2005/2006 Semester 2 MA1102R Calculus Tutorial 2.

Solutions for Students

Assignment 2

COMMENT. In order to show two sets A and B are equal, we need to show that $A \subseteq B$ **AND** $B \subseteq A$. In most questions, where you are asked to **determine the range of a function**, do as follows. Having guessed(??, Should you?) that as in question 1 the range(f) is **R**−{ 5/3}, you need to check that the range (f) is indeed **R**−{5/3}. This is done by first arguing that range(f)⊆ **R**−{5/3}. (You should see that we do this by showing that $f(x) \neq 5/3$ for those x which lies in the domain of f.) Next, we show that **R**−{5/3} ⊆ *range*(f) . This is done as follows: start with a general element $y \in R - \{5/3\}$, we then proceed to show that eventually $y \in range(f)$ by obtaining a choice of x in the domain of f such that $y = f(x)$. Having studied Question 1 closely and understood it well, try to work out question 2 and compare your workings. Try to find out whether you have missed any key point. Questions 3 and 4 are similar.

To show that a function f is injective, one starts with any two elements a and b with the same image under f and try to argue (this requires your ability in algebraic manipulation) that $a = b$. If in the process of arguing $a = b$ you obtain other possibility (for example in Question 4(b) $a = 6-b$), this would suggests that f may not be injective. For Question 4(b), you may choose $a = 0$ and $b = 6$ and check that $f(0) = f(6)$, which clearly means that f is not injective.

To show that a function f is surjective, one begins with an arbitrary chosen element y from the codomain(f), and try (working backwards so to speak to obtain a choice for x in the domain of f) to find an x such that $y = f(x)$.

There are different ways of solving inequalities that you probably have learnt in pre-university mathematics. Now try to see why your method works and at the same time, understand the arguments provided in the solution.

Finally, the solution presented here should not be taken as a '**standard' solution.**

1. a. The expression $f(x) = \frac{5x-19}{3x-6}$ is meaningful if and only if $x \neq 2$.
   Thus the largest subset D for which f defines a function is **R** − {2}.

   b. The range of $f : D \to \boldsymbol{R}$ by definition is Range (f) $= \{ f(x) : x \in D\}$. Thus an element y is in Range (f) if there exists an element x in the domain of f such that $f(x) = y$. Thus we ask: For what values of y can we solve the equation

   $$f(x) = y \qquad (1)$$

   for x in the domain D ?

   Equation (1) can be written

   $$f(x) = \frac{5x-19}{3x-6} = \frac{5}{3}\left[\frac{x-\frac{19}{5}}{x-2}\right] = \frac{5}{3}\left[\frac{x-2+\frac{10-19}{5}}{x-2}\right] = \frac{5}{3}\left[1 - \frac{9}{5(x-2)}\right] = y. \qquad (2)$$

   Note that $\frac{9}{5(x-2)} \neq 0$ for any $x \neq 2$. Thus, when $x \neq 2, y = f(x) = \frac{5}{3}[1 - \frac{9}{5(x-2)}] \neq \frac{5}{3}$.

   Thus, **range(f) ⊆ R−{ $\frac{5}{3}$ }.**

   Solving (2) gives $x = \frac{9}{5-3y} + 2$. (3)

   Thus, given any $y \neq \frac{5}{3}$ (i.e. $y \in$ **R**−{ $\frac{5}{3}$ }), we have $y = f(x)$, where $x = \frac{9}{5-3y} + 2$. This means that $y \in$ range(f). So, $\boldsymbol{R} - \{\frac{5}{3}\} \subseteq range(f)$.

   In conclusion, we have shown that the **range(f) = R−{ $\frac{5}{3}$ }.**

   c. Let E = Range (f). Now $f' : D \to E$ is defined by $f'(x) = f(x)$ for all x in D. Thus $f'(D) = f(D)$ = Range (f) $= E$. So f' is surjective. f' is injective. This is shown as follows.

   $f'(x) = f'(y) \Rightarrow f(x) = f(y) \Rightarrow \frac{5}{3}[1 - \frac{9}{5(x-2)}] = \frac{5}{3}[1 - \frac{9}{5(y-2)}] \Rightarrow x = y$.

