National University of Singapore

Department of Mathematics

level 1000 Semester 2 (2005/2006) MA1102R Calculus Tutorial set 3

> **Cauchy:** *When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the limit of all the others.*

Unfortunately, the phrase "a variable approach a limit" somehow suggests time and motion. Thus Weierstrass in trying to remove this vague and intuitive meaning of "approaching a limit" and making precise the meaning of "differing by as little as one wishes" gave the now accepted definition of a limit. Bernhard Bolzano (1781-1848), a priest, philosopher, and mathematician of Bohemia, gave the proper definition of continuity, although the language used was not very precise. Thus our modern definition of continuity owes much to Bolzano, Cauchy, Weierstrass and also to Cantor for providing the existence of the irrationals and the real number system as we know it today. Mistakes had been made. For example, Cauchy asserted that if a function of several variables is continuous in each one separately, then it is a continuous function of all the variables. This is not correct. Also the existence of the irrational numbers was taken for granted. However, Bolzano, Cauchy and Weierstrass pioneered the method of investigation of subsets of the reals and continuity of a function in what is now commonly referred to as the "Cauchy sequence" technique. What came out of this is Cantor's construction of the real number system using Cauchy sequences. Mathematics did not evolve in a logical way. Looking back to the complaint of Abel in 1826 "the tremendous obscurity which one unquestionably finds in analysis (calculus)." " It lacks so completely all plan and system that it is peculiar that so many men could have studied it. The worst of it is, it has never been treated stringently. There are very few theorems in advance analysis which have been demonstrated in a logical tenable manner. **Everywhere one finds this miserable way of concluding from the special to the general** and it is extremely peculiar that such a procedure has led to so few of the so-called paradoxes." Does it still stands in our way of learning calculus **today? I hope not. Don't base your understanding on "concluding from the special to the general" ? Don't mistake it together with intuition to be the way to understanding. Intuition must give way to precision.**

Definition 3. We say the *limit* of $f(x)$ as x tends to a exists and equals L if given any $\varepsilon > 0$, $\exists\, \delta > 0$ such that $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$ and we write $\lim_{x\to a} f(x) = L$.

(When does a function NOT have a limit at the point a ? Think!)

Activity 4. Show that $\lim_{x\to 7} x = 7$.

Theorem 1. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then 1. $\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$,

2. $\lim_{x\to a} (f(x)\cdot g(x)) = (\lim_{x\to a} f(x)) \cdot (\lim_{x\to a} g(x))$ and

3. if further $\lim_{x\to a} g(x) \neq 0$, $\lim_{x\to a} \frac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.

Activity 5. Find the following limits. 1. $\lim_{x\to 2} (x^3 - 3x^2 + 7)$. 2. $\lim_{x\to 1} \dfrac{2x+5}{x+3}$.

Theorem 2. If n is a positive integer and $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} (f(x))^{\frac{1}{n}} = L^{\frac{1}{n}}$ $(= (\lim_{x\to a} f(x))^{\frac{1}{n}})$, where if n is even we assume that $L \geq 0$.

Activity 6. Find $\lim_{x\to 2} [x^3 + 3]^{\frac{1}{2}}$.

Definition 4. Let f be a function defined on some open interval (a, c) with $a < c$. Then we say L is the *limit from the right* or *right limit* of $f(x)$ as x tends to a if given $\varepsilon > 0$, $\exists\, \delta > 0$ such that $a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon$. We write $\lim_{x\to a^+} f(x) = L$.

Definition 5. Let f be a function defined on some open interval (d, a) with $d < a$. Then we say L is the *limit from the left* or *left limit* of $f(x)$ as x tends to a if
given $\varepsilon > 0$, $\exists\, \delta > 0$ such that $a - \delta < x < a \Rightarrow |\, f(x) - L| < \varepsilon$. We write $\lim_{x \to a^-} f(x) = L$.

Activity 7. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \begin{cases} 2x - 1 & \text{if } x > 1 \\ 3 - x & \text{if } x \le 1 \end{cases}$. Find $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$.

Theorem 3. Let f be a function defined on an open interval containing a except possibly at a. Then $\lim_{x \to a} f(x)$ exists and is equal to $L \Leftrightarrow \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$.

Activity 8. 1. Explain why for the function f defined in Activity 7 above $\lim_{x \to 1} f(x)$ does not exist.

