National University of Singapore

Department of Mathematics

level 1000 (2005/2006) Semester 2 MA1102R Calculus Tutorial set 7

Definition 1. We say a function f is *increasing* on the interval I if for any x, y in I, $x < y \Rightarrow f(x) < f(y)$. Likewise, we say f is *decreasing* on I if for any x, y in I, $x < y \Rightarrow f(x) > f(y)$ f is said to be *monotonic* on I if f is either *increasing* or *decreasing* on I.

Activity 1. Show that the function $f(x) = \frac{1}{x^2}$ is a decreasing function on (0, ∞).
(Do not use derivative. Start with $0 < x < y$, try to argue and arrive at $f(x) > f(y)$, using properties of positive real numbers.)

Theorem 1. Suppose f is continuous on $[a, b]$ with $a < b$ and differentiable on (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.

[Here is a proof of part (2) of this without using the Mean Value Theorem, as a consequence of Theorem 3 of Http://www.math.nus.edu.sg/~matngtb/Calculus/whymean/whymean.htm :

Do we need Mean Value Theorem to prove $f'(x) = 0$ on (a, b) implies that f = constant on (a, b)?
Theorem 3 there says if $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b). Then for any $k > a$ in (a, b), $f(x) > f(k)$ for all x with $a < x < k$. Therefore, by continuity of f at $x = a$, $f(a) = \lim_{x \to a^+} f(x) \geq f(k') > f(k)$, by choosing any k' such that $a < k' < k$. Therefore f is decreasing on $[a, b)$. Likewise for any $j < b$ in $[a, b)$, $f(j) > f(x)$ for all x with $j < x < b$. Thus by the continuity of f at $x = b$, $f(b) = \lim_{x \to b^-} f(x) \leq f(j') < f(j)$, by choosing j' such that $j < j' < b$.. Hence f is decreasing on $[a, b]$. Part (1) can be proven by a similar argument. We have avoided the use of the notion of supremum and infimum here]

Caution. *The notion of 'increasing' is one involving a non-trivial interval. A local information like derivative of f at a point a is positive does not guarantee that the function is increasing upto and including a* or *after a and including a* or *in any interval containing a.*

[For instance take the fiunction $f(x) = \begin{cases} x + 4x^2 \cos(\frac{1}{x}), x \neq 0 \\ 0, x = 0 \end{cases}$. Then $f'(0) = 1 > 0$ but f is neither increasing nor decreasing on any interval containing 0. This is because for any integer $n > 0$, $1/(2n\pi+\pi) > 1/(2n\pi+2\pi)$ but $f(1/(2n\pi+\pi)) = 1/(2n\pi+\pi) - 4/(2n\pi+\pi)^2 < 1/(2n\pi+2\pi) + 4/(2n\pi+2\pi)^2 = f(1/(2n\pi+2\pi))$ and that when $1/(2n\pi+\pi/2) > 1/(2n\pi+3\pi/2)$, $f(1/(2n\pi+\pi/2)) = 1/(2n\pi+\pi/2) > 1/(2n\pi+3\pi/2) = f(1/(2n\pi+3\pi/2))$.]

Remark. The domain $[a, b]$ above may be replaced by any unbounded interval of the type $(-\infty, b]$, $[a, \infty)$, $(-\infty, \infty)$ or the bounded interval of the type $[a, b)$, $(a, b]$ or (a, b) and the conclusion is still valid with the corresponding statement.

Activity 2. Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x^4$. Show that f is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$. Hence deduce that $f(0) = 0$ is the absolute minimum. (Is there any easier way to obtain the absolute minimum?)

Theorem 2 (First Derivative Test for Relative Extrema). Suppose f is continuous on the open interval (a, b) containing x_0 and that f is differentiable at all points of (a, b) except possibly at x_0.

1. If there exists $\delta > 0$ such that or all x with $x_0 - \delta < x < x_0$, $f'(x) > 0$ and for all x with $x_0 < x < x_0 + \delta$, $f'(x) < 0$, then f has a relative maximum value at x_0 .

2. If there exists $\delta > 0$ such that or all x with $x_0 - \delta < x < x_0$, $f'(x) < 0$ and for all x with $x_0 < x < x_0 + \delta$, $f'(x) > 0$, f has a relative minimum value at x_0 .

Definition 2. The graph of a function f is said to be *concave upward* (respectively. *concave downward*) at a point $(c, f(c))$ if (1) $f'(c)$ exists and (2) if there is an open interval I containing c such that, for all $x \neq c$ in I, the point $(x, f(x))$ on the graph is above (respectively below) the tangent line to the graph at $(c, f(c))$.

Theorem 3. Let f be a function differentiable on an open interval containing x_0. Then

1. if $f''(x_0) > 0$, the graph of f is concave upward at $(x_0, f(x_0))$,
2. if $f''(x_0) < 0$, the graph of f is concave downward at $(x_0, f(x_0))$.

