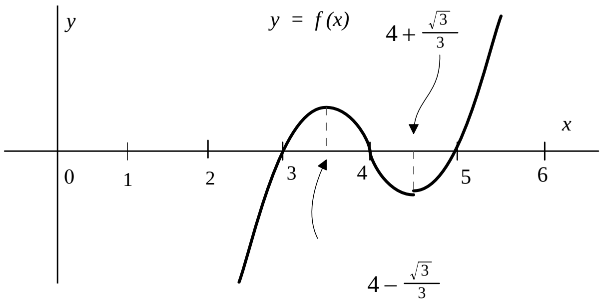
Students should try to sketch the graph of each function in Questions 1 and 2 in order to have a feel for Curve Sketching. This will enable students to be aware of how to obtain necessary and useful information for curve sketching.

1. (i) $f(x) = (x-3)(x-4)(x-5) = x^3 - 12x^2 + 47x - 60$. Since f is a polynomial function, f is continuous and differentiable on **R**..
Note that $f'(x) = 3x^2 - 24x + 47 = 3(x - 4 + \frac{\sqrt{3}}{3})(x - 4 - \frac{\sqrt{3}}{3})$ so that $f'(x) = 0 \Leftrightarrow x = 4 \pm \frac{\sqrt{3}}{3}$.
Thus, f is increasing on the interval $(-\infty, 4 - \frac{\sqrt{3}}{3}]$ and on the interval $[4 + \frac{\sqrt{3}}{3}, \infty)$ and decreasing on $[4 - \frac{\sqrt{3}}{3}, 4 + \frac{\sqrt{3}}{3}]$.
By the first derivative test, $f(4 - \frac{\sqrt{3}}{3}) = \frac{2\sqrt{3}}{9}$ is a relative maximum and $f(4 + \frac{\sqrt{3}}{3}) = -\frac{2\sqrt{3}}{9}$ is a relative minimum.
Now we have $f''(x) = 6x - 24$. Thus, $f''(x) = 6x - 24 > 0 \Leftrightarrow x > 4$. and the graph of f is concave upward on $(4, \infty)$.
$f''(x) = 6x - 24 < 0 \Leftrightarrow x < 4$. Thus the graph of f is concave downward on $(-\infty, 4)$.



(ii) . $f(x) = \begin{cases} 3(x-3)^2, & x \le 3 \\ (3-x)^3, & x > 3 \end{cases}$

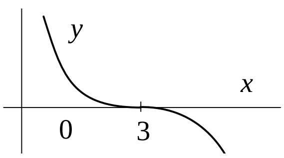
Note that f *is **continuous at*** $x = 3$. **(Students should check this!)**
Moreover, f is differentiable at $x = 3$ and $f'(3) = 0$.**(Students should check this via definition of derivative! Otherwise take caution, see the article on my Calculus site:** http://www.math.nus.edu.sg/~matngtb/Calculus/Derivedfunction/Derivative.htm**)** Therefore, we have

$$f'(x) = \begin{cases} 6(x-3), & x \le 3 \\ -3(3-x)^2, & x > 3 \end{cases} \qquad (1)$$

Thus, $f'(x) = 0 \Leftrightarrow x = 3$.
From (1), $x < 3 \Rightarrow f'(x) < 0$. Thus f is decreasing on the interval $(-\infty, 3]$ since f is also continuous at 3.
Also from (1) we have $x > 3 \Rightarrow f'(x) < 0$. Hence this and the continuity of f at $x = 3$ says that f is decreasing on $[3, \infty)$.
Therefore, f is decreasing on **R** and so f does not have any relative extrema.

Note that $f''(x) = \begin{cases} 6, & x < 3 \\ 6(3-x), & x > 3 \end{cases}$ and $f''(3)$ does not exist.

Thus, $f''(x) > 0$ for $x < 3$ and the graph of f is concave upward on $(-\infty, 3)$.
When $x > 3$, $f''(x) < 0$ and the graph of f is concave downward on $(3, \infty)$.
There is a point of inflection at $x = 3$.



2. i. The domain of f is $\mathbf{R}-\{4\}$. For $x \neq 4$, $f'(x) = \dfrac{-8}{(x-4)^2} < 0.$ (1)

a. Thus f is decreasing on the interval $(-\infty, 4)$ and also on $(4, \infty)$.

