National University of Singapore

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level 1000 (2005/2006) Semester 2 MA1102R Calculus Tutorial set 6

Definition 1. Let (a, b) be an open interval, f a function defined on (a, b). Let $x_0 \in (a, b)$. We say f has a *relative maximum* (*local maximum*) at x_0 if $f(x) \leq f(x_0)$ for all x in some open interval I containing x_0 . Similarly we say f has a *relative minimum* (*local minimum*) at x_0 if $f(x) \geq f(x_0)$ for all x in some open interval containing x_0 .
If the function f has either a relative maximum or a relative minimum at x_0 , then we say f has a *relative extremum* (*local extremum*) at x_0 .

Activity 1. 1. Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$. Show that f has a relative minimum at $x = 0$.

2. Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$. Explain why f does not have a relative extremum at $x = 0$.

Theorem 1. Let $f: (a, b) \to \mathbf{R}$ be a function. Let x_0 be a point in (a, b). Suppose f is differentiable at x_0 and has a relative extremum at x_0 . Then $f'(x_0) = 0$.

Definition 2. Let $f: (a, b) \to \mathbf{R}$ be a function. A point x_0 in (a, b) is called a *stationary point* of f if f is differentiable at x_0 and $f'(x_0) = 0$. A point x_0 is called a *critical point* if either $f'(x_0)$ does not exist or x_0 is a stationary point.

Note that a stationary point is a critical point but not the converse. As an example $x = 0$ is a critical point of $f(x) = |x|$ but it is not a stationary point. In fact f is not differentiable at $x = 0$ (Why?). .

Definition 3. Let c be in the domain of f. Then $f(c)$ is said to be the *absolute maximum* value of f if $f(x) \leq f(c)$ for all x in the domain of f. $f(c)$ is said to be the *absolute minimum* value of f if $f(x) \geq f(c)$ for all x in the domain of f.

Activity 2. Show that the function $f(x) = x^5$ does not have a relative extremum although $f'(0) = 0$.

Theorem 2 (Extreme Value Theorem). Let $f: [a, b] \to \mathbf{R}$ be a continuous function defined on a closed interval $[a,b]$. Then f has an absolute maximum and an absolute minimum. (i.e. We can find c and d in $[a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all x in $[a, b]$.)

(For the proof see the following article in my Calculus site,

http://www.math.nus.edu.sg/~matngtb/Calculus/Extreme Value/bound.htm,

you will need to be familiar with the ideas of supremum and infimum of a bounded set in **R**.)

Procedure 1 (for finding the absolute extrema of a continuous function on [*a*, *b*]).

Suppose f is a continuous function on $[a, b]$.

1. Find the critical points of f *in* (a, b), i.e., all the points where the derivatives of f are 0 and the points where f is not differentiable.

2. Include the end points a and b.

3. Compute the values of f at all the points obtained i.e. those found in (1) and (2) above. The maximum and minimum of the values of f at these points are the absolute maximum and absolute minimum of f on $[a, b]$.

Activity 3. Let $f : [-1, 2] \to \mathbf{R}$ be defined by $f(x) = 2x^3 - 3x^2$. Find the absolute maximum and absolute minimum of f on $[-1, 2]$.

Theorem 3 (Rolle's Theorem). Suppose (1) $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$, (2) f is differentiable on (a, b) and (3) $f(a) = f(b)$. Then there exists a point c in (a, b) such that $f'(c) = 0$.

Activity 4 To understand Rolle's Theorem, draw a smooth curve on the closed interval [1,5] representing a continuous function f on [1,5], which is also differentiable on the open interval (1,5), and the end points of the curve are of the same height (i.e. $f(1) = f(5)$). Look at the curve drawn, can you find a point c in (1, 5) whose tangent is horizontal (i.e. $f'(c) = 0$) ?

Theorem 4 (Mean Value Theorem). Let $a < b$. Suppose (1) $f : [a, b] \to \mathbf{R}$ is continuous (2) f is differentiable on (a, b). Then we can find a real number $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Activity 5. To understand Mean Value Theorem, as in Activity 4, draw a smooth curve over $[a,b]$, whose end points need not be of the same height. Now join end points $(a, (f(a))$ and $(b, f(b))$ of the curve to obtain a straight line l. Can you find a point c in (a, b) whose tangent is parallel to the line l (i.e. $f'(c) =$ gradient of the line $l = \dfrac{f(b) - f(a)}{b - a}$) ?

Remark Rolle's Theorem is a special case of Mean Value Theorem (when $f(a) = f(b)$). However, to prove Mean Value Theorem, one first prove Rolle's theorem, then apply Rolle's theorem to $g(x) = f(x) - \dfrac{f(b) - f(a)}{b - a}(x - a)$ on $[a, b]$.

