National University of Singapore

Department of Mathematics

level 1000 (2005/2006) Semester 2 MA1102R Calculus Tutorial set 4

If the doors of perception were cleansed, every thing would appear to man as it is, infinite.
For man has closed himself up, till he sees all things through narrow chinks of his cavern.
- William Blake

To see the world in a grain of sand, and a heaven in a wild flower.
Hold infinity in the palm of your hand, and eternity in an hour.
- William Blake

Definition 1. We say that the *limit* of $f(x)$ as x tends to a is $+\infty$, if for every real number $K > 0$, there exists $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow f(x) > K$. We write formally $\lim_{x\to a} f(x) = +\infty$.

Definition 2. We say the *right limit* of $f(x)$ as x tends to a from the right is $+\infty$ if for any $K > 0$, there exists a $\delta > 0$ such that $0 < x - a < \delta \Rightarrow f(x) > K$, i.e., $a < x < a + \delta \Rightarrow f(x) > K$. We write $\lim_{x\to a^+} f(x) = +\infty$. Similarly, we say the *left limit* of $f(x)$ as x tends to a from the left is $+\infty$, if for any $K >$ 0, there exists $\delta > 0$ such that $a - \delta < x < a \Rightarrow f(x) > K$. We write $\lim_{x\to a^-} f(x) = +\infty$.

Theorem 1. $\lim_{x\to 0} \frac{1}{|x|} = +\infty$, $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

Activity 1. Find the following limits. 1. $\lim_{x\to 5^+} \frac{1}{|x-5|}$. 2. $\lim_{x\to 7^+} \frac{1}{x-7}$.

Definitions 3. 1. We write $\lim_{x\to +\infty} f(x) = L \in \boldsymbol{R}$ if, given $\varepsilon > 0$, there exists $K > 0$ such that $x > K \Rightarrow | f(x) - L| < \varepsilon$.

2. We write $\lim_{x\to -\infty} f(x) = L$ if, given $\varepsilon > 0$, there exists $K < 0$ such that $x < K \Rightarrow | f(x) - L| < \varepsilon$.

Activity 2. Find the following limits. 1. $\lim_{x\to +\infty} \frac{1}{x-3}$. 2. $\lim_{x\to -\infty} \frac{1}{x+1}$.

Definitions 4. 1. We say the *limit* of $f(x)$ as x tends to a is $-\infty$ if, for any $K < 0$, there exists a $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow f(x) < K$ and we write formally $\lim_{x\to a} f(x) = -\infty$.

2. We write $\lim_{x\to a^+} f(x) = -\infty$ if, given any $K < 0$, there exists a $\delta > 0$ such that $0 < x - a < \delta \Rightarrow f(x) < K$.

3. We write $\lim_{x\to a^-} f(x) = -\infty$ if, given any $K < 0$, there exists a $\delta > 0$ such that $0 < a - x < \delta \Rightarrow f(x) < K$
$0 < a - x < \delta \Rightarrow f(x) < K$.

4. Similar definitions for $\lim_{x\to +\infty} f(x) = \pm\infty$, $\lim_{x\to -\infty} f(x) = \pm\infty$.

Activity 3. Find the following limits. 1. $\lim_{x\to 1} \frac{-1}{(x-1)^2}$. 2. $\lim_{x\to 1^-} \frac{1}{x-1}$. 3. $\lim_{x\to 2^+} \frac{1}{2-x}$.

Theorem 2. 1. If $\lim_{x\to a} f(x) = +\infty$ and $\lim_{x\to a} g(x) = c$, then $\lim_{x\to a} [\, f(x) + g(x)] = +\infty$.

2. If $\lim_{x\to a} f(x) = -\infty$ and $\lim_{x\to a} g(x) = c$, then $\lim_{x\to a} [\, f(x) + g(x)] = -\infty$.
Here $x \to a$ may be replaced by $x \to a^+$ or $x \to a^-$.

Theorem 3. Suppose $\lim_{x\to a} f(x) = +\infty$ and $\lim_{x\to a} g(x) = c \neq 0$.

1. If $c > 0$, then $\lim_{x \to a} f(x)g(x) = +\infty$.

2. If $c < 0$, then $\lim_{x \to a} f(x)g(x) = -\infty$.
Here $x \to a$ may be replaced by $x \to a^+$ or $x \to a^-$.

Theorem 4. Suppose $\lim_{x \to a} f(x) = -\infty$ and $\lim_{x \to a} g(x) = c \neq 0$.

1. If $c > 0$, then $\lim_{x \to a} f(x)g(x) = -\infty$.

2. If $c < 0$, then $\lim_{x \to a} f(x)g(x) = +\infty$.

Here $x \to a$ may be replaced by $x \to a^+$ or $x \to a^-$.

Activity 4. Find 1. $\lim_{x \to 7} [\frac{1}{(x-7)^2} + 2 + 3x]$. 2. $\lim_{x \to 7} \frac{2+3x}{(x-7)^2}$. 3. $\lim_{x \to 7} \frac{x-9}{(x-7)^2}$.

