2005/2006 Semester 2 **MA1102R Calculus** **Solution to Tutorial 3**

See **Guide and Comment for Tutorial Assignment** in the web-page for the learning objective of Tutorial 3.

For Questions 1 and 2. Be aware of the techniques and consideration used to evaluate each of the limit. (See Example Sessions page on Prof. Ng Calculus web site.)

1. a. $\lim_{x\to 2} \frac{9x^2+5x-2}{x-4} = -22.$

   b. $\lim_{x\to 1} \frac{x^3+4x^2-4x-1}{x-1} = 7.$

   c. $\lim_{x\to 4} \frac{\sqrt[3]{x^2+1}-\sqrt[3]{17}}{x-4} = \lim_{x\to 4} \frac{(\sqrt[3]{x^2+1}-\sqrt[3]{17})((\sqrt[3]{x^2+1})^2+(\sqrt[3]{x^2+1})(\sqrt[3]{17})+(\sqrt[3]{17})^2)}{(x-4)((\sqrt[3]{x^2+1})^2+(\sqrt[3]{x^2+1})(\sqrt[3]{17})+(\sqrt[3]{17})^2)}$

   $= \lim_{x\to 4} \frac{x^2+1-17}{(x-4)((\sqrt[3]{x^2+1})^2+(\sqrt[3]{x^2+1})(\sqrt[3]{17})+(\sqrt[3]{17})^2)}$

   $= \lim_{x\to 4} \frac{(x-4)(x+4)}{(x-4)((\sqrt[3]{x^2+1})^2+(\sqrt[3]{x^2+1})(\sqrt[3]{17})+(\sqrt[3]{17})^2)}$

   $= \lim_{x\to 4} \frac{(x+4)}{((\sqrt[3]{x^2+1})^2+(\sqrt[3]{x^2+1})(\sqrt[3]{17})+(\sqrt[3]{17})^2)} = \frac{8}{3(\sqrt[3]{17})^2} = \frac{8}{51}(\sqrt[3]{17})$

   d. $\lim_{x\to 5} \frac{\sqrt{x^2+1}-\sqrt{26}}{x-5} = 5/\sqrt{26} = \frac{5\sqrt{26}}{26}.$

   e. $\lim_{t\to 0} \frac{(k+t)^3-k^3}{t} = \lim_{t\to 0} \frac{k^3+3k^2t+3kt^2+t^3-k^3}{t} = \lim_{t\to 0} (3k^2+3kt+t^2) = 3k^2.$

2. a. $\lim_{x\to 7^+} \frac{x-6}{1-\sqrt{x-7}} = \frac{\lim_{x\to 7^+}(x-6)}{\lim_{x\to 7^+}(1-\sqrt{x-7})} = \frac{1}{1-0} = 1.$

   b. The main point is to rewrite the function on the left of the point 5 in a form that we know its limit. Start with an open interval (small enough) on the left of 5, say (4,5). If we can obtain a polynomial then we are done. If not take an even smaller interval say (41/2, 5). If after several trials, it looks as if nothing familiar can be obtained in this way, then other properties of the function would have to be used. Observe that $4 < x < 5 \Rightarrow |x-7| - |x-2| = -(x-7)-(x-2) = 9-2x$. Therefore, $\lim_{x\to 5^-} (|x-7|-|x-2|) = \lim_{x\to 5^-} (9-2x) = 9-10 = -1.$

Exercise. Try to work out $\lim_{x\to 5^+} (|x-7|-|x-2|)$. You may use Derive to check your answer.

Remark: If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} |f(x)| = |L|$. Use this result and part b would be an easy consequence. **Try proving this.** (Use the Epsilon-Delta definition of limit and the property (15) of inequality, page 25, Calculus, An introduction.)

3. As in 2(b), work on suitable open intervals to simplify the function.

   a. We begin by observing that $2 < x < 3 \Rightarrow 1 < x/2 < 1.5 \Rightarrow [x/2] = 1$
   and $2 < x < 3 \Rightarrow -1.5 < -x/2 < -1 \Rightarrow [-x/2] = -2.$
   Hence, $2 < x < 3 \Rightarrow g(x) = 1+[2x]+[-2x] = 1+1-2 = 0.$ Therefore,
   $\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} 0 = 0.$

   b. Now $1 < x < 2 \Rightarrow 0.5 < x/2 < 1 \Rightarrow [x/2] = 0$ and
   $1 < x < 2 \Rightarrow -1 < -x/2 < -0.5 \Rightarrow [x/2] = -1$
   Thus, $1 < x < 2 \Rightarrow g(x) = 1+[x/2]+[-x/2] = 1+0-1 = 0.$

Therefore, $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} 0 = 0.$

c. Since the left and the right limits at $x = 2$ exist and are both equal to 0, $\lim_{x\to 2} g(x) = 0.$

4. **(THE MAIN IDEA IS TO CHOOSE A SUITABLE OPEN INTERVAL ON THE LEFT OF 3)**

To evaluate $\lim_{x\to 3^-} [x^2 - 3]$, we look at $[x^2 - 3]$, on $(2, 3)$.
For $2 < x < 3$, we have $1 < x^2 - 3 < 6$.
Thus, possible values for $[x^2 - 3]$ are $1, 2, 3, 4, 5$.
We note that $[x^2 - 3] = 5 \Leftrightarrow 5 \le x^2 - 3 < 6 \Leftrightarrow 8 \le x^2 < 9$. ....(1)
This suggests that it is enough to consider $x \in (\sqrt{8}, 3)$.
For $x \in (\sqrt{8}, 3)$, $[x^2 - 3] = 5$ (by (1)). Hence $\lim_{x\to 3^-} [x^2 - 3] = 5.$

Work out $\lim_{x\to 3^+}[x^2 - 3]$ as an exercise, by choosing a suitable open interval on the right of 3.