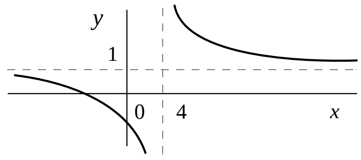
b. Differentiating f twice gives $f''(x) = \dfrac{16}{(x-4)^3}$ for $x \neq 4$. (2)

Therefore, $x > 4 \Rightarrow f''(x) > 0$. Thus the graph of f is concave upward on the interval $(4, \infty)$. Also from (2), $x < 4 \Rightarrow f''(x) < 0 \Rightarrow$ the graph of f is concave downward on the interval $(-\infty, 4)$.

c. Now f is decreasing and *differentiable* on its domain of definition and so f does not have any relative extrema. From (2) $f''(x) \neq 0$. Hence f does not have any point of inflection.

$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \dfrac{x+4}{x-4} = 1$. And so the line $y = 1$ is a horizontal asymptote.

$\lim_{x\to 4^-} f(x) = \lim_{x\to 3^-} \dfrac{x+4}{x-4} = -\infty$ and $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \dfrac{x+4}{x-4} = \infty$. Thus the line $x = 4$ is a vertical asymptote.



ii. $f(x) = \begin{cases} 2 - x^3 + 3x, & x \leq 0 \\ \sqrt[3]{x+8}, & x > 0 \end{cases}$. Observe $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2 - x^3 + 3x = 2$ and also $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt[3]{x+8} = 2$ so that $\lim_{x\to 0} f(x) = 2 = f(0)$ Hence f is continuous at $x = 0$.

f is not differentiable at $x = 0$. This is seen as follows.

$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{2-x^3+3x-2}{x-0} = \lim_{x\to 0^-} (-x^2+3) = 3$ whereas

$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{\sqrt[3]{x+8}-2}{x} = \frac{1}{12}$ and they are not the same.

Therefore f is not differentiable at $x = 1$ and hence $f'(0)$ does not exist. Now,

$$f'(x) = \begin{cases} 3 - 3x^2, & x < 0 \\ \frac{1}{3}(x+8)^{\frac{-2}{3}}, & x > 0 \end{cases} \qquad (1)$$

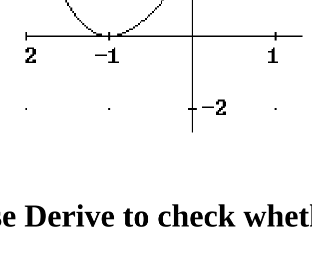
and

$$f''(x) = \begin{cases} -6x, & x < 0 \\ \frac{-2}{9}(x+8)^{\frac{-5}{3}}, & x > 0 \end{cases}. \qquad (2)$$

a. From (1), when $x < -1$, $f'(x) < 0$ and when $-1 < x < 0$, $f'(x) > 0$. Also when $x > 0$, $f'(x) > 0$. Therefore, f is decreasing on $(-\infty, -1]$ and f is increasing on $[-1, 0]$ and on $[0, \infty)$.

b. From (2), $x < 0 \Rightarrow f''(x) > 0$. Therefore , the graph of f is concave upward on $(-\infty, 0)$. Now $x > 0 \Rightarrow f''(x) < 0$. Therefore, the graph of f is concave downward on $(0, \infty)$.

c. From part a, by the first derivative test, $f(-1) = 0$ is a relative minimum and from part b, the point $(0, f(0)) = (0, 2)$ is a point of inflection.



Now sketch the graph and use Derive to check whether your sketch is correct.

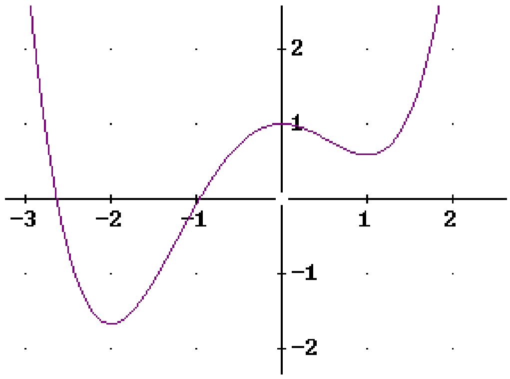
3. $f'(x) = x(x-1)(x+2) = x^3 + x^2 - 2x$ Since f is differentiable, f is continuous on $\mathbf{R}$. At $x = -2$, 0 and 1 $f(x) = 0$.