Definition 3. A point $(c, f(c))$ is a *point of inflection* of the graph of the function f if f is continuous at c and there is an open interval containing c such that the graph of f changes from concave upward before c to concave downward after c or from concave downward before c to concave upward after c .

This means that ***either*** there is a small interval before c , $(c-\delta, c)$ such that the graph of f is concave upward and a small interval after c, $(c, c + \delta)$ on which the graph of f is concave downward ***or*** there is a small interval before c , $(c-\delta, c)$ such that the graph of f is concave downward and a small interval after c, $(c, c + \delta)$ on which the graph of f is concave upward**.**

Activity 3. Let $f(x) = x^5$. Show that the graph of f is concave upward on the interval $(0, \infty)$ and concave downward on the interval$(-\infty, 0)$. Hence deduce that the point $(0, 0)$ is a point of inflection.

Theorem 4. Suppose f is differentiable on some open interval containing c and $(c, f(c))$ is a point of inflection of the graph of f . If $f''(c)$ exists, then $f''(c) = 0$.

Theorem 5 (Second Derivative Test for Relative Extremum). Let x_0 be a stationary point of a function f , i.e., $f'(x_0) = 0$. Suppose $f'(x)$ exists for all values of x in some open interval I containing x_0. Suppose $f''(x_0)$ exists.

1. If $f''(x_0) < 0$, then f has a relative maximum value at x_0.
2. If $f''(x_0) > 0$, then f has a relative minimum value at x_0 .

Activity 4. Sketch the graph of a function that satisfies all of the given conditions:

$$f(1) = f'(1) = 0,\ \lim_{x\to 2^+} f(x) = \infty,\ \lim_{x\to 2^-} f(x) = -\infty,\ \lim_{x\to 0} f(x) = -\infty,\ \lim_{x\to -\infty} f(x) = \infty,\ \lim_{x\to \infty} f(x) = 0,$$

$$f''(x) > 0 \text{ for } x > 2,\ f''(x) < 0 \text{ for } x < 0 \text{ and for } 0 < x < 2.$$

Read Definition of antiderivative, indefinite integral and its properties.

Definition 4. A function F is called an antiderivative of f on an open interval I, if $F'(x) = f(x)$ for all x in I.

Activity 5.

Show that an antiderivative of $f(x) = 4x^3 - 8x + \cos(5x)$ is given by $F(x) = x^4 - 4x^2 + \frac{1}{5}\sin(5x) - 2001$.

Assignment 7

1. For each of the function f given below in (i) and (ii),
   a. find the relative extrema of f;
   b. determine the intervals on which f is increasing and decreasing;
   c. find the intervals on which the graph of f is concave upward;
   d. find the x-coordinate of the point of inflection (if there is);
   e. sketch the graph of the function.

   i. $f(x) = (x-3)(x-4)(x-5)$. ii. $f(x) = \begin{cases} 3(x-3)^2, & x \le 3 \\ (3-x)^3, & x > 3 \end{cases}$.

2. Sketch the graph of each of the following functions

   i. $f(x) = \dfrac{x+4}{x-4}$; domain = $\mathbf{R} - \{4\}$. ii. $f(x) = \begin{cases} 2 - x^3 + 3x, & x \le 0 \\ \sqrt[3]{x+8}, & x > 0 \end{cases}$.

   (You may use DfW to *help* you with the answer.)

3. Given that $f'(x) = x(x-1)(x+2)$, sketch a possible graph of f.

4. Let $f(x) = x^7 + rx + 7$. Prove that if $r < 0$, then f has both a relative maximum value and a relative minimum value. If $r > 0$, how many relative extrema does f have?

5. Find the most general anti-derivative of each of the following:

   (a) $107x^9 + 5x^5 - 51x^3 - 3x + 29$ (b) $\frac{3x}{\sqrt{2x+5}}$

   (c) $\sqrt[7]{x} + \dfrac{9}{\sqrt[9]{x}}$ (e) $x^3\sqrt{7-x}$

6. If $f(x) = x + |x-3|$ and $F(x) = \begin{cases} 3x - 9/2, & x < 3 \\ x^2 - 3x + 9/2, & x \ge 3 \end{cases}$. Show that F is an antiderivative of f on $\mathbf{R}$.
   Hence, deduce $\int f(x)\, dx$.

7. Evaluate, by the change of variable (substitution), each of the following integrals (antiderivatives).

   a. $\int (x^9 - 5x^4)\sqrt{x^5+5}\, dx.$ b. $\int \dfrac{\tan^2(\sqrt{t}\,)}{\sqrt{t}} dt.$

8. Evaluate $\int (7\cot^2(\theta) - 6\tan^2(\theta) + \theta) d\theta$

9. **(Optional)** Show that if f is a function differentiable on the open interval (a, b) with $a < b$ and if the derivative f' is increasing on (a, b), then the graph of f is concave upward at $(x, f(x))$ for all x in (a, b).

   (If you are having difficulty in this question check out the following article

   Http://www.math.nus.edu.sg/~matngtb/Calculus/Concavity_line/concave.htm)