5. a. Now $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (7x - 5) = 21 - 5 = 16$ and
$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x^2 + x - 5) = 9 + 3 - 5 = 7.$ Thus $\lim_{x\to 3^-} f(x) \ne \lim_{x\to 3^+} f(x).$
Therefore, $\lim_{x\to 3} f(x)$ does not exist.

6. It is important to state the reason of obtaining equations relating a and b. (The existence of limit at a point requires the existence and the equality of the left and the right limits.)

a $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} (5x + a) = -10 + a$ and $\lim_{x\to -2^+} f(x) = \lim_{x\to -2^-} (ax^2 + 4b) = 4a + 4b.$
For the limit $\lim_{x\to -2} f(x)$ to exist, $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x)$. Hence, we must have
$-10 + a = 4a + 4b$, i.e., $3a + 4b = -10$ (1)
Also $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax^2 + 4b) = 9a + 4b$ and $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (4b - 12x) = 4b - 36.$ For the limit at $x = 3$ to exist, the left and the right limits must be the same. So we must have $9a + 4b = 4b - 36$, i.e., $9a = -36$ so that $a = -4$. Hence from equation (1) we have $4b = -3a - 10 = 12 - 10 = 2$. Thus b = 1/2.

b. With the found values of a and b above we have $\lim_{x\to -2} f(x) = -10 + a = -10 - 4 = -14$ and $\lim_{x\to 3} f(x) = 4b - 36 = 2 - 36 = -34.$

c. Now $f(-2) = -14$. So $\lim_{x\to -2} f(x) = -14 = f(-2)$. Also $f(3) = -11 \ne \lim_{x\to 3} f(x).$

7. False. Take $f(x) = |x|$ or $f(x) = x^2$.

8. (OPTIONAL) (a) Given $\varepsilon > 0, \exists \delta > 0$ such that

$$0 < |x - 3| < \delta \Rightarrow |f(x) - 5| < \varepsilon/2 \Rightarrow 2|f(x) - 5| < \varepsilon.$$

Thus, $0 < |x - 3| < \delta \Rightarrow |2f(x) - 10| = 2|f(x) - 5| < \varepsilon.$
This says then $\lim_{x\to 3} 2f(x) = 10$.

(b) (i) $|x^2 - 1| = |x - 1|.|x + 1| < |x - 1|.5$ if $|x + 1| < 5$
< 0.1 if $5|x - 1| < 0.1$.

Now $|x + 1| < 5 \Leftrightarrow -6 < x < 4$ and
$5|x - 1| < 0.1 \Leftrightarrow 1 - 0.02 < x < 1 + 0.02 \Leftrightarrow 0.98 < x < 1.02.$
Taking intersection gives $0.98 < x < 1.02$ I.e. $|x - 1| < 0.02.$

Choose $\delta = 0.02$. Need to check that this choice does work:
$|x-1| < 0.02 \Leftrightarrow 0.98 < x < 1.02 \Leftrightarrow 1.98 < x+1 < 2.02$
Thus, $|x-1| < 0.02 \Rightarrow |x^2-1| = |x-1|.|x+1| < 2.02 \times 0.02 < 0.1$.

(ii) Given $\varepsilon > 0$, you want to find a $\delta > 0$, such that $|x^2-1| = |x-1||x+1| < \varepsilon$.
Notice that for x near $x = 1$, $x+1$ cannot be large and would lie in a small interval.
Start with $0<x<2$ (i.e. $|x-1|<1$), then $1<x+1<3$ and so $|x+1| = x+1<3$
And so when $|x-1|<1$, $|x^2-1| = |x-1||x+1| < 3|x-1|$ --------- --- (1)
Now given $\varepsilon > 0$, if $|x-1| < \varepsilon/3$, then $3|x-1| < \varepsilon$ ------------ (2)
Choose $\delta = \min(1, \frac{\varepsilon}{3})$. Then $\delta \le 1, \varepsilon/3$. Thus both (1) and (2) holds when $|x-1| < \delta$.
$|x-1| < \delta \Rightarrow |x-1| < 1$ and $|x-1| < \frac{\varepsilon}{3}$
$\Rightarrow |x^2-1| = |x-1|.|x+1| < 3|x-1|$ by (1)
$< \varepsilon$ by (2)
Here is another way to get δ for part (i). Take $\delta = \min(1, (0.1)/3) = 1/30$.