1. ***COMMENT*. From the first fundamental theorem of calculus, if $f(x)$ is continuous on [*a, b*] we have $\int_a^x f(t)dt$ is an antiderivative of $f(x)$ for $a \le x \le b$.**
For any *c* in [*a, b*], using $\int_c^x f(t)dt = (\int_c^a f(t)dt + \int_a^x f(t)dt) = -\int_a^c f(t)dt + \int_a^x f(t)dt$, we have

$$\frac{d}{dx}\int_c^x f(t)dt = \frac{d}{dx}\int_a^x f(t)dt = f(x) \text{ for } x \in [a,b].$$

It follows that if f is continuous on R, then $\int_c^x f(t)dt$ is an antiderivative of $f(x)$ for any choice of *c* in R.

Note that f is continuous on **R**. It is obviously continuous at x for x not equal to 2. f is continuous at $x = 2$. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{2x}{\sqrt{12+x^2}} + 5x + 26 = 1 + 10 + 26 = 37$ and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 4x^3 + x^2 + 1 = 32 + 4 + 1 = 37$ and so $\lim_{x\to 2} f(x) = 37 = f(2)$ and so f is continuous at $x = 2$.
Since $f(x)$ is continuous on **R**, by the Fundamental Theorem of Calculus, the function $\int_a^x f(t)dt$ is an antiderivative of $f(x)$ for any chosen fixed point a. In particular, we have $F(x) = \int_0^x f(t)\,dt$ is an antiderivative of $f(x)$. It thus remains to find $F(x)$:

$$F(x) = \begin{cases} \int_0^x f(t)dt, x \le 2 \\ \int_0^2 f(t)dt + \int_2^x f(t)dt, x > 2 \end{cases}$$

$$= \begin{cases} \int_0^x (\frac{2t}{\sqrt{12+t^2}} + 5t + 26)dt, x \le 2 \\ \int_0^2 (\frac{2t}{\sqrt{12+t^2}} + 5t + 26)dt + \int_2^x (4t^3 + t^2 + 1)dt, x > 2 \end{cases}$$

$$= \begin{cases} \left[2\sqrt{12+t^2} + \frac{5}{2}t^2 + 26t\right]_0^x, x \le 2 \\ \left[2\sqrt{12+t^2} + \frac{5}{2}t^2 + 26t\right]_0^2 + \left[t^4 + \frac{1}{3}t^3 + t\right]_2^x, x > 2 \end{cases}$$

$$= \begin{cases} 2\sqrt{12+x^2} + \frac{5}{2}x^2 + 26x - 4\sqrt{3}, x \le 2 \\ 49\tfrac{1}{3} - 4\sqrt{3} + x^4 + \frac{1}{3}x^3 + x, x > 2 \end{cases}.$$

Therefore $\int f(x)dx = F(x) + C$.
(Note that you can always define $F(x)$ for any Riemann integrable function f this way but there is no guarantee that F will de differentiable. For instance take the signum function for f. Then F so defined will not be differentiable at $x = 0$. We have this assurance for this question because our f is continuous so that we can use the Fundamental Theorem of Calculus.)

2. Now the derivative of f is given by

$$f'(x) = \frac{\sin(5x)}{\sqrt{5 - 2\cos(5x)}} \qquad (1)$$

and

$$f(0) = 1 \qquad (2)$$

To find f we shall have to integrate (1). Let $u = 5 - 2\cos(5x)$. Then $\frac{du}{dx} = 10\sin(5x)$.
Therefore,

$$f(x) = \int \frac{\sin(5x)}{\sqrt{5-2\cos(5x)}}dx = \frac{1}{10}\int \frac{1}{\sqrt{5-2\cos(5x)}}(10\sin(5x))dx = \frac{1}{10}\int \frac{1}{\sqrt{5-2\cos(5x)}}\frac{du}{dx}dx$$

$$= \frac{1}{10}\int \frac{1}{\sqrt{u}}du = \frac{1}{10}2\sqrt{u} + C = \frac{1}{5}\sqrt{5-2\cos(5x)} + C \qquad (3)$$

Evaluating the function f so obtained at $x = 0$, we have

$$f(0) = \frac{1}{5}\sqrt{5-2\cos(0)} + C = \frac{\sqrt{3}}{5} + C.$$

But by (1), this is equal to 1. Therefore, we must have $\frac{\sqrt{3}}{5} + C = 1$. Hence $C = 1 - \frac{\sqrt{3}}{5}$ so the function f is given by $f(x) = 1 - \frac{\sqrt{3}}{5} + \frac{1}{5}\sqrt{5-2\cos(5x)}$.