   Thus by definition f' is injective and so f' is bijective.

The inverse function $(f')^{-1} : E \to D$ is defined by $(f')^{-1}(y) = x$ such that $f'(x) = y$. This is the same as equation (1) and x is given by (3). Therefore,

$$(f')^{-1}(y) = \tfrac{9}{5-3y} + 2 .$$

2. $g : \mathbf{R} - \{\tfrac{2}{9}\} \to \mathbf{R}$ is a function defined by $g(x) = \tfrac{1}{9x-2}$ for all x in $\mathbf{R} - \{\tfrac{2}{9}\}$.

a. As in Question 1b we look at the equation

$$g(x) = \tfrac{1}{9x-2} = y. \qquad (1)$$

Since $\tfrac{1}{9x-2} \neq 0$, if $y = 0$, then (1) has no solution. So a necessary condition for (1) to be solvable is $y \neq 0$. Now if $y \neq 0$, then we can take reciprocals on both sides of the equation (1) and obtain $9x - 2 = \tfrac{1}{y}$ and so $x = \tfrac{1}{9}[2 + \tfrac{1}{y}]$. Thus the range of g is $\mathbf{R} - \{0\}$.

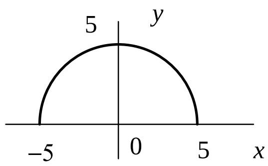
b. When g is considered as a function $g : \mathbf{R} - \{\tfrac{2}{9}\} \to range(g)$, g is surjective since its codomain is already its range. g is also injective for $g(x) = g(y)$ implies that $\tfrac{1}{9x-2} = \tfrac{1}{9y-2}$ which in turn implies that $x = y$. Thus g is bijective.

g^{-1} : Range$(g) \to \mathbf{R} - \{\tfrac{2}{9}\}$ is defined by $g^{-1}(y) = x$ such that $g(x) = y$. The value x is given by $\tfrac{1}{9}[2 + \tfrac{1}{y}]$ in part (a) above. Thus $g^{-1}(y) = \tfrac{1}{9}[2 + \tfrac{1}{y}]$.

3. a. f is to be a real-valued function. $f(x) = \sqrt{25 - x^2}$ is a real number if and only if $25 - x^2 \geq 0$. Now $25 - x^2 \geq 0 \Leftrightarrow x^2 \leq 25 \Leftrightarrow |x|^2 \leq 25 \Leftrightarrow |x| \leq 5 \Leftrightarrow -5 \leq x \leq 5$. Thus the largest domain for f is the closed interval $[-5, 5]$. For the range of f, we consider the equation

$$f(x) = \sqrt{25 - x^2} = y. \qquad (1)$$

Since $\sqrt{25 - x^2} \geq 0$, a necessary condition for the solvability of (1) is $y \geq 0$. Now $\sqrt{25 - x^2} = y \Leftrightarrow 25 - x^2 = y^2 \Leftrightarrow x^2 = 25 - y^2$. Since for any real x, $x^2 \geq 0$, so $25 - y^2 \geq 0 \Leftrightarrow y^2 \leq 25 \Leftrightarrow -5 \leq y \leq 5$. Thus, since $y \geq 0$, the values of y for which (1) is solvable is $\{y : 0 \leq y \leq 5\} = [0, 5]$.



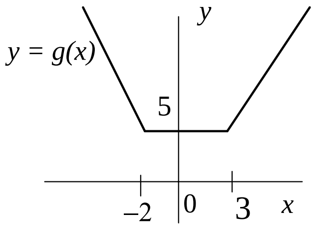
b. Since the modulus function is defined on the whole of $\mathbf{R}$, the largest domain for g is $\mathbf{R}$. To simplify the function $g(x)$, we first note that

$$|x-3| = \begin{cases} -(x-3), & x < 3 \\ x-3, & 3 \leq x \end{cases} \quad \text{and} \quad |x+2| = \begin{cases} -(x+2), & x < -2 \\ x+2, & x \geq -2 \end{cases}$$

Then we have

$$g(x) = |x-3| + |x+2| = \begin{cases} -(x-3) - (x+2), & x < -2 \\ -(x-3) + x + 2, & -2 \leq x < 3 \\ x - 3 + x + 2, & 3 \leq x \end{cases} = \begin{cases} -2x+1, & x < -2 \\ 5, & -2 \leq x < 3 \\ 2x-1, & 3 \leq x \end{cases} .$$

