1. Since for $x \neq 0$, $f(x) = x^3 \sin(\frac{1}{x^2})$, $f'(x) = 3x^2 \sin(\frac{1}{x^2}) + x^3 \cos\left(\frac{1}{x^2}\right) \cdot (\frac{-2}{x^3})$, for $x \neq 0$.
   Thus $f'(x) = 3x^2 \sin(\frac{1}{x^2}) - 2\cos\left(\frac{1}{x^2}\right)$, for $x \neq 0$.
   Now $\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^3 \sin(\frac{1}{x^2}) - 0}{x} = \lim_{x\to 0} x^2 \sin(\frac{1}{x^2}) = 0$ by the *Squeeze Theorem*, since $-|x|^2 \leq x^2 \sin(\frac{1}{x^2}) \leq |x|^2$ and $\lim_{x\to 0} |x|^2 = 0$. Therefore, $f'(0) = 0$.
   For any $\delta > 0$ choose an integer $k > 0$, such that $\frac{1}{\sqrt{k\pi}} < \delta$. Then, if we take $x_\delta = \frac{1}{\sqrt{k\pi}}$, we have $-\delta < x_\delta < \delta$ and $|f'(x_\delta)| = \left|3x_\delta^2 \sin(k\pi) - 2\cos(k\pi)\right| = |0 - 2\cos(k\pi)| = 2 \geq 1$. This means for $\varepsilon = 1$ and for any $\delta > 0$, we can find a $x_\delta \neq 0$ such that $-\delta < x_\delta < \delta$ but $|f'(x_\delta) - 0| \geq 1$. Hence $\lim_{x\to 0} f'(x) \neq 0 = f'(0)$ Therefore, f' is not continuous at $x = 0$.

2. a. Derivative of $\frac{1}{(x-1)^2}$ at $x = 2$.
   b. Derivative of $\sin(x^2)$ at $x = 0$.
   c. Derivative of $\sqrt{x+1}$ at $x = 0$.

3. a. Obviously $f(x) = |x|^3 = \begin{cases} -x^3, & x < 0 \\ x^3, & x \geq 0 \end{cases}$. Therefore, $f'(x) = \begin{cases} -3x^2, & x < 0 \\ 3x^2, & x > 0 \end{cases}$. We have $\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{|x|^3}{x} = 0$. Therefore, $f'(0) = 0$.
   b. Since $f(x) = \begin{cases} x^4 + 3, & x \leq 1 \\ 3x^2, & x > 1 \end{cases}$, $f'(x) = \begin{cases} 4x^3, & x < 1 \\ 6x, & x > 1 \end{cases}$. Now $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^4 + 3 = 4$ but $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 3x^3 = 3$. Thus $\lim_{x\to 1} f(x)$ does not exist and so f is not continuous at $x = 1$. Therefore, f is not differentiable at $x = 1$.

4. For the function f to be differentiable at $x = 1$, f must be continuous at $x = 1$.
   Now $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (ax^2 + 3) = a + 3$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} bx^3 = b = f(1)$. Therefore, the condition for continuity is translated into
   $$a + 3 = b. \qquad (1)$$
   Also $\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{b(1+h)^3 - b}{h} = \lim_{h\to 0^+} (3b + 3bh + bh^2) = 3b$ and
   $\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{a(1+h)^2+3-b}{h} = \lim_{h\to 0^-} \frac{ah^2 + 2ah + a + 3 - b}{h} = \lim_{h\to 0^-} \frac{ah^2 + 2ah}{h}$ by (1) $= 2a$.
   Thus, for differentiability at $x = 1$,
   $$2a = 3b. \qquad (2)$$
   Solving (1) and (2) gives $a = -9$ and $b = -6$.

5. a. $f'(x) = \frac{d}{dx}(\cot(x^2) + \sec(x)) = -\csc^2(x^2)2x + \sec(x)\tan(x)$.
   b. $f'(x) = \frac{d}{dx}\frac{\cos(3x)+1}{\cos(2x)-1} = \frac{-3\sin(3x)(\cos(2x)-1) - (\cos(3x)+1)(-2\sin(2x))}{(\cos(2x)-1)^2}$ by the quotient rule
   $$= \frac{3\sin(3x) + 2\sin(2x) - 3\sin(3x)\cos(2x) + 2\cos(3x)\sin(2x)}{(\cos(2x)-1)^2}$$
   $$= \frac{3\sin(3x) + 2\sin(2x) - \sin(3x)\cos(2x) - 2\sin(x)}{(\cos(2x)-1)^2}.$$
   c. $f'(x) = \frac{d}{dx}(\sin^2(7x)\cos^3(x))$
   $$= 2\sin(7x)(7\cos(7x)) \cdot \cos^3(x) + \sin^2(7x) \cdot 3\cos^2(x)(-\sin(x))$$
   $$= 14\sin(7x)\cos(7x)\cos^3(x) - 3\sin^2(7x)\cos^2(x)\sin(x).$$
   d. $\frac{d}{dx}\left[\frac{7x+1}{x^2+x+1}\right]^4 = 4\left[\frac{7x+1}{x^2+x+1}\right]^3 \frac{7(x^2+x+1) - (7x+1)(2x+1)}{(x^2+x+1)^2}$
   $$= 4\frac{(7x+1)^3(-7x^2 - 2x + 6)}{(x^2+x+1)^5}.$$

e. $\frac{d}{dx}\cos^3(\cos(5x)) = 3\cos^2(\cos(5x))\frac{d}{dx}\cos(\cos(5x))$
$= 3\cos^2(\cos(5x))(-\sin(\cos(5x)))(-5\sin(5x))$
$= 15\cos^2(\cos(5x))\sin(\cos(5x))\sin(5x)$.

f. $f(x) = \sqrt{x+\sqrt{x+\sqrt{x}}} = \left(x+(x+x^{\frac{1}{2}})^{\frac{1}{2}}\right)^{\frac{1}{2}}$. Thus,

$f'(x) = \frac{1}{2}\left(x+(x+x^{\frac{1}{2}})^{\frac{1}{2}}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(x+(x+x^{\frac{1}{2}})^{\frac{1}{2}}\right) = \frac{1}{2}\left(x+(x+x^{\frac{1}{2}})^{\frac{1}{2}}\right)^{-\frac{1}{2}}\left(1+\frac{d}{dx}(x+x^{\frac{1}{2}})^{\frac{1}{2}}\right)$.

