Factorization of Steenrod Squares Sq^{16}, Sq^{32} and Sq^{64}

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Introduction.

The late Frank Adams in his celebrated paper "*On the Non-existence of Elements of Hopf Invariant One*" gave many formulae, though not explicitly, relating to secondary cohomology operations. He gave the celebrated formula for the decomposition of the primary cohomology operations, the Steenrod square $SQ(2^k)$ for $k \geq 4$ by secondary cohomology operations, which are now known as the Adams operations $\phi_{i,j}$, $j \geq 0$, $i \leq j$ and $i \neq j-1$. Here we write $SQ(n)$ for Sq^n in the usual notation. Although he did not give explicit complete formula, the decomposition is good enough to solve the Hopf invariant one problem. However, for some application, the explicit decomposition is required. The explicit decomposition for $SQ(16)$ and $SQ(32)$ has been used in solving problems in H spaces and deciding whether a truncated polynomial algebra can be realised as the cohomology algebra for an appropriate space. It is possible to give explicitly a computable formula for the relevant coefficients in the most relevant operations in the decomposition of $SQ(2^{k+1})$, they are precisely the coefficients of $SQ(2^i)$, $0 \leq i \leq k$. If

$\phi_{0,k} = \sum_{0 \leq i \leq k} \beta_i SQ(2^i)$, then in the decomposition,

$$SQ(2^{k+1}) = \sum_{0 \leq i \leq k, i \neq k-1} \alpha_i \phi_{i,k} + \sum_{0 \leq j \leq k-1, 0 \leq i \leq j, i \neq j-1} \gamma_{i,j} \phi_{i,j} ,$$

$\alpha_i = \beta_i$ for $0 \leq i \leq k$. I note that except for the relation for $\phi_{0,2}$, the choice of the relations for $\phi_{0,k} = \sum_{0 \leq i \leq k} \beta_i SQ(2^i)$ for $k \geq 3$ can be chosen so that $\beta_{k-1} = 0$. An explicit description of $\phi_{0,k} = \sum_{0 \leq i \leq k} \beta_i SQ(2^i)$ is available and based on the description, the decomposition of $SQ(2^{k+1})$ for $k \geq 3$ can be easily verified by evaluation on the appropriate class in the cohomology of the infinite complex projective space $\mathbb{C}P(\infty)$ or the infinite quaternionic projective space, $QP(\infty)$.

1. The Decomposition for $SQ(16)$, $SQ(32)$ and $SQ(64)$.

Below I shall give the decomposition for $SQ(16)$, $SQ(32)$ and $SQ(64)$. The coefficients do get longer and longer. The coefficients, for $SQ(32)$ and $SQ(64)$ are computed and verified using REDUCE, whereas that for $SQ(16)$ is easily verified by hand..

The relevant Adams operations are listed below by listing the defining relations.

$\phi_{0,0}$: $SQ(1)SQ(1) = 0$;

$\phi_{1,1}$: $SQ(2)SQ(2) + SQ(3)SQ(1) = 0;$

$\phi_{0,2}$: $SQ(1)SQ(4) + (SQ(2)SQ(1))SQ(2) + SQ(4)SQ(1) = 0;$

$\phi_{2,2}$: $SQ(4)SQ(4) + SQ(6)SQ(2) + SQ(7)SQ(1) = 0;$

$\phi_{0,3}$: $(SQ(8)+SQ(4)SQ(4))SQ(1) + (SQ(4)SQ(2)SQ(1)+SQ(7))SQ(2) + SQ(1)SQ(8) = 0;$

$\phi_{1,3}$: $(SQ(9)+SQ(4)SQ(5))SQ(1) + SQ(8)SQ(2) + (SQ(4)SQ(2))SQ(4) + SQ(2)SQ(8) = 0;$

$\phi_{3,3}$: $SQ(15)SQ(1) + SQ(14)SQ(2) + SQ(12)SQ(4) + SQ(8)SQ(8) = 0;$

$\phi_{0,4}$: $(SQ(16)+SQ(8)SQ(8)+SQ(8)(SQ(4)SQ(4))SQ(1) +$

$(SQ(15)+SQ(8)SQ(7)+SQ(8)SQ(4)SQ(2)SQ(1))SQ(2)$

$+ SQ(13)SQ(4) + SQ(1)SQ(16) = 0;$

$\phi_{1,4}$: $(SQ(17)+SQ(4)SQ(13)+SQ(4)SQ(2)SQ(11))SQ(1)$

$+ (SQ(16)+\mathrm{Q}(4)SQ(2)SQ(10))SQ(2) + (SQ(4)SQ(2)SQ(4))SQ(8) + SQ(2)SQ(16) = 0;$