Activity 6. Tin cans come in various shapes and sizes. A manufacturer wishes to enclosed a fixed volume V, using a cylindrical can. The height of the cylinder is denoted by h and the radius of the circular can by r. Express the surface S of the cylindrical can as a function of r. What advice would you give the manufacturer regarding minimising the cost of making the cylindrical cans?

The next result has been used by many students implicitly. We now record it as a theorem and leave the proof to students. (Hint: use Mean Value Theorem and some ε–δ argument.) But beware, even though when the limit $\lim_{x \to a} f'(x)$ does not exist, the function f may be differentiable at $x = a$. Theorem 6 cannot be used as an alternative to the definition of derivative. If in doubt always use the limit of the difference quotient to consider differentiability.

Theorem 6. If the function f is continuous at $x = a$ and differentiable on the interval $(a-\delta, a)$ and on the interval $(a, a+\delta)$ for some $\delta > 0$ and if $\lim_{x \to a} f'(x) = L$, then f is differentiable at $x = a$ and $f'(a) = L$.

[Hint: Write down what $\lim_{x \to a} f'(x) = L$ means. Then consider $\left| \dfrac{f(x) - f(a)}{x - a} - L \right|$, apply the Mean Value Theorem to get $\left| \dfrac{f(x) - f(a)}{x - a} - L \right| = |f'(c) - L|$ for some c between x and a (this is where you will need the continuity of f at $x = a$) and proceed.]

See also the article at my Calculus web site
http://www.math.nus.edu.sg/~matngtb/Calculus/Derivedfunction/Derivative.htm

Tutorial Assignment 6 **MA1102R**

1. Find the absolute extrema of the given function on the indicated interval. Draw a sketch of the function on the interval. (Caution: Don't apply Procedure 1 blindly. Analyse the interval carefully.)

   a. $g(x) = x^5 - 60x^2 + 3$; $[-6, 8]$.

   b. $h(x) = |x - 3| + 3$; $(0,5)$.

   c. $k(x) = 2x + [3x-1]$, where $[z]$ denotes the greatest integer less than or equal to z; $(1, 2]$.

   d. $f(x) = (x+3)^{\frac{2}{3}}$; $[-5, 0]$.

2. Find the absolute extrema of the given function on the indicated interval.

   a. $f(x) = \dfrac{x^2-6}{2x-5}$; $[0, \frac{9}{4}]$.

   b. $g(x) = \begin{cases} x^3 - 3x + 5, & 0 \le x \le 2 \\ x^2 - 5x + 13, & 2 < x \le 3 \end{cases}$; $[0, 3]$.

3. Test the conditions of *Rolle's Theorem*, and determine a point at which there is a horizontal tangent line.

   a. $f(x) = x^{\frac{3}{5}} - 2x^{\frac{2}{5}}$; domain $= [0, 32]$.

   b. $g(x) = \begin{cases} x^2 - 4, & x < 1 \\ 4x - 7, & x \ge 1 \end{cases}$; domain $= [-\sqrt{3}, \frac{7}{4}]$.

4. Without computing $\sqrt[3]{28}$ show that $\dfrac{1}{28} < \sqrt[3]{28} - 3 < \dfrac{1}{27}$.

   (Hint : Apply the *Mean Value Theorem* to $f(x) = \sqrt[3]{x}$ on $[27,28]$.)

5. (a) Use *Rolle's Theorem* to prove that the equation $\dfrac{x^6}{6} - \dfrac{x^5}{5} + \dfrac{x^4}{2} - \dfrac{2x^3}{3} + \dfrac{x^2}{2} - x - 4 = 0$ has **at most** one root that lies in the interval (3/2, 2).

   (b) Use Intermediate Value Theorem to show that the equation $\dfrac{x^6}{6} - \dfrac{x^5}{5} + \dfrac{x^4}{2} - \dfrac{2x^3}{3} + \dfrac{x^2}{2} - x - 4 = 0$ has **at least** one root that lies in the interval (3/2, 2).

   (c) Using (a) and (b), deduce that the equation $\dfrac{x^6}{6} - \dfrac{x^5}{5} + \dfrac{x^4}{2} - \dfrac{2x^3}{3} + \dfrac{x^2}{2} - x - 4 = 0$ has **exactly one root** that lies in the interval (3/2, 2).

6. Is there a differentiable function f such that $f(0) = -1$, $f(2) = 4$ and $f'(x) \le 2$ for all real numbers x?

7. Let f be a function whose $f'(x) = 4$ for all real numbers x, and $f(1) = 3$. Use Mean Value Theorem to show that $f(x) = 4x - 1$.