2. Let $f(x) = \begin{cases} 5x^2 - 2 & x \ge 1 \\ x^3 + 2 & x \le 1 \end{cases}$. Find $\lim_{x \to 1} f(x)$.

Discuss: If there is a sequence $\{a_n\}$ such that a_n tends to a as n tends to infinity and $\lim_{n \to \infty} f(a_n) = L$, can we conclude in general that $\lim_{x \to a} f(x) = L$? If not, why? What other or additional condition do we need to impose so that we can conclude that $\lim_{x \to a} f(x) = L$?

(A sequence $\{a_n\}$ tends to the point a if given $\varepsilon > 0$, we can find an integer N such that for all integer $n > N$, $|a_n - a| < \varepsilon$. We can also write $\lim_{n \to \infty} a_n = a$. Thus $\lim_{n \to \infty} f(a_n) = L$ means given any $\varepsilon > 0$, we can find an integer N such that for all integer $n > N$, $|\, f(a_n) - L| < \varepsilon$.)

Suggested Reading: Article on Rigour of Calculus in, A Century of Calculus Volume 1 and Volume 2 by TM Apostol et al.

> ***Who gave you the epsilon? Cauchy and the origins of rigorous calculus, Judith V Grabiner, American Mathematical Monthly Vol 90 p185-194.***

1. Evaluate the following limits. (Examples in the course site example sessions page.)

a. $\lim\limits_{x\to 2}\dfrac{9x^2+5x-2}{x-4}$. b. $\lim\limits_{x\to 1}\dfrac{x^3+4x^2-4x-1}{x-1}$. c. $\lim\limits_{x\to 4}\dfrac{\sqrt[3]{x^2+1}-\sqrt[3]{17}}{x-4}$.

d. $\lim\limits_{x\to 5}\dfrac{\sqrt{x^2+1}-\sqrt{26}}{x-5}$ e. $\lim\limits_{t\to 0}\dfrac{(k+t)^3-k^3}{t}$.

2. Evaluate the following limits.

a. $\lim\limits_{x\to 7^+}\dfrac{x-6}{1-\sqrt{x-7}}$. b. $\lim\limits_{x\to 5^-}(|x-7|-|x-2|)$.

3. Let $g(x) = 1 + [x/2] + [-x/2]$, where as usual $[t]$ denotes the greatest integer less than or equal to t. Find the following limits if they exist. (Hint: Examine values of g near 2 on both sides of 2.)

a. $\lim\limits_{x\to 2^+} g(x)$. b. $\lim\limits_{x\to 2^-} g(x)$. c. $\lim\limits_{x\to 2} g(x)$.

4. Evaluate $\lim\limits_{x\to 3^-}[x^2-3]$, where [t] is as defined in Question 3.

5. Let the function f be defined by $f(x)=\begin{cases} 7x-5, & \text{if } x<3 \\ 7, & \text{if } x=3 \\ x^2+x-5, & \text{if } x>3 \end{cases}$.

a. Determine if $\lim\limits_{x\to 3} f(x)$ exists. b. Sketch the graph of $f(x)$.

6. Let $f(x)=\begin{cases} 5x+a, & x<-2 \\ -14, & x=-2 \\ ax^2+4b, & -2<x<3 \\ -11, & x=3 \\ 4b-12x, & x>3 \end{cases}$, where a and b are constants.

a. Find the values of a and b for which both $\lim\limits_{x\to -2} f(x)$ and $\lim\limits_{x\to 3} f(x)$ exist.

b. Evaluate $\lim\limits_{x\to -2} f(x)$ and $\lim\limits_{x\to 3} f(x)$. c.

Is $\lim\limits_{x\to -2} f(x)$ equal to $f(-2)$? Is $\lim\limits_{x\to 3} f(x)$ equal to $f(3)$?

(When the answer is yes, the function f is continuous at the point where the limit is taken.)

7. State if the following statement is true or false. If you think it is true, then prove it. If you think it is false, then give a counterexample. (Hint: Start with sketches of a function whose values between -1 and 1 are positive except perhaps at 0 and think about the possibilities of left and right limits being the same.)

If f is a function and $f(x)>0$ on $(-1,0)\cup(0,1)$ and $\lim\limits_{x\to 0} f(x)$ exists, **then** $\lim\limits_{x\to 0} f(x)>0$.

8. **Optiona**l. (a) Use the epsilon-delta definition (i.e. from first principle) to show that

If $\lim\limits_{x\to 3} f(x)=5$, then $\lim\limits_{x\to 3} 2f(x)=10$.

(b) For each of the following parts, find a positive number $\delta > 0$ such that

(i) $|x-1|<\delta \Rightarrow |x^2-1|<0.1$

(ii) $|x-1|<\delta \Rightarrow |x^2-1|<\varepsilon$, where $\varepsilon > 0$. ($\delta > 0$ will be in terms of ε).