$\phi_{2,4}$: $(SQ(19)+SQ(8)SQ(11))SQ(1) + (SQ(18)+\mathrm{Q}(8)SQ(10))SQ(2)$

$+ SQ(16)SQ(4)+(SQ(8)SQ(4))SQ(8) + SQ(4)SQ(16) = 0;$

$\phi_{4,4}$: $SQ(31)SQ(1) + SQ(30)SQ(2) + SQ(28)SQ(4)+SQ(24)SQ(8) + SQ(16)SQ(16) = 0;$

$\phi_{0,5}$: $(SQ(32)+SQ(16)SQ(16)+SQ(16)SQ(8)SQ(8)+SQ(16)SQ(8)SQ(4)SQ(4))SQ(1) +$

$(SQ(31)+SQ(16)SQ(15)+SQ(16)SQ(8)SQ(7)+SQ(16)SQ(8)SQ(4)SQ(2)SQ(1))SQ(2)$

$+(SQ(29)+SQ(16)SQ(13))SQ(4) + SQ(25)SQ(8) + SQ(1)SQ(32) = 0;$

$\phi_{1,5}$: $(SQ(33)+SQ(4)SQ(29)+SQ(4)SQ(2)SQ(27)+SQ(4)SQ(2)SQ(4)SQ(23))SQ(1)$

$+ (SQ(32)+SQ(4)SQ(2)SQ(26)+SQ(4)SQ(2)SQ(4)SQ(22))SQ(2)$

$+ (SQ(4)SQ(2)SQ(4)SQ(20))SQ(4) + (SQ(4)SQ(2)SQ(4)SQ(8))SQ(16)$

$+ SQ(2)SQ(32) = 0;$

$\phi_{2,5}$: $(SQ(35)+SQ(8)SQ(27)+SQ(8)SQ(4)SQ(23))SQ(1)$

$+ (SQ(34)+SQ(8)SQ(26)+SQ(8)SQ(4)SQ(22))SQ(2)$

$+ (SQ(32)+SQ(8)SQ(4)SQ(20))SQ(4) + (SQ(8)SQ(4)SQ(8))SQ(16)$

$+ SQ(4)SQ(32) = 0;$

$\phi_{3,5}$: $(SQ(39)+SQ(16)SQ(23))SQ(1) + (SQ(38)+SQ(16)SQ(22))SQ(2)$

$+\ (SQ(36)+SQ(16)SQ(20))SQ(4) + SQ(32)SQ(8) + (SQ(16)SQ(8))SQ(16)$

$+\ SQ(8)SQ(32) = 0;$

$\phi_{5,5}$: $SQ(63)SQ(1) + SQ(62)SQ(2) + SQ(60)SQ(4) + SQ(56)SQ(8) + SQ(48)SQ(16)$

$+\ SQ(32)SQ(32) = 0;$

The decompositions are as follows:

***SQ*(16):**

$SQ(16) = C1\phi_{0,0} + C2\,\phi_{1,1} + C3\,\phi_{0,2} + C4\,\phi_{2,2} + C5\,\phi_{3,3} + C6\,\phi_{1,3} + C7\,\phi_{0,3}$, where

$C1 = SQ(15) + SQ(10)SQ(5) + SQ(13)SQ(2) + SQ(12)SQ(3),$

$C2 = SQ(13) + SQ(12)SQ(1),$

$C3 = SQ(12) + SQ(8)SQ(4),$

$C4 = SQ(8)SQ(1) + SQ(6)SQ(3),$

$C5 = SQ(1),$

$C6 = SQ(7) + SQ(4)SQ(2)SQ(1),$

$C7 = SQ(8) + SQ(4)SQ(4).$

***SQ*(32):**

$SQ(32) = A1\phi_{0,0} + A2\,\phi_{1,1} + A3\,\phi_{0,2} + A4\,\phi_{2,2} + A5\,\phi_{3,3} + A6\,\phi_{1,3}$

$+ A7\,\phi_{0,3} + A8\,\phi_{4,4} + A9\,\phi_{2,4} + A10\,\phi_{1,4} + A11\,\phi_{0,4}$, where

$A1 = SQ(31) + SQ(29)SQ(2) + SQ(28)SQ(3) + SQ(27)SQ(4) + SQ(23)SQ(6)SQ(2)$

$+ SQ(21)SQ(7)SQ(3) + SQ(20)SQ(9)SQ(2) + SQ(20)SQ(8)SQ(3)$

$+ SQ(18)SQ(9)SQ(4),$

$A2 = SQ(29) + SQ(27)SQ(2) + SQ(25)SQ(3)SQ(1) + SQ(25)SQ(4) + SQ(24)SQ(5)$

$+ SQ(21)SQ(5)SQ(2)SQ(1) + SQ(21)SQ(7)SQ(1) + SQ(20)SQ(7)SQ(2)$

$+ SQ(19)SQ(9)SQ(1),$

$A3 = SQ(25)SQ(3) + SQ(22)SQ(4)SQ(2) + SQ(21)SQ(7) + SQ(21)SQ(5)SQ(2)$

$+ SQ(19)SQ(6)SQ(3),$

$A4 = SQ(23)SQ(2) + SQ(21)SQ(4) + SQ(19)SQ(6),$

$A5 = SQ(17) + SQ(15)SQ(2) + SQ(13)SQ(4) + SQ(12)SQ(5),$

$A6 = SQ(19)SQ(4) + SQ(16)SQ(7) + SQ(15)SQ(7)SQ(1),$

$A7 = SQ(24) + SQ(21)SQ(3) + SQ(18)SQ(4)SQ(2) + SQ(16)SQ(8)$

$+ SQ(16)SQ(6)SQ(2) + SQ(15)SQ(7)SQ(2),$

$A8 = SQ(1),$

$A9 = SQ(13),$

$A10 = SQ(15) + SQ(8)SQ(7) + SQ(8)SQ(4)SQ(2)SQ(1),$

$A11 = SQ(8)SQ(8) + SQ(8)SQ(4)SQ(4) + SQ(16).$

***SQ*(64):**

$SQ(64) = B1\phi_{0,0} + B2\,\phi_{1,1} + B3\,\phi_{0,2} + B4\,\phi_{2,2} + B5\,\phi_{3,3} + B6\,\phi_{1,3} + B7\,\phi_{0,3}$

