16. Integration using trigonometrical identities

1. (a) $-\frac{1}{3}\cos 3x + c$ (b) $\frac{1}{8}\sin 8x + c$ (c) $-\frac{1}{7}\cos 7t + c$ (d) $\frac{\sin 6x}{6} + c$

2. $\frac{x}{2} - \frac{\sin 2x}{4} + c.$

3. $-\frac{1}{4}\cos 2t + c.$

4. (a) $-\frac{1}{10}\cos 10t - \frac{1}{4}\cos 4t + c$ (b) $\frac{4}{5}\sin 5x + \frac{4}{13}\sin 13x + c$
   (c) $\frac{1}{12}\sin 6t - \frac{1}{16}\sin 8t + c$

5. $\tan t - t + c.$

6. $\frac{3}{8}t + \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t + c.$

7. $-\cos t + \frac{2}{3}\cos^3 t - \frac{1}{5}\cos^5 t + c.$

Section 17. Miscellaneous integration exercises

1. $\dfrac{(9x-2)^6}{54} + c.$

2. $2\sqrt{t} - 2\ln|\sqrt{t}+1| + c.$ (Hint: let $u = \sqrt{t}+1$.)

3. $\frac{1}{5}t^5 \ln|t| - \frac{1}{25}t^5 + c.$

4. $\frac{10}{3}t^{3/2} - \frac{3t^4}{4} + 2t + c.$

5. $\frac{1}{3}\sin 3t - 3\cos t + c.$

6. $\frac{1}{\ln a}\mathrm{e}^{x\ln a} + c = \frac{a^x}{\ln a} + c.$

7. $\mathrm{e}^{3x+1}\left(\frac{x}{3} - \frac{1}{9}\right) + c$

8. $\ln|x+2| + \ln|x-2| + c.$

9. $\frac{1}{2}\tan^2\theta + c.$

10. $2/3.$

11. $\frac{1}{2}\arcsin x + \frac{x\sqrt{1-x^2}}{2} + c.$

12. $\frac{1}{5}\mathrm{e}^x(\sin 2x - 2\cos 2x) + c$

13. $3\ln|x| - 2\ln|x-1| - \ln|x+3| + c.$

14. $\frac{1}{2}x^2 + 2x - 2\ln|x+3| + \ln|x-3| + c.$

15. $\frac{1}{4}\mathrm{e}^{2x}[2x^4 - 4x^3 + 6x^2 - 6x + 3] + c.$

16. $\frac{\sqrt{3}}{12}.$

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