3. $\int_1^2 \frac{1}{x^2}dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{(1+\frac{i}{n})^2}\left(\frac{1}{n}\right)$. (Maybe some justification is needed here. The function is integrable and the above sum is the lower sum w.r.t. the regular partition and it will be the same form as the upper sum and the limits of both are the same. The less obvious reason comes from the definition of the Riemann integral, see supplementary notes on the web : http://www.math.nus.edu.sg/~matngtb/Calculus/Riemannsum/Riemannsum.htm.) Here, we have taken the regular partition of the interval [1, 2] to be

$$1 = x_0 < x_1 < x_2 < \cdots < x_n = 2, \text{ where } x_i = 1 + \frac{i}{n}.$$

Thus, $\sum_{i=1}^{n}\frac{1}{(1+\frac{i}{n})^2}\left(\frac{1}{n}\right)$ is a Riemann sum of $\int_1^2 \frac{1}{x^2}dx$.

Now we have the following inequality

$$\sum_{i=1}^{n}\frac{1}{(1+\frac{i}{n})(1+\frac{i+1}{n})}\left(\frac{1}{n}\right) \le \sum_{i=1}^{n}\frac{1}{(1+\frac{i}{n})^2}\left(\frac{1}{n}\right) \le \sum_{i=1}^{n}\frac{1}{(1+\frac{i}{n})(1+\frac{i-1}{n})}\left(\frac{1}{n}\right). \qquad (1)$$

Also $\sum_{i=1}^{n}\frac{1}{(1+\frac{i}{n})(1+\frac{i+1}{n})}\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left(\frac{1}{n+i} - \frac{1}{n+i+1}\right)n = \left(\frac{1}{n+1} - \frac{1}{2n+1}\right)n \to \frac{1}{2}$ as $n \to \infty$.

and $\sum_{i=1}^{n}\frac{1}{(1+\frac{i}{n})(1+\frac{i-1}{n})}\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left(\frac{1}{n+i-1} - \frac{1}{n+i}\right)n = \left(\frac{1}{n} - \frac{1}{2n}\right)n = \frac{1}{2} \to \frac{1}{2}$ as $n \to \infty$.

Therefore, by (1) and the *Squeeze Theorem* $\int_1^2 \frac{1}{x^2}dx = \frac{1}{2}$.

Remark: After much effort in finding the value $\int_1^2 \frac{1}{x^2}dx$, you should now appreciate the power of the second fundamental theorem of Calculus in the evaluation of $\int_1^2 \frac{1}{x^2}dx$ via antiderivative (indefinite integral).

4. **This question can be thought of as the reverse problem of Q5 in the following sense: We are given an infinite sum we would like to rewrite its partial sum (if possible) as a Riemann sum, i.e. we need to find a function $f(x)$ (necessarily Riemann integrable) and a suitable interval [a, b] so that the given partial sum is the Riemann sum of f with respect to the regular partition of [a, b] and so its limit is the definite integral $\int_a^b f(x)dx$.**

a. $\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\frac{n+5i}{n^3}} = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+\frac{5i}{n}}\left(\frac{1}{n}\right) = \int_1^6 \frac{1}{5}\sqrt{x}\,dx = \left[\frac{2}{15}x^{\frac{3}{2}}\right]_1^6 = \frac{2}{15}(6^{\frac{3}{2}} - 1)$.

b. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sin\left(\frac{\pi i}{7n}\right) = 7\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{7n}\sin\left(\frac{\pi i}{7n}\right) = 7\int_0^{\frac{1}{7}}\sin(\pi x)dx = 7\left[-\frac{1}{\pi}\cos(\pi x)\right]_0^{\frac{1}{7}}$

$= -\frac{7}{\pi}(\cos(\frac{\pi}{7}) - 1)$.

5. We first note that when $0 \le t \le 2$, then $2 \le t^3 + 2 \le 10$. We shall then partition the range [2, 10] into intervals of unit length. Corresponding to the range we would be partitioning the domain into 9 intervals $[0,1]$, $[1, \sqrt[3]{2}]$, $[\sqrt[3]{2}, \sqrt[3]{3}]$, $[\sqrt[3]{3}, \sqrt[3]{4}]$, $[\sqrt[3]{4}, \sqrt[3]{5}]$, $[\sqrt[3]{5}, \sqrt[3]{6}]$, $[\sqrt[3]{6}, \sqrt[3]{7}]$, $[\sqrt[3]{7}, 2]$ Note that

$$[t^3+2] = \begin{cases} 2, & 0 \le t < 1 \\ 3, & 1 \le t < \sqrt[3]{2} \\ 4, & \sqrt[3]{2} \le t < \sqrt[3]{3} \\ 5, & \sqrt[3]{3} \le t < \sqrt[3]{4} \\ 6, & \sqrt[3]{4} \le t < \sqrt[3]{5} \\ 7, & \sqrt[3]{5} \le t < \sqrt[3]{6} \\ 8, & \sqrt[3]{6} \le t < \sqrt[3]{7} \\ 9, & \sqrt[3]{7} \le t < 2 \end{cases}$$