Note that $f'(x) > 0 \Leftrightarrow -2 < x < 0, x > 1$. Thus f is increasing on $[-2, 0]$ and on $[1, \infty)$.
And f is decreasing on $(-\infty, -2]$ and $[0,1]$.
By the first derivative test, we have a relative minimum at $x = -2$ and $x = 1$, and relative maximum at $x = 0$.

$f''(x) = 3x^2 + 2x - 2 > 0 \Leftrightarrow x < -\frac{1}{3}(1+\sqrt{7})$ or $x > -\frac{1}{3}(1-\sqrt{7})$

Thus, graph of f is concave upward on the intervals $(-\infty, -\frac{1}{3}(1+\sqrt{7}))$ and $(-\frac{1}{3}(1-\sqrt{7}), \infty)$ and concave downward on the interval $(-\frac{1}{3}(1+\sqrt{7}), -\frac{1}{3}(1-\sqrt{7}))$
Points of inflection occur at $x = -\frac{1}{3}(1+\sqrt{7})$ and at $x = -\frac{1}{3}(1-\sqrt{7})$.

Using the above information, give a rough sketch of the graph. The shape of the graph is like that of $g(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2 + 1$**.** (check with Derive.)



4. $f(x) = x^7 + rx + 7$. Then, $f'(x) = 7x^6 + r = 7(x^6 + \frac{r}{7})$. (1)

If $r < 0$, then we can write $f'(x) = 7(x^3 - \frac{\sqrt{|r|}}{\sqrt{7}})(x^3 + \frac{\sqrt{|r|}}{\sqrt{7}})$. Then the stationary points of f are $-\sqrt[6]{\frac{|r|}{7}}$ and $\sqrt[6]{\frac{|r|}{7}}$.

Differentiating $f'(x)$: $f''(x) = 42x^5$. (2)

Therefore, by (2), $f''(-\sqrt[6]{\frac{|r|}{7}}) < 0$ and $f''(\sqrt[6]{\frac{|r|}{7}}) > 0$.

Thus by the second derivative test, $f(-\sqrt[6]{\frac{|r|}{7}})$ is a relative maximum and $f(\sqrt[6]{\frac{|r|}{7}})$ is a relative minimum. (We may use the first derivative test here.)

If $\boldsymbol{r > 0}$, then we deduced from (1) $f'(x) > 0$ since $(x^6 + \frac{r}{7}) \geq \frac{r}{7} > 0$. Note that f is differentiable on the whole of **R**. Therefore, f has **no relative extrema** for if f were to have a relative extrema at c, then we can conclude that $f'(c) = 0$ giving us a contradiction.

5. (a) $\frac{107}{10}x^{10} + \frac{5}{6}x^6 - \frac{51}{4}x^4 - \frac{3}{2}x^2 + 29x + C$
   (b) $\frac{1}{2}(2x+5)^{3/2} - \frac{15}{2}\sqrt{2x+5} + C$.
   (c) $\frac{7}{8}x^{8/7} + \frac{81}{8}x^{8/9} + C$
   (d) $\frac{2}{45}(7-x)^{3/2}(5x^3 + 30x^2 + 168x + 784) + C$
   (Suggestion : (b) use $u = 2x + 5$ (d) use $u = 7 - x$)

6. Note that $F(x) = \begin{cases} 3x - 9/2, x < 3 \\ x^2 - 3x + 9/2, \ x \geq 3 \end{cases}$ is a continuous function on **R. (Students should check this.)**

Differentiating we get $F'(x) = \begin{cases} 3, & x < 3 \\ 2x - 3, & x > 3 \end{cases}$. At $x = 3$ we have

$$\lim_{x\to 3^-} \frac{F(x)-F(3)}{x-3} = \lim_{x\to 3^+} \frac{F(x)-F(3)}{x-3} = 3.$$

Thus, F is differentiable at $x = 3$ and $F'(3) = 3$. .

Therefore $F'(x) = \begin{cases} 3, & x \le 3 \\ 2x - 3, & x > 3 \end{cases} = f(x)$. That is, F is an antiderivative of f .