$+ B8\,\phi_{4,4} + B9\,\phi_{2,4} + B10\,\phi_{1,4} + B11\,\phi_{0,4} + B12\,\phi_{5,5} + B13\,\phi_{3,5}$

$+ B14\,\phi_{2,5} + B15\,\phi_{1,5} + B16\,\phi_{0,5}$, where

$B1 = SQ(63) + SQ(61)SQ(2) + SQ(60)SQ(3) + SQ(59)SQ(4) + SQ(56)SQ(7)$

$+ SQ(55)SQ(8) + SQ(53)SQ(7)SQ(3) + SQ(51)SQ(10)SQ(2) + SQ(49)SQ(11)SQ(3)$

$+ SQ(48)SQ(13)SQ(2) + SQ(48)SQ(12)SQ(3) + SQ(48)SQ(11)SQ(4)$

$+ SQ(47)SQ(16) + SQ(47)SQ(14)SQ(2) + SQ(47)SQ(12)SQ(4)$

$+ SQ(47)SQ(11)SQ(5) + SQ(47)SQ(10)SQ(4)SQ(2) + SQ(45)SQ(18)$

$+ SQ(45)SQ(14)SQ(4) + SQ(45)SQ(13)SQ(5) + SQ(45)SQ(11)SQ(5)SQ(2)$

$+ SQ(44)SQ(17)SQ(2) + SQ(44)SQ(15)SQ(4) + SQ(44)SQ(14)SQ(5)$

$+ SQ(44)SQ(13)SQ(4)SQ(2) + SQ(44)SQ(12)SQ(5)SQ(2) + SQ(43)SQ(20)$

$+ SQ(43)SQ(13)SQ(5)SQ(2) + SQ(43)SQ(12)SQ(6)SQ(2) + SQ(42)SQ(17)SQ(4)$

$+ SQ(42)SQ(16)SQ(5) + SQ(42)SQ(15)SQ(6) + SQ(42)SQ(14)SQ(7)$

$+ SQ(42)SQ(13)SQ(6)SQ(2) + SQ(41)SQ(18)SQ(4) + SQ(41)SQ(17)SQ(5)$

$+ SQ(41)SQ(14)SQ(6)SQ(2) + SQ(40)SQ(17)SQ(4)SQ(2)$

$+ SQ(40)SQ(16)SQ(5)SQ(2) + SQ(40)SQ(15)SQ(6)SQ(2)$

$+ SQ(40)SQ(14)SQ(6)SQ(3) + SQ(39)SQ(19)SQ(5) + SQ(39)SQ(18)SQ(6)$

$+ SQ(39)SQ(17)SQ(7) + SQ(39)SQ(15)SQ(6)SQ(3) + SQ(39)SQ(14)SQ(7)SQ(3)$

$$+ SQ(38)SQ(17)SQ(8) + SQ(38)SQ(17)SQ(6)SQ(2) + SQ(38)SQ(15)SQ(7)SQ(3)$$

$$+ SQ(37)SQ(18)SQ(6)SQ(2) + SQ(37)SQ(16)SQ(7)SQ(3)$$

$$+ SQ(36)SQ(17)SQ(8)SQ(2) + SQ(36)SQ(16)SQ(8)SQ(3)$$

$$+ SQ(35)SQ(17)SQ(8)SQ(3) + SQ(34)SQ(17)SQ(8)SQ(4),$$

$$\mathrm{B2} = SQ(61) + SQ(59)SQ(2) + SQ(57)SQ(4) + SQ(57)SQ(3)SQ(1) + SQ(56)SQ(5)$$