Useful results:

1. If $\lim_{x \to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x \to a} \frac{1}{f(x)} = 0$.

2. If $\lim_{x \to a} f(x) = 0$ and for some $\delta > 0$, $f(x) > 0$ on $(a\text{-}\delta, a\text{+}\delta)$ except possibly at a, then $\lim_{x \to a} \frac{1}{f(x)} = +\infty$.

3. If $\lim_{x \to a} f(x) = 0$ and for some $\delta > 0$, $f(x) < 0$ on $(a\text{-}\delta, a\text{+}\delta)$ except possibly at a, then $\lim_{x \to a} \frac{1}{f(x)} = -\infty$.

Definition 5. The line $x = a$ is said to be a *vertical asymptote* of the graph of the function f if $\lim_{x \to a^+} f(x) = +\infty$ or $-\infty$ or if $\lim_{x \to a^-} f(x) = +\infty$ or $-\infty$. The line $y = b$ is said to be a *horizontal asymptote* of the graph of f if $\lim_{x \to +\infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$.

Activity 5. What are the vertical and horizontal asymptotes of the graph of the function $f(x) = \frac{1}{x-9}$?

Theorem 5 (Squeeze Theorem). Suppose f, g and h are functions defined on an open interval I except possibly at a in I. Suppose for all $x \neq a$ in I, $f(x) \leq g(x) \leq h(x)$. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} h(x)$ exist and are both equal to L , then $\lim_{x \to a} g(x) = L$.

Activity 6. Show that $\lim_{x \to 0} x^2 \cos(\frac{1}{x^2}) = 0$.

Theorem 6. $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ and $\lim_{x \to 0} \frac{1-\cos(x)}{x} = 0$.

Activity 6. Find the following limits. 1. $\lim_{x \to 0} \frac{x}{\sin(x)}$. 2. $\lim_{t \to 0} \frac{\sin(t^3)}{t^3}$.

Definition 6. Let f be a function defined on some open interval I containing a. Then we say f is *continuous* at a if 1. $f(a)$ is defined, 2. $\lim_{x \to a} f(x)$ exists and 3. $\lim_{x \to a} f(x) = f(a)$.

We say f is *continuous* on I if f is continuous at x for all x in I.

Discuss: When is a function defined at $x = a$ not continuous at $x = a$? Is there a function which is no where continuous?

Activity 7. Show that the function $f(x) = x^2$ is continuous at $x = a$ for all a in **R**.

Theorem 7. Suppose f and g are continuous at a. Then 1. $f \pm g$ is continuous at a,

2. $f \cdot g$ is continuous at a, and 3. if $g(a) \neq 0$, $\frac{f}{g}$ is continuous at a.

Theorem 8. Any polynomial function is continuous on the whole of **R.**

Theorem 9 (Composite Function). Suppose f and g are two functions such that the range of f is contained in the domain of g so that the composite $g \circ f$ is defined on *domain*(f). Let a be in *domain*(f). If f is *continuous* at a and g is *continuous* at $f(a)$, then $g \circ f$ is *continuous* at a.

Activity 8. Explain why $\sqrt{x^2+2}$ is continuous at $x = 0$.

Definitions 7. 1. The function f is said to be *continuous from the right* at a point a if
a. $f(a)$ is defined and
b. the right limit $\lim_{x \to a^+} f(x)$ exists and is equal to $f(a)$.

2. The function f is said to be *continuous from the left* at a if
a. $f(a)$ is defined and
b. the left limit $\lim_{x \to a^-} f(x)$ exists and is equal to $f(a)$.

Definitions 8. 1. A function f is said to be *continuous* on $[a, b]$ if and only if f is continuous on (a, b), f is continuous from the right at a and continuous from the left at b.
2. f is said to be *continuous* on $[a, b)$ if and only if f is continuous on (a, b) and continuous from the right at a.
3. f is said to be *continuous* on $(a, b]$ if and only if f is continuous on (a,b) and continuous from the left at b.

Activity 9. Explain why the function h defined by $h(x) = \sqrt{x}$ with domain(h) = $[0, \infty)$ is continuous at $x = 0$.

Read the following theorem carefully. Think of the graph of a continuous function on $[a, b]$ as a deformed but yet uncut flexible "wire". Make sure you read this before you attempt questions 8 and 9.

Theorem 10 (Intermediate Value Theorem). Let $[a, b]$ be a closed interval with $a < b$. Suppose $f:[a, b] \to \mathbf{R}$ is a continuous function such that $f(a) = \alpha$, $f(b) = \beta$ and $\alpha \neq \beta$. Then for any real number γ between α and β, there is a point c in $[a, b]$ such that $f(c) = \gamma$.

For a proof of this theorem see the following article **"The Boundedness Theorem, Extreme Value Theorem and Intermediate Value Theorem**" at
http://www.math.nus.edu.sg/~matngtb/Calculus/Extreme%20Value/bound.htm

You will need to know the meaning of the completeness for **R**, supremum and infimum of a set.

