The laws of logarithms

There are a number of rules which enable us to rewrite expressions involving logarithms in different, yet equivalent, ways. These rules are known as the **laws of logarithms**. You will find that your lecturers use these laws to present answers in different forms, and so you should make yourself aware of them and how they are used.

The laws apply to logarithms of any base but the same base must be used throughout a calculation.

The laws of logarithms

The three main laws are stated here:

First Law

$$\log A + \log B = \log AB$$

This law tells us how to add two logarithms together. Adding $\log A$ and $\log B$ results in the logarithm of the product of A and B, that is $\log AB$.

For example, we can write

$$\log_{10} 6 + \log_{10} 2 = \log_{10}(6 \times 2) = \log_{10} 12$$

The same base, in this case 10, is used throughout the calculation. You should verify this by evaluating both sides separately on your calculator.

Second Law

$$\log A^n = n \log A$$

So, for example

$$\log_{10} 6^4 = 4 \log_{10} 6$$

You should verify this by evaluating both sides separately on your calculator.

Third Law

$$\log A - \log B = \log \frac{A}{B}$$

So, subtracting $\log B$ from $\log A$ results in $\log \frac{A}{B}$.

For example, we can write

$$\log_e 15 - \log_e 3 = \log_e \tfrac{15}{3} = \log_e 5$$

The same base, in this case e, is used throughout the calculation. You should verify this by evaluating both sides separately on your calculator.

Four other useful results are

$$\log 1 = 0, \qquad \log_m m = 1$$
$$\log_{10} 10^n = n \qquad \log_{\mathrm{e}} \mathrm{e}^n = n$$

The logarithm of 1 to any base is always 0.

The logarithm of a number to the same base is always 1. In particular,

$$\log_{10} 10 = 1, \qquad \text{and} \qquad \log_{\mathrm{e}} \mathrm{e} = 1$$

Exercises

1. Use the first law to simplify the following.

   (a) $\log_{10} 8 + \log_{10} 5$,
   (b) $\log x + \log y$,
   (c) $\log 5x + \log 3x$,
   (d) $\log a + \log b^2 + \log c^3$.

2. Use the third law to simplify the following.

   (a) $\log_{10} 12 - \log_{10} 4$,
   (b) $\log x - \log y$,
   (c) $\log 4x - \log x$.

3. Use the second law to write each of the following in an alternative form.

   (a) $3 \log_{10} 5$,
   (b) $2 \log x$,
   (c) $\log(4x)^2$,
   (d) $5 \ln x^4$,
   (e) $\ln 1000$.

4. Simplify $7 \log x - \log x^5$.

Answers

1. (a) $\log_{10} 40$, (b) $\log xy$, (c) $\log 15x^2$, (d) $\log ab^2c^3$.
2. (a) $\log_{10} 3$, (b) $\log \dfrac{x}{y}$, (c) $\log 4$.
3. (a) $\log_{10} 5^3$ or $\log_{10} 125$, (b) $\log x^2$, (c) $2 \log(4x)$,
   (d) $20 \ln x$ or $\ln x^{20}$, (e) $1000 = 10^3$ so $\ln 1000 = 3 \ln 10$.
4. $\log x^2$ or equivalently $2 \log x$.