$$+ SQ(55)SQ(6) + SQ(55)SQ(4)SQ(2) + SQ(54)SQ(6)SQ(1) + SQ(53)SQ(7)SQ(1)$$

$$+ SQ(53)SQ(6)SQ(2) + SQ(52)SQ(9) + SQ(52)SQ(8)SQ(1) + SQ(51)SQ(10)$$

$$+ SQ(51)SQ(9)SQ(1) + SQ(51)SQ(8)SQ(2) + SQ(51)SQ(6)SQ(3)SQ(1)$$

$$+ SQ(49)SQ(11)SQ(1) + SQ(49)SQ(9)SQ(2)SQ(1) + SQ(48)SQ(12)SQ(1)$$

$$+ SQ(48)SQ(11)SQ(2) + SQ(47)SQ(12)SQ(2) + SQ(47)SQ(11)SQ(2)SQ(1)$$

$$+ SQ(47)SQ(10)SQ(3)SQ(1) + SQ(45)SQ(15)SQ(1) + SQ(45)SQ(13)SQ(2)SQ(1)$$

$$+ SQ(45)SQ(11)SQ(5) + SQ(44)SQ(15)SQ(2) + SQ(44)SQ(13)SQ(3)SQ(1)$$

$$+ SQ(44)SQ(11)SQ(4)SQ(2) + SQ(43)SQ(15)SQ(3) + SQ(43)SQ(15)SQ(2)SQ(1)$$

$$+ SQ(43)SQ(14)SQ(4) + SQ(43)SQ(12)SQ(6) + SQ(43)SQ(12)SQ(4)SQ(2)$$

$$+ SQ(43)SQ(10)SQ(5)SQ(2)SQ(1) + SQ(41)SQ(20) + SQ(41)SQ(18)SQ(2)$$

$$+ SQ(41)SQ(17)SQ(2)SQ(1) + SQ(41)SQ(16)SQ(4) + SQ(41)SQ(14)SQ(6)$$

$$+ SQ(41)SQ(12)SQ(6)SQ(2) + SQ(41)SQ(12)SQ(5)SQ(2)SQ(1)$$

$$+ SQ(40)SQ(20)SQ(1) + SQ(40)SQ(16)SQ(5) + SQ(40)SQ(15)SQ(6)$$

$$+ SQ(40)SQ(15)SQ(4)SQ(2) + SQ(39)SQ(15)SQ(7)$$

$$+ SQ(39)SQ(12)SQ(6)SQ(3)SQ(1) + SQ(38)SQ(18)SQ(5) + SQ(37)SQ(18)SQ(6)$$

$$+ SQ(37)SQ(18)SQ(4)SQ(2) + SQ(37)SQ(17)SQ(7) + SQ(37)SQ(17)SQ(6)SQ(1)$$

$$+ SQ(37)SQ(16)SQ(6)SQ(2) + SQ(37)SQ(15)SQ(7)SQ(2)$$

$$+ SQ(36)SQ(18)SQ(4)SQ(2)SQ(1) + SQ(36)SQ(17)SQ(7)SQ(1)$$

$$+ SQ(36)SQ(17)SQ(5)SQ(2)SQ(1) + SQ(35)SQ(17)SQ(8)SQ(1)$$

$$+ SQ(35)SQ(17)SQ(7)SQ(2) + SQ(35)SQ(16)SQ(6)SQ(3)SQ(1),$$

$$\mathrm{B3} = SQ(51)SQ(6)SQ(3) + SQ(47)SQ(13) + SQ(47)SQ(11)SQ(2)$$

$+ SQ(46)SQ(8)SQ(4)SQ(2) + SQ(45)SQ(15) + SQ(45)SQ(12)SQ(3)$

$+ SQ(45)SQ(10)SQ(5) + SQ(45)SQ(9)SQ(4)SQ(2) + SQ(44)SQ(13)SQ(3)$

$+ SQ(44)SQ(11)SQ(5) + SQ(43)SQ(15)SQ(2) + SQ(43)SQ(14)SQ(3)$

$+ SQ(43)SQ(12)SQ(5) + SQ(43)SQ(10)SQ(5)SQ(2) + SQ(42)SQ(18)$

$+ SQ(42)SQ(16)SQ(2) + SQ(42)SQ(14)SQ(4) + SQ(41)SQ(17)SQ(2)$

$+ SQ(41)SQ(16)SQ(3) + SQ(40)SQ(17)SQ(3) + SQ(39)SQ(19)SQ(2)$

$+ SQ(39)SQ(18)SQ(3) + SQ(39)SQ(17)SQ(4) + SQ(39)SQ(17)SQ(3)SQ(1)$

$+ SQ(39)SQ(16)SQ(5) + SQ(39)SQ(15)SQ(6) + SQ(39)SQ(15)SQ(5)SQ(1)$

$+ SQ(39)SQ(12)SQ(6)SQ(3) + SQ(38)SQ(16)SQ(4)SQ(2) + SQ(37)SQ(17)SQ(6)$

$+ SQ(37)SQ(17)SQ(4)SQ(2) + SQ(37)SQ(16)SQ(7) + SQ(37)SQ(14)SQ(6)SQ(3)$

$+ SQ(36)SQ(15)SQ(6)SQ(3) + SQ(35)SQ(15)SQ(7)SQ(3),$

$\mathrm{B4} = SQ(57) + SQ(55)SQ(2) + SQ(52)SQ(5) + SQ(51)SQ(6) + SQ(49)SQ(8)$

$+ SQ(49)SQ(7)SQ(1) + SQ(49)SQ(6)SQ(2) + SQ(48)SQ(9) + SQ(48)SQ(7)SQ(2)$

$+ SQ(45)SQ(9)SQ(3) + SQ(43)SQ(14) + SQ(43)SQ(12)SQ(2) + SQ(43)SQ(10)SQ(4)$

$+ SQ(41)SQ(16) + SQ(41)SQ(12)SQ(4) + SQ(40)SQ(17) + SQ(40)SQ(15)SQ(2)$

$+ SQ(40)SQ(13)SQ(4) + SQ(39)SQ(18) + SQ(39)SQ(15)SQ(3)$

$+ SQ(39)SQ(14)SQ(4) + SQ(39)SQ(13)SQ(4)SQ(1) + SQ(39)SQ(12)SQ(5)SQ(1)$

$+ SQ(39)SQ(11)SQ(5)SQ(2) + SQ(37)SQ(18)SQ(2) + SQ(37)SQ(17)SQ(3)$

$+ SQ(37)SQ(16)SQ(4) + SQ(37)SQ(15)SQ(4)SQ(1) + SQ(37)SQ(14)SQ(6)$