Now $x < -2 \Rightarrow g(x) = -2x + 1 > 5$. Also for any $y > 5$, we can find $x < -2$, such that $g(x) = y$. Thus g maps $(-\infty, -2)$ onto $(5, +\infty)$. Also, $-2 \leq x < 3 \Rightarrow g(x) = 5$. Thus $g([-2, 3)) = \{5\}$. Since $x \geq 3 \Rightarrow g(x) = 2x - 1 \geq 5$ and for any $y \geq 5$, we can find an $x \geq 3$ such that $g(x) = y$, g maps $[3, +\infty)$ onto $[5, +\infty)$. Therefore, the range of g is $[5, +\infty)$.



4. a. *f is injective.* For any x, y in **R**, $f(x) = f(y) \Rightarrow 7x + 5 = 7y + 5 \Rightarrow x = y$. Thus f is injective.
f is surjective. Take any y in **R**, consider the equation $f(x) = y$, i.e., $7x + 5 = y$. Then $x = \dfrac{y-5}{7}$. For this value of x, f maps x onto y. Thus the range of f is the whole of **R**. Therefore, f is surjective. Hence f is bijective.

b. Now we can write $f(x) = x^2 - 6x + 5 = (x-3)^2 - 4$. f is not injective for obviously $f(2) = f(4)$ but $2 \neq 4$. Note that $f(x) \geq -4$. Thus there does not exist an x in **R** such that $f(x) = -5$. Therefore, f is not surjective and so f can never be bijective.

5. a. Take any c in C. Then since $h \circ g$ is surjective, there exists an element a in A such that $h \circ g(a) = c$, i.e., $h(g(a)) = c$. Take $b = g(a)$ in B, then $h(b) = h(g(a)) = c$. Thus h is surjective.

b. Suppose x and y are in dom(g) such that g(x) = g(y). Then
$g(x) = g(y) \Rightarrow h(g(x)) = h(g(y)) \Rightarrow h \circ g(x) = h \circ g(y) \Rightarrow x = y$ since $h \circ g$ is injective.
Therefore, g is injective.

6. a. $x^2 - x - 6 > 0 \Leftrightarrow (x-3)(x+2) > 0 \Leftrightarrow (x > 3 \text{ and } x > -2) \text{ or } (x < 3 \text{ and } x < -2)$
$\Leftrightarrow x > 3 \text{ or } x < -2.$
Therefore, the solution set is $(-\infty, -2) \cup (3, +\infty)$.

b. $\frac{4x+1}{x-3} < 1 \Leftrightarrow \frac{4x+1-x+3}{x-3} < 0 \Leftrightarrow \frac{3x+4}{x-3} < 0$
$\Leftrightarrow ((3x+4) < 0 \text{ and } x - 3 > 0) \text{ or } ((3x+4) > 0 \text{ and } x - 3 < 0)$
$\Leftrightarrow (x < -\frac{4}{3} \text{ and } x > 3) \text{ or } (x > -\frac{4}{3} \text{ and } x < 3) \Leftrightarrow (-\frac{4}{3} < x < 3).$
Thus the solution set is $(-\frac{4}{3}, 3)$.

c. $\frac{5}{x} - 4 \geq \frac{3}{x} - 10 \Leftrightarrow \frac{2}{x} + 6 \geq 0 \Leftrightarrow \frac{1}{x} + 3 \geq 0 \Leftrightarrow \frac{1+3x}{x} \geq 0$
$\Leftrightarrow ((1+3x) \geq 0 \text{ and } x > 0) \text{ or } ((1+3x) \leq 0 \text{ and } x < 0)$
$\Leftrightarrow (x \geq -\frac{1}{3} \text{ and } x > 0) \text{ or } (x \leq -\frac{1}{3} \text{ and } x < 0) \Leftrightarrow x > 0 \text{ or } x \leq -\frac{1}{3}.$
Therefore, the solution set is $(-\infty, -\frac{1}{3}] \cup (0, +\infty)$.