. Note also that two integrals are the same if the integrands differ only on finite number of points in the domain. (In our case only at one end point of each interval.) Therefore,

$$\begin{aligned}\int_0^2 [t^3+2]\,dt &= \int_0^1 2dt + \int_1^{\sqrt[3]{2}} 3dt + \int_{\sqrt[3]{2}}^{\sqrt[3]{3}} 4dt + \int_{\sqrt[3]{3}}^{\sqrt[3]{4}} 5dt \\ &\quad + \int_{\sqrt[3]{4}}^{\sqrt[3]{5}} 6\,dt + \int_{\sqrt[3]{5}}^{\sqrt[3]{6}} 7dt + \int_{\sqrt[3]{6}}^{\sqrt[3]{7}} 8dt + \int_{\sqrt[3]{7}}^{2} 9dt \\ &= 2 + 3(\sqrt[3]{2} - 1) + 4(\sqrt[3]{3} - \sqrt[3]{2}) + 5(\sqrt[3]{4} - \sqrt[3]{3}) \\ &\quad + 6(\sqrt[3]{5} - \sqrt[3]{4}) + 7(\sqrt[3]{6} - \sqrt[3]{5}) + 8(\sqrt[3]{7} - \sqrt[3]{6}) + 9(2 - \sqrt[3]{7}) \\ &= 17 - \sqrt[3]{2} - \sqrt[3]{3} - \sqrt[3]{4} - \sqrt[3]{5} - \sqrt[3]{6} - \sqrt[3]{7}.\end{aligned}$$

6. Let $t = \pi - x$. Then $\frac{dt}{dx} = -1$. Therefore,

$$\begin{aligned}\int_0^\pi x f(\sin(x))dx &= -\int_0^\pi x f(\sin(x))\frac{dt}{dx}dx = -\int_\pi^0 (\pi - t) f(\sin(\pi - t))dt \\ &= \int_0^\pi (\pi - t) f(\sin(\pi - t))dt \\ &= \int_0^\pi (\pi - t) f(\sin(t))dt \text{ since } \sin(\pi - t) = \sin(t) \\ &= \pi \int_0^\pi f(\sin(t))dt - \int_0^\pi t\, f(\sin(t))dt \\ &= \pi \int_0^\pi f(\sin(x))dx - \int_0^\pi x\, f(\sin(x))dx \text{ by renaming the variable } t \text{ as } x.\end{aligned}$$

Therefore, $\int_0^\pi x\, f(\sin(x))dx = \frac{\pi}{2}\int_0^\pi f(\sin(x))dx$.

Thus, $\int_0^\pi x \sin^4(x)dx = \frac{\pi}{2}\int_0^\pi \sin^4(x)dx = \frac{\pi}{2}\left[\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x)\right]_0^\pi = \frac{3}{16}\pi^2$.

where we use $\int \sin^4(x)dx = \frac{1}{4}\int(1 - \cos(2x))^2 dx = \frac{1}{4}\int(1 - 2\cos(2x) + \cos^2(2x))dx$
$= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{8}\int(1 + \cos(4x))dx = \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.10$

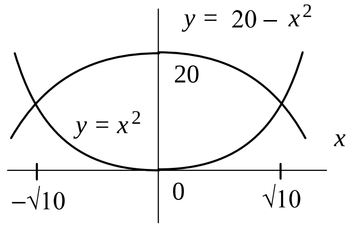
7. a. By the *Fundamental Theorem of Calculus*, $\frac{d}{dx}\int_1^x \frac{1}{1+t^3+t^6}dt = \frac{1}{1+x^3+x^6}$.