$+ SQ(37)SQ(13)SQ(6)SQ(1) + SQ(37)SQ(13)SQ(5)SQ(2) + SQ(36)SQ(15)SQ(6)$

$+ SQ(35)SQ(17)SQ(5) + SQ(35)SQ(16)SQ(6) + SQ(35)SQ(14)SQ(7)SQ(1),$

$\mathrm{B5} = SQ(47)SQ(2) + SQ(45)SQ(4) + SQ(40)SQ(9) + SQ(39)SQ(10)$

$+ SQ(39)SQ(9)SQ(1) + SQ(39)SQ(8)SQ(2) + SQ(37)SQ(12) + SQ(37)SQ(11)SQ(1)$

$+ SQ(37)SQ(10)SQ(2) + SQ(35)SQ(13)SQ(1) + SQ(35)SQ(12)SQ(2)$

$+ SQ(35)SQ(11)SQ(3) + SQ(33)SQ(12)SQ(4) + SQ(32)SQ(13)SQ(4)$

$+ SQ(31)SQ(15)SQ(3) + SQ(31)SQ(14)SQ(4) + SQ(31)SQ(13)SQ(5),$

$\mathrm{B6} = SQ(51)SQ(4) + SQ(49)SQ(6) + SQ(45)SQ(9)SQ(1) + SQ(43)SQ(12)$

$+ SQ(43)SQ(11)SQ(1) + SQ(43)SQ(8)SQ(4) + SQ(41)SQ(14) + SQ(41)SQ(13)SQ(1)$

$+ SQ(41)SQ(10)SQ(4) + SQ(39)SQ(15)SQ(1) + SQ(39)SQ(13)SQ(2)SQ(1)$

$+ SQ(39)SQ(12)SQ(4) + SQ(37)SQ(18) + SQ(37)SQ(16)SQ(2)$

$+ SQ(37)SQ(15)SQ(2)SQ(1) + SQ(37)SQ(14)SQ(4) + SQ(37)SQ(13)SQ(5)$

$+ SQ(37)SQ(12)SQ(6) + SQ(36)SQ(15)SQ(4) + SQ(35)SQ(17)SQ(3)$

$+ SQ(35)SQ(17)SQ(2)SQ(1) + SQ(35)SQ(15)SQ(5) + SQ(32)SQ(15)SQ(7)SQ(1),$

B7 = $SQ(53)SQ(3) + SQ(47)SQ(9) + SQ(47)SQ(8)SQ(1) + SQ(44)SQ(8)SQ(4)$

$+ SQ(42)SQ(8)SQ(4)SQ(2) + SQ(41)SQ(9)SQ(4)SQ(2) + SQ(40)SQ(13)SQ(3)$

$+ SQ(39)SQ(17) + SQ(39)SQ(14)SQ(3) + SQ(39)SQ(12)SQ(5)$

$+ SQ(39)SQ(11)SQ(4)SQ(2) + SQ(37)SQ(17)SQ(2)$

$+ SQ(37)SQ(15)SQ(4) + SQ(37)SQ(13)SQ(6) + SQ(35)SQ(16)SQ(5)$

$+ SQ(35)SQ(15)SQ(6) + SQ(35)SQ(14)SQ(5)SQ(2) + SQ(35)SQ(12)SQ(6)SQ(3)$

$+ SQ(33)SQ(16)SQ(7) + SQ(33)SQ(14)SQ(6)SQ(3) + SQ(32)SQ(15)SQ(7)SQ(2),$

B8 = $SQ(33) + SQ(31)SQ(2) + SQ(29)SQ(4) + SQ(25)SQ(8) + SQ(24)SQ(9),$

B9 = $SQ(43)SQ(2) + SQ(37)SQ(8) + SQ(35)SQ(10) + SQ(32)SQ(13)$

$+ SQ(29)SQ(14)SQ(2),$

B10 = $SQ(41)SQ(6) + SQ(40)SQ(7) + SQ(39)SQ(8) + SQ(35)SQ(11)SQ(1)$

$+ SQ(35)SQ(8)SQ(4) + SQ(33)SQ(13)SQ(1) + SQ(32)SQ(15) + SQ(32)SQ(11)SQ(4)$

$+ SQ(31)SQ(15)SQ(1) + SQ(31)SQ(11)SQ(5) + SQ(29)SQ(14)SQ(4)$

$+ SQ(29)SQ(13)SQ(5),$

B11 = $SQ(48) + SQ(41)SQ(7) + SQ(41)SQ(5)SQ(2) + SQ(39)SQ(9)$

$+ SQ(39)SQ(7)SQ(2) + SQ(39)SQ(6)SQ(3) + SQ(38)SQ(8)SQ(2)$

$+ SQ(38)SQ(7)SQ(3) + SQ(37)SQ(9)SQ(2) + SQ(36)SQ(9)SQ(3)$

$+ SQ(36)SQ(8)SQ(4) + SQ(35)SQ(9)SQ(4) + SQ(34)SQ(8)SQ(4)SQ(2)$

$+ SQ(32)SQ(16) + SQ(32)SQ(14)SQ(2) + SQ(32)SQ(13)SQ(3)$

$+ SQ(32)SQ(12)SQ(4) + SQ(32)SQ(11)SQ(5) + SQ(32)SQ(10)SQ(4)SQ(2)$

$+ SQ(31)SQ(15)SQ(2) + SQ(31)SQ(11)SQ(4)SQ(2) + SQ(29)SQ(14)SQ(5),$

B12 = $SQ(1)$,

B13 = $SQ(25)$,

B14 = $SQ(29) + SQ(25)SQ(4) + SQ(24)SQ(5) + SQ(23)SQ(6) + SQ(21)SQ(8)$,

B15 = $SQ(31) + SQ(29)SQ(2) + SQ(28)SQ(3) + SQ(27)SQ(4) + SQ(25)SQ(4)SQ(2)$
$+ SQ(24)SQ(7) + SQ(24)SQ(5)SQ(2) + SQ(23)SQ(8) + SQ(23)SQ(6)SQ(2)$
$+ SQ(22)SQ(6)SQ(3) + SQ(21)SQ(8)SQ(2) + SQ(21)SQ(7)SQ(3) + SQ(20)SQ(8)SQ(3)$
$+ SQ(19)SQ(8)SQ(4) + SQ(16)SQ(8)SQ(4)SQ(2)SQ(1)$,

B16 = $SQ(32) + SQ(31)SQ(1) + SQ(30)SQ(2) + SQ(29)SQ(2)SQ(1) + SQ(28)SQ(4)$