Hence $\int f(x)dx = F(x) + C = \begin{cases} 3x - 9/2 + C, & x < 3 \\ x^2 - 3x + 9/2 + C, & x \ge 3 \end{cases}$.

7. a. $\int (x^9 - 5x^4)\sqrt{x^5+5}\,dx$. Let $u = x^5 + 5$. Then $\frac{du}{dx} = 5x^4$ and $x^5 = u - 5$. Therefore,

$$\int(x^9 - 5x^4)\sqrt{x^5+5}\,dx = \int \tfrac{1}{5}(x^5-5)\sqrt{x^5+5}\,5x^4dx = \int \tfrac{1}{5}(x^5-5)\sqrt{x^5+5}\,\frac{du}{dx}dx$$

$$= \int \tfrac{1}{5}(u-5-5)\sqrt{u}\,du = \tfrac{1}{5}\int(u^{\frac{3}{2}} - 10u^{\frac{1}{2}})du = \tfrac{1}{5}\left(\tfrac{2}{5}u^{\frac{5}{2}} - 10\cdot\tfrac{2}{3}u^{\frac{3}{2}}\right) + C$$

$$= \frac{2}{25}(x^5+5)^{\frac{5}{2}} - \frac{4}{3}(x^5+5)^{\frac{3}{2}} + C.$$

b. $\int \frac{\tan^2(\sqrt{t})}{\sqrt{t}}dt$.. The domain of the integrand is the set of positive real numbers.

Let $u = \sqrt{t}$. Then $\frac{du}{dt} = \frac{1}{2\sqrt{t}}$. Therefore,

$$\int \frac{\tan^2(\sqrt{t})}{\sqrt{t}}dt = 2\int \tan^2(\sqrt{t})\frac{1}{2\sqrt{t}}dt = 2\int \tan^2(\sqrt{t})\frac{du}{dt}dt = 2\int \tan^2(u)du$$

$$= 2\int(\sec^2(u) - 1) + C = 2\tan(u) - 2u + C = 2\tan(\sqrt{t}) - 2\sqrt{t} + C$$

8. We shall use the identities $1 + \tan^2(\theta) = \sec^2(\theta)$ and $1 + \cot^2(\theta) = \csc^2(\theta)$.

$$\int(7\cot^2(\theta) - 6\tan^2(\theta) + \theta)d\theta = \int[7(\csc^2(\theta) - 1) - 6(\sec^2(\theta) - 1) + \theta]d\theta$$

$$= \int[-1 + 7\csc^2(\theta) - 6\sec^2(\theta) + \theta]d\theta$$

$$= \tfrac{1}{2}\theta^2 - \theta + 7\int\csc^2(\theta)d\theta - 6\int\sec^2(\theta)d\theta$$

$$= \tfrac{1}{2}\theta^2 - \theta - 7\cot(\theta) - 6\tan(\theta) + C.$$

9. Let c be any fixed point in the interval (a, b). Then since f is differentiable, the tangent line to the graph of f at $(c, f(c))$ is given by the equation $y = f(c) + (x - c)f'(c)$. We need to show that for any $x \ne c$ in a suitable open interval containing c , $f(x) > y = f(c) + (x - c)f'(c)$. Now for any $x < c$ and x in (a, b), by the Mean Value Theorem, there exists d such that $x < d < c$ with $\frac{f(c) - f(x)}{c - x} = f'(d)$. But since $d < c$ and f' is increasing, $f'(d) < f'(c)$ and so $\frac{f(c) - f(x)}{c - x} = f'(d) < f'(c)$. Now multiply the last inequality by $(c-x)$ and noting that $(c-x) > 0$, we get $f(c) - f(x) < f'(c)(c - x)$ and so $f(x) > f(c) + (x - c)f'(c)$. It now remains to check if this is also the case for any $x > c$. Again by the Mean Value Theorem, there exists e such that $x > e > c$ with $\frac{f(x) - f(c)}{x - c} = f'(e) > f'(c)$. The last inequality is because f' is increasing. Therefore multiplying by $(x - c) > 0$ we obtain again $f(x) - f(c) > f'(c)(x - c)$ and it follows once more that $f(x) > f(c) + (x - c)f'(c)$. Hence the graph of f is concave upward at $(c, f(c))$ for any c in (a, b). This completes the proof.