Activity 10. Let $f(x) = x^2 + 3x - 2$. Show that f has a zero at some point between 0 and 1.

MA1102R **Calculus** **Assignment 4**

1. Evaluate the following limits.

a. $\lim_{x\to+\infty} \sqrt[3]{\dfrac{8x^2+7}{27x^2-1}}$. b. $\lim_{x\to-\infty} \dfrac{\sqrt{x^2+9}}{1+4x}$. c. $\lim_{x\to-6^-} \dfrac{3x}{36-x^2}$. d. $\lim_{x\to4^+} \dfrac{x-4}{\sqrt{8x-x^2}-4}$.

2. Evaluate the following limits.

a. $\lim_{x\to+\infty} \dfrac{23x^2-5x^3+7}{13x^3+3}$. b. $\lim_{x\to-\infty} \left(\dfrac{x^3}{4x^2-2} - \dfrac{x^2}{4x+3}\right)$. c. $\lim_{x\to-\infty} \dfrac{\sqrt{9x^2-2}}{3-x}$. d. $\lim_{x\to+\infty} \left(\sqrt{x^2+25000} - x\right)$.

3. Find the vertical and horizontal asymptotes of the graphs of the following functions:
   a. $f(x) = \dfrac{5x}{x-3}$. b. $f(x) = \dfrac{1}{2x^2-x-17}$.

4. Show that the function f defined by $f(x) = \begin{cases} 2x^2+1\,, & x \le -1 \\ 2-x\,, & x > -1 \end{cases}$ is continuous at $x = -1$.

5. Find all values of x at which the given function is continuous.

a. $f(x) = \begin{cases} 2x^2+3\,, & x \le 1 \\ 5-3x\,, & 1 < x < 3 \\ x-7\,, & x \ge 3 \end{cases}$. b. $g(x) = \begin{cases} \dfrac{|x-4|}{x-4}\,, & x \ne 4 \\ 2\,, & x = 4 \end{cases}$.

6. Evaluate the following limits.

a. $\lim_{x\to0} \dfrac{\sin(7x)}{\sin(5x)}$. b. $\lim_{x\to0} \dfrac{\tan^4(2x)}{4x^4}$. c. $\lim_{x\to0} \dfrac{1-\cos(8x)}{\sin(8x)}$. d. $\lim_{x\to0} \dfrac{\sin(\sin(x^2))}{23x}$. e. $\lim_{x\to0} x^3\cos(\dfrac{1}{x^7})$.

7. Let g be a function that satisfies $1-4x^2 \le g(x) \le \cos(2x)$ for all x in the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Is g continuous at $x = 0$? Justify your answer. [Hint: What is the limit at $x = 0$?]

8. Find a non-zero value for the constant k so that the function h defined by
   $h(x) = \begin{cases} \dfrac{\tan(kx)}{x}\,, & x < 0 \\ 7x+5k^2\,, & x \ge 0 \end{cases}$ will be continuous at $x = 0$.

9. a. Apply the *Intermediate Value Theorem* to show that there is a root of the polynomial equation $2x^3 + x^2 + 2 = 0$ in the interval (-2, -1).
   b. Show that the equation $x - 2\sin(x) = 1$ has a solution in the open interval $(0, \frac{3\pi}{2})$.

10. Let f be a function such that 1. f is continuous on [0,2] and 2. $0 \le f(x) \le 2$ for all x in [0, 2]. Show that there is a number c in [0,2] such that $f(c) = c$. (Hint: Apply the *Intermediate Value Theorem* to the function g defined on [0, 2] by $g(x) = f(x) - x$. Use the inequality in 2.)

11. **(Optional)** True or false? If f is continuous on $[a, b]$, $f(a) < 0$ and $f(b) > 0$, then $f(x)$ cannot have *exactly two* zeros in $[a, b]$. Why? [Hint: Consider graphs as flexible wires.]

12. **(Optional)*** Suppose that the function $f : \mathbf{R} \to \mathbf{R}$ is continuous and that $f(x) = 0$ if x is rational. Prove that $f(x) = 0$ for all x in **R**. (Hint: Think of the meaning of continuity. The following might be useful. If f is continuous at $x = a$ and $\{x_n\}$ is a sequence that converges to a, then $\{f(x_n)\}$ converges to $f(a)$. Try proving this. Refer to Discussion after Activity 8 in Tutorial 3.)

13. **(Optional)*** Suppose that the function $f : \mathbf{R} \to \mathbf{R}$ satisfies $f(x + y) = f(x) + f(y)$ for all x and y in R.

    (1) Prove that if $f(1) = k$, then $f(x) = kx$ for all rational numbers x.
    (2) Use part (1) to prove that if $f : \mathbf{R} \to \mathbf{R}$ is continuous, then $f(x) = kx$ for all x in **R**.