$+ SQ(28)SQ(3)SQ(1) + SQ(27)SQ(4)SQ(1) + SQ(26)SQ(4)SQ(2)$
$+ SQ(25)SQ(4)SQ(2)SQ(1) + SQ(24)SQ(8) + SQ(24)SQ(7)SQ(1)$
$+ SQ(24)SQ(6)SQ(2) + SQ(24)SQ(5)SQ(2)SQ(1) + SQ(23)SQ(8)SQ(1)$
$+ SQ(23)SQ(6)SQ(2)SQ(1) + SQ(22)SQ(8)SQ(2) + SQ(22)SQ(7)SQ(3)$
$+ SQ(22)SQ(6)SQ(3)SQ(1) + SQ(21)SQ(8)SQ(2)SQ(1) + SQ(21)SQ(7)SQ(3)SQ(1)$
$+ SQ(20)SQ(8)SQ(4) + SQ(20)SQ(8)SQ(3)SQ(1) + SQ(19)SQ(8)SQ(4)SQ(1)$
$+ SQ(18)SQ(8)SQ(4)SQ(2)$.

2. Comments

There are many applications and uses for the decomposition. The evaluation of these operations on the appropriate classes of the infinite complex or quaternionic projective spaces are as follows:

$$\phi_{0,s+1}(\omega^{2^{s+1}}) = \omega^{2^{s+1}+2^s} \;,\; \phi_{0,s+1}(\omega^{h2^{s+1}}) = h\omega^{(h-1)2^{s+1}}\phi_{0,s+1}(\omega^{2^{s+1}}) = h\omega^{h2^{s+1}+2^s} \;,\quad s \geq 1,$$

where $\omega \in H^2(\mathbb{C}P(\infty);\mathbb{Z}_2)$ is the generator.

$$\phi_{0,s+2}(u^{2^{s+1}}) = u^{2^{s+1}+2^s} \;,\; \phi_{0,s+2}(u^{h2^{s+1}}) = hu^{(h-1)2^{s+1}}\phi_{0,s+2}(u^{2^{s+1}}) = hu^{h2^{s+1}+2^s} \;,\quad s \geq 0,$$

where $u \in H^4(QP(\infty);\mathbb{Z}_2)$ is the generator.

With this information, one can show that the secondary obstruction to immersing the quaternionic projective space $QP(n)$ for n even and a power of 2 in $\mathbb{R}^{8n-4}$ is non zero and hence $QP(n)$ does not immerse in $\mathbb{R}^{8n-4}$. We shall give a proof of this statement in the next section.

The decomposition of $SQ(16)$ had been used by James Lin and Frank Williams ([6], [7] and [8]) in the investigation of 6-connected finite H spaces with 2 torsion, namely that there is no finite H-space *X* with $H^*(X;\mathbb{Z}_2) = \frac{\mathbb{Z}_2[x_7]}{x_7{}^4} \otimes \Lambda(x_{11}, x_{13})$ and the decomposition of *SQ*(32) had been used by them to prove that there is no mod 2 H-space *X* with $H^*(X;\mathbb{Z}_2) = \frac{\mathbb{Z}_2[x_{15}]}{x_{15}{}^4} \otimes \Lambda(x_{23}, x_{27}, x_{29})$. A variation of the decomposition of *SQ*(16), which can be verified by evaluation of the relation of u^4, where $u \in H^4(QP(\infty);\mathbb{Z}_2)$, had been used by them on their work on 6-connected finite H-spaces. Another form of the decomposition of *SQ*(16), which can be verified on $u^6 \in H^{24}(QP(\infty);\mathbb{Z}_2)$ or $\omega^{12} \in H^{24}(\mathbb{C}P(\infty);\mathbb{Z}_2)$ is used by Daciberg Lima Goncalves [4] in his work on mod 2 homotopy associative H-spaces. Indeed, based on the formula for $\phi_{0,s+1}(\omega^{2^{s+1}}) = \omega^{2^{s+1}+2^s}$ any specific decomposition of $SQ(2^{s+1})$ by the Adams operations can be verified. The problem is to obtain the coefficients of the relations.