b. By the *Fundamental Theorem of Calculus* and the *Chain Rule*,

$$\frac{d}{dx}\int_1^{\sin(2x)} \frac{1}{1+t^5+t^{10}} = \frac{1}{1+\sin^5(2x)+\sin^{10}(2x)}2\cos(2x) = \frac{2\cos(2x)}{1+\sin^5(2x)+\sin^{10}(2x)}.$$

c. $$\begin{aligned}\frac{d}{dx}\int_{-3x}^{\sin(x^2)} \frac{1}{1+t^3+t^6}dt &= \frac{d}{dx}\left[\int_{-3x}^{0} \frac{1}{1++t^3+t^6}dt + \int_0^{\sin(x^2)} \frac{1}{1+t^3+t^6}dt\right] \\ &= \frac{d}{dx}\left[\int_0^{\sin(x^2)} \frac{1}{1+t^3+t^6}dt - \int_0^{-3x} \frac{1}{1+t^3+t^6}dt\right] \\ &= \frac{2x\cos(x^2)}{1+\sin^3(x^2)+\sin^6(x^2)} + \frac{3}{1-27x^3+729x^6}.\end{aligned}$$

8. a. The intersection of the two curves is given by $20 - x^2 = x^2$. Solving this equation gives $x = \pm\sqrt{10}$ and $y = 10$. Therefore, the area of the region bounded by the two curves is $\int_{-\sqrt{10}}^{\sqrt{10}} (20 - x^2 - x^2)dx = [20x - \frac{2}{3}x^3]_{-\sqrt{10}}^{\sqrt{10}} = \dfrac{80\sqrt{10}}{3}$ sq units.

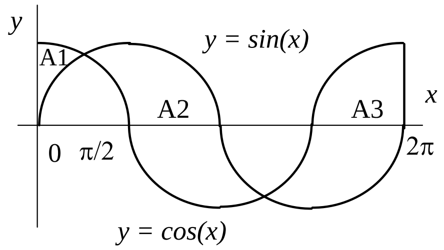


b. The area between the curves =

$$A1 + A2 + A3 = \int_0^{\frac{\pi}{4}} (\cos(x) - \sin(x))dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin(x) - \cos(x))dx + \int_{\frac{5\pi}{4}}^{2\pi} (\cos(x) - \sin(x))dx$$

$$= [\sin(x) + \cos(x)]_0^{\frac{\pi}{4}} + [-\cos(x) - \sin(x)]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} + [\sin(x) + \cos(x)]_{\frac{5\pi}{4}}^{2\pi}$$

$$= \sqrt{2} - 1 + 2\sqrt{2} + 1 + \sqrt{2} = 4\sqrt{2} \text{ sq units.}$$



9. We have, by the Mean Value Theorem for integral, a point c in [3, 5] such that

$$\int_3^5 \frac{1}{x^5 + x^3 + 1}dx = (5-3)\frac{1}{c^5 + c^3 + 1} = \frac{2}{c^5 + c^3 + 1}.$$

Also, for $3 \le c \le 5$, we have that $271 \le c^5 + c^3 + 1 \le 3251$ and consequently $\frac{2}{3251} \le \frac{2}{c^5 + c^3 + 1} \le \frac{2}{271}$. Therefore, $\frac{2}{3251} \le \int_3^5 \frac{1}{x^5 + 5}dx \le \frac{2}{271}$.

10. a. Let $f(c) = 1$. Then $f(c) = \sqrt{10c + 1} = 1$. Thus $c = 0$. Therefore, $f^{-1}(1) = 0$. Thus $(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{5}$, since $f'(x) = \frac{5}{\sqrt{10x+1}}$.

b. $f'(x) = \frac{1}{2}\sec(x)\tan(x)$. $f(c) = \frac{1}{\sqrt{3}} \Leftrightarrow \frac{1}{2}\sec(c) = \frac{1}{\sqrt{3}}$. Therefore, since the domain of f is $[0, \frac{\pi}{2})$, $c = \frac{\pi}{6}$. Thus $f^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$. Hence $(f^{-1})'(\frac{1}{\sqrt{3}}) = \frac{1}{f'(f^{-1}(\frac{1}{\sqrt{3}}))} = \frac{1}{f'(\frac{\pi}{6})} = 3$ since $f'(\frac{\pi}{6}) = \frac{1}{2}\sec(\frac{\pi}{6})\tan(\frac{\pi}{6}) = \frac{1}{2}\frac{2}{\sqrt{3}}\frac{1}{\sqrt{3}} = \frac{1}{3}$.

c. $f'(x) = 2x - 5$. Now $f(c) = 4 \Leftrightarrow c^2 - 5c - 10 = 4 \Rightarrow c = 7$ since the domain of f is $\{x : x \ge \frac{5}{2}\}$. Therefore, $(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(7)} = \frac{1}{9}$.