d. $|x+2| + |x-5| \geq 6.$ (1)
For $x < -2$, the inequality becomes $-(x+2) - (x-5) \geq 6$ i.e. $-2x + 3 \geq 6 \Leftrightarrow x \leq -\frac{3}{2}$.
Therefore, the solution set for this part is $(-\infty, -2)$.
For $-2 \leq x < 5$, the inequality becomes $(x+2) - (x-5) \geq 6$ i.e. $7 \geq 6$ which is true.
Therefore, the solution set for this part is $[-2, 5)$.
For $x \geq 5$, the inequality becomes $(x+2) + (x-5) = 2x - 3 \geq 6$, i.e. $x \geq \frac{9}{2}$.
Therefore, the solution set for this part is $[5, +\infty)$. Thus the solution set for (1) is **R**.

OR $|x+2| + |x-5| = |x+2| + |x+2-7| \geq 7 > 6$ *which is always true. Here we are using* $|a| + |a-b| \geq |b|$.

7. If $b, d > 0$, then $b + 101d > 0$.

So $\frac{a}{b} < \frac{a+101c}{b+101d} \Leftrightarrow a(b+101d) < b(a+101c) \Leftrightarrow ad < bc$. This is true since we are given that $\frac{a}{b} < \frac{c}{d}$ which is equivalent to $ad < bc$ Similarly, we can show that $\frac{a+101c}{b+101d} < \frac{c}{d}$.

8. a. Since $x^2 \geq 0$, $5x^2 + 9 \geq 9 > 4$ and so $|5x^2 + 9| = 5x^2 + 9 > 4$. Therefore the solution set here is the whole of the real numbers.

   b. Note that $x = 0$ is not a solution.
   For $x > 0$, the inequality $\left|\frac{2}{x}\right| \geq 7$ becomes $\frac{2}{x} \geq 7$. Therefore, multiplying by x on both sides and noting that x is positive, we have $7x \leq 2$ so that $x \leq \frac{2}{7}$,i.e. $0 < x \leq \frac{2}{7}$.
   Now for the case that $x < 0$. The inequality becomes $-\frac{2}{x} \geq 7$ where upon multiplying by x gives $-2 \leq 7x$ since x is negative. Hence we have $0 > x \geq -\frac{2}{7}$.
   Thus the solution set is $[-\frac{2}{7}, \frac{2}{7}] - \{0\}$.

9. For positive a_1, a_2 and a_3, and since we are given that $a_1a_2a_3 = 1$, by the *Arithmetic-Geometric Mean Inequality* $a_1 + a_2 + a_3 \geq 3a_1a_2a_3 = 3$ and $a_1a_2 + a_1a_3 + a_2a_3 \geq 3a_1^2a_2^2a_3^2 = 3$. Therefore ,

$$(1+a_1)(1+a_2)(1+a_3) = 1 + (a_1 + a_2 + a_3) + (a_1a_2 + a_1a_3 + a_2a_3) + a_1a_2a_3$$
$$\geq 1 + 3 + 3 + 1 = 8.$$

   You can also use: $(1 + a_i) \geq 2\sqrt{a_i}$, and so $(1 + a_1)(1 + a_2)(1 + a_3) \geq 2\sqrt{a_1}2\sqrt{a_2}\,2\sqrt{a_3} = 8$.

10. (Optional) Show first $1 \in P$. If on the contrary $1 \notin P$, then $1 \in -P$ since $1 \neq 0$. Therefore $-1 \in P$ and so by Property 2, $1 = (-1)(-1) \in P$ contradicting $1 \notin P$. Then by property 2, $2 = 1+1 \in P$. Now assume that $n \in P$, then $n + 1 \in P$ by property 2 . Thus by the principle of mathematical induction, the set of counting numbers $\subseteq P$. Now it is easily seen that for any natural number n, $1/n \in P$. Hence for any rational number m/n with m,n in **N**, $m/n = m \,.\, 1/n \in P$ by property 2. Thus the set of positive rational numbers is a subset of P. Clearly no negative rational number q belongs to P for if it did then both q and $-q$ would belong to P and so by Property 2, $0 = q + -q \in P$ contradicting $0 \notin P$. Therefore, P is precisely the set of positive rational numbers. Properties 1-10 then follow easily.