3. Non-Immersion of $QP(m)$, $m = 2^s, s \geq 1$ in $\mathbb{R}^{8m-4}$.

We now prove our assertion on the non-immersion result for the *4m* dimensional quaternionic projective space $QP(m)$, $m = 2^s, s \geq 1$.

Suppose *m* is an even integer and is a power of 2. Then the quaternionic projective space *QP*(*m*) immerses in $\mathbb{R}^{8m-3}$. (See for example Davis and Mahowald [3].)

We can deduce this by taking a 4*m*-MPT for the fibration $p: BSpin(4m-3) \to BSpin(\infty)$ and by using the method of Ng [12] with a generating class theorem for the k-invariants to deduce that the relevant top dimensional obstructions are realized as stable secondary and tertiary cohomology operations and that the normal bundle ν of *QP*(*m*) has no obstruction below the top dimension by connectivity condition and no obstruction in the top dimension since any cohomological operation into the top cohomology group of the Thom space of the normal bundle is zero as the top class is spherical. Therefore, *QP*(*m*) of 4*m* real dimension immerses in $\mathbb{R}^{8m-3}$. It actually embeds in $\mathbb{R}^{8m-3}$ according to James [10].

Take an embedding of *QP*(*m*) in $\mathbb{R}^{8m}$. Let ν be the normal bundle of this embedding. Since *QP*(*m*) is 3-connected, the normal bundle is a 4*m* dimensional spin bundle. The normal bundle ν is classified by a map $f: QP(m) \to BSpin(4m)$, where $BSpin(4m)$ is the classifying space for 4*m* dimensional spin bundles and its stable normal bundle is classified by inclusion of $BSpin(4m)$ in $BSpin(\infty)$. Therefore, we start with the classifying space for spin bundles and consider the fibration, $p: BSpin(4m-4) \to BSpin(\infty)$. The normal bundle ν is classified by a map $f: QP(m) \to BSpin(4m)$ and the stable normal bundle is classified by inclusion of $BSpin(4m)$ in $BSpin(\infty)$. $QP(m)$ immerses in $\mathbb{R}^{8m-4}$ if, and only if, the classifying map $f: QP(m) \to BSpin(4m) \to BSpin(\infty)$ of the stable normal bundle ν lifts to $BSpin(4m-4)$.

If this classifying map lifts to $BSpin(4m-4)$ we say the bundle ν has stable geometric dimension less than or equal to $4m$–4. We can just stabilise the normal bundle of the embedding in $\mathbb{R}^{8m}$ by just adding one trivial line bundle. We shall show that the obstruction to lifting is non-trivial and consequently, $QP(m)$ does not immerse in $\mathbb{R}^{8m-4}$.

We take the 4-stage $4m$-MPT (modified Postnikov tower) for $p: BSpin(4m-4) \to BSpin(\infty)$. We list the k-invariants for the various stages in the following table.

4m-MPT for the fibration $p: BSpin(4m-4) \to BSpin(\infty)$

	k-invariant	Dimension	Defining relation
Stage 1	$k_1^1 = \delta w_{4m-4}$ $k_2^1 = w_{4m-2}$ $k_3^1 = w_{4m}$	$4m-3$ $4m-2$ $4m$	
Stage 2	k_1^2 k_2^2 k_3^2 k_4^2	$4m-2$ $4m-1$ $4m$ $4m$	$Sq^2 k_1^1 = 0$ $Sq^2 k_2^1 = 0$ $(Sq^4 + w_4 \cdot)k_1^1 + Sq^2 Sq^1 k_2^1 = 0$ $Sq^2 Sq^1 k_2^1 + Sq^1 k_3^1 = 0$
Stage 3	k_1^3 k_2^3 k_3^3	$4m-1$ $4m$ $4m$	$Sq^2 k_1^2 = 0$ $Sq^2 Sq^1 k_1^2 + Sq^1 k_3^2 = 0$ $Sq^2 k_2^2 + Sq^1 k_4^2 = 0$
Stage 4	k^4	$4m$	$Sq^2 k_1^3 + Sq^1 k_2^3 = 0$

We shall show that $k_3^2(\nu)$ is non-zero and consequently the classifying map p cannot lift pass stage 2 of the m-MPT for p and so it cannot lift to $BSpin(4m$–4).

Plainly for $QP(m)$, $\delta w_{4m-4}(\nu) = 0, w_{4m-2}(\nu) = 0$. Now $w_{4m}(\nu) = 0$.(See Massey [11]).

Let Γ be the stable secondary cohomology operation of Hughes-Thomas type with the defining relation:

$$Sq^4(\delta Sq^{4m-4}) + Sq^{4m-1}Sq^2 + (Sq^2 Sq^1)Sq^{4m-2} = 0.$$

Note that on the fundamental class b_{4m-4} of the principal bundle $Y_{4m-4} \to K(\mathbb{Z}_2, 4m-4)$, with classifying map $(\delta Sq^{4m-4}, Sq^2, Sq^{4m-2})$, where $K(\mathbb{Z}_2, 4m-4)$ is the Eilenberg Maclane space, we have that $b_{4m-4} \cup Sq^4 b_{4m-4} \in \Gamma(b_{4n-4})$. (k_1^1, k_2^1, k_3^1) is admissible for k_3^2 in the sense of the Admissible Class Theorem of Ng [12]. In particular we have

$$U(T(\nu)) \cdot (k_3^2(\nu) + w_4(\nu) \cdot w_{4m-4}(\nu) \in \Gamma(U(T(\nu))) \text{ modulo zero indeterminacy,}$$

where $T(\nu)$ is the Thom space of the normal bundle and $U(T(\nu))$ the Thom class of the normal bundle. Observe that the indeterminacy of Γ on the Thom class $U(T(\nu))$ is zero.