Now $\frac{d}{dx}(x+x^{\frac{1}{2}})^{\frac{1}{2}} = \frac{1}{2}(x+x^{\frac{1}{2}})^{-\frac{1}{2}}(1+\frac{1}{2}x^{-\frac{1}{2}})$. Therefore,

$f'(x) = \frac{1}{2}\left(x+(x+x^{\frac{1}{2}})^{\frac{1}{2}}\right)^{-\frac{1}{2}}\left(1+\frac{1}{2}(x+x^{\frac{1}{2}})^{-\frac{1}{2}}(1+\frac{1}{2}x^{-\frac{1}{2}})\right)$

$= \frac{1}{2(x+(x+x^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}}\left[1+\frac{1}{2(x+x^{\frac{1}{2}})^{\frac{1}{2}}}\left(1+\frac{1}{2x^{\frac{1}{2}}}\right)\right]$.

6. Since $f(x) = x\,g(x^2)$, by the *Product Rule*, $f'(x) = g(x^2) + x\,g'(x^2)2x = g(x^2) + 2x^2 g'(x^2)$.
Therefore, $f''(x) = 2x\,g'(x^2) + 4x\,g'(x^2) + 2x^2 g''(x^2)2x$
$= 6x\,g'(x^2) + 4x^3 g''(x^2)$

7. a. Differentiating $x^6+y^6=1$ implicitly with respect to x, we get

$$6x^5+6y^5\frac{dy}{dx}=0. \qquad (1)$$

Differentiating implicitly again, we get

$$30x^4+30y^4\frac{dy}{dx}\frac{dy}{dx}+6y^5\frac{d^2y}{dx^2}=0. \qquad (2)$$

Therefore, $y^5\frac{d^2y}{dx^2} = -5(x^4+y^4\left(\frac{dy}{dx}\right)^2)$. From (1), for $y\neq 0$, $\frac{dy}{dx} = -\frac{x^5}{y^5}$. Thus,
$y^5\frac{d^2y}{dx^2} = -5(x^4+y^4\frac{x^{10}}{y^{10}})$ so that $\frac{d^2y}{dx^2} = -5(\frac{x^4y^6+x^{10}}{y^{11}}) = -5\frac{x^4}{y^{11}}$ for $y\neq 0$.

b. $y^3+x=\sin(y)$. Differentiating implicitly, we get $3y^2\frac{dy}{dx}+1=\cos(y)\frac{dy}{dx}$, that is,
$(\cos(y)-3y^2)\frac{dy}{dx}=1$. Differentiating this equation implicitly, we get
$(-\sin(y)-6y)\left(\frac{dy}{dx}\right)^2+(\cos(y)-3y^2)\frac{d^2y}{dx^2}=0$. Therefore, $\frac{d^2y}{dx^2} = \frac{\sin(y)+6y}{(\cos(y)-3y^2)^3}$ for $\cos(y)-3y^2\neq 0$.

8. (a) $p(t) = \frac{50000e^{0.6t}}{9+e^{0.6t}} = \frac{50000}{\frac{9}{e^{0.6t}}+1} < 50000$. Since
$\lim_{t\to\infty} p(t) = \lim_{t\to\infty}\frac{50000}{\frac{9}{e^{0.6t}}+1} = \frac{50000}{0+1} = 50000$, the upper limit for the population is 50000.

$p(t) = \frac{50000e^{0.6t}}{9+e^{0.6t}}\frac{50000}{\frac{9}{e^{0.6t}}+1} = \frac{9}{10}50000 \Leftrightarrow \frac{9}{e^{0.6t}} = \frac{10}{9}-1 = \frac{1}{9} \Leftrightarrow e^{0.6t} = 81$

$$\Leftrightarrow 0.6t = \ln(81) = 4\ln(3) \Leftrightarrow t = \frac{20}{3}\ln(3) \approx 7.3241 \text{ years.}$$

(b) The population index would be $\sqrt{50000} = 100\sqrt{5} \approx 223.61$.
$I(p) = 100 \Leftrightarrow \sqrt{p} = 100 \Leftrightarrow p = 10000$. Therefore a population of 10000 would give an index of 100.
$p(t) = \frac{50000e^{0.6t}}{9+e^{0.6t}} = \frac{50000}{\frac{9}{e^{0.6t}}+1} = 10000 \Leftrightarrow \frac{9}{e^{0.6t}} = 4 \Leftrightarrow e^{0.6t} = \frac{9}{4} \Leftrightarrow 0.6t = 2\ln(\frac{3}{2})$
$\Leftrightarrow t = \frac{10}{3}\ln(\frac{3}{2}) \approx 1.3516$.
Therefore a population index of 100 would be reached in approximately 1.35 years.

(c) $I(t) = \sqrt{p(t)} = \sqrt{\frac{50000e^{0.6t}}{9+e^{0.6t}}}$. Therefore, using *the Chain Rule* and *the Quotient rule*, we have
$I'(t) = \frac{270\sqrt{5}\,e^{0.3t}}{(9+e^{0.6t})^{3/2}}$ assuming that the derivative of $e^x = e^x$.