At this point, we assert that $\Gamma(U(T(\nu))) = 0$ modulo zero indeterminacy. It is now folk lore that the top cohomology class or homology class of the normal bundle of an embedding is spherical and so any cohomology operation into the top cohomology group of the Thom space of ν is trivial. As an exercise of not using this fact we can proceed to examine the operation Γ.

We shall change the operation Γ to the stable secondary cohomology operation ϕ defined by the relation

$$Sq^4(\delta Sq^{4m-4}) + Sq^1(Sq^{4m-2}Sq^2) + Sq^2Sq^{4m-1} = 0.$$

We can choose ϕ such that $\Gamma \subseteq \phi$. Moreover, $\Gamma(U(T(\nu))) = \phi(U(T(\nu)))$ modulo zero indeterminacy. Plainly, $\Gamma(U(T(\nu))) = \eta(U(T(\nu)))$, where η is associated with the relation

$$Sq^4(Sq^{4m-3}) + Sq^1(Sq^{4m-2}Sq^2) + Sq^2Sq^{4m-1} = 0.$$

We are going to use an S dual operation and so we introduce an operation, ζ, associated with the relation, $Sq^1(Sq^{4m-3}) = 0$. Plainly, $\zeta(U(T(\nu))) = 0$. Let $\Lambda = \eta + Sq^3\varsigma$. Then Λ is associated with the relation,

$$\chi Sq^4(Sq^{4m-3}) + Sq^1(Sq^{4m-2}Sq^2) + Sq^2Sq^{4m-1} = 0$$

and $\Lambda(U(T(\nu))) = \Gamma(U(T(\nu)))$ modulo zero indeterminacy.

According to Atiyah [1], $\Sigma^{4c-4m-1}T(\nu) \approx Q_{c,m+1}$, where c is a multiple of the quaternionic James number c_{m+1} and $Q_{c,m+1}$ is the stunted quasi-projective space of James [10], where the first nonzero cohomology class γ_{c-m} is of dimension $4(c-m)-1$ and so Λ will take this class to a class in dimension $4c-1$. Also, by Atiyah [1], $Q_{c,m+1}$ is S-dual to the S type of the stunted quaternionic projective space, $QP_{m+1-c,m+1}$. Hence, we may take the S dual to be $QP_{c_{m+1}+m+1,m+1}$. Now, if $m = 2^s$ and $s \geq 1$, then the 2 exponent of c_{m+1} is given by

$$\nu_2(c_{m+1}) = \nu_2(c_{2^s+1}) = \max_{1\leq j\leq 2^s}\left\{2(2^s+1)-1, 2j+\nu_2(j)\right\} = 2^{s+1}+s. \text{ (See [13]).}$$

Hence, we may write $c_{m+1} = 2^{2^{s+1}+s} + g2^{2^{s+1}+s+1}$ for some positive integer g. Note that $QP_{c_{m+1}+m+1,m+1} = \dfrac{QP_{c_{m+1}+m+1}}{QP_{c_{m+1}}}$, where $QP_{c_{m+1}+m+1} = QP(c_{m+1}+m)$ and $QP_{c_{m+1}} = QP(c_{m+1}-1)$. The first non-zero cohomology class in $H^*(QP_{c_{m+1}+m+1,m+1};\mathbb{Z}_2) = H^*\left(\dfrac{QP_{c_{m+1}+m+1}}{QP_{c_{m+1}}};\mathbb{Z}_2\right)$ is in dimension

$4c_{m+1}$ and is given by $u^{c_{m+1}} = u^{2^{2^{s+1}+s}+g2^{2^{s+1}+s+1}} = u^{2(2^{2^{s+1}+s-1}+g2^{2^{s+1}+s})}$. Therefore, $\Lambda(U(T(\nu))) = 0$ if, and only if, $\chi\Lambda(u^{2(2^{2^{s+1}+s-1}+g2^{2^{s+1}+s})}) = 0$, where $\chi\Lambda$ is the dual operation to Λ associated with the dual relation,

$$\chi\Lambda : \chi(Sq^{4m-3})Sq^4 + \chi(Sq^{4m-2}Sq^2)Sq^1 + \chi Sq^{4m-1}Sq^2 = 0.$$

Observe that $\chi Sq^{4m-1} = \chi(Sq^3 Sq^{4m-4}) = \chi(Sq^{4m-4})\chi(Sq^3) = \chi(Sq^{4m-4})Sq^2Sq^1$, $\chi Sq^{4m-3} = \chi(Sq^1 Sq^{4m-4}) = \chi Sq^{4m-4}Sq^1$ and since $\chi Sq^4 Sq^{4m-4} = Sq^{4m-2}Sq^2$, $\chi\left(Sq^{4m-2}Sq^2\right) = \chi Sq^{4m-4}Sq^4$. Therefore, the defining relation is in the form $\chi\Lambda : \left(\chi(Sq^{4m-4})Sq^1\right)Sq^4 + \left(\chi(Sq^{4m-4})Sq^4\right)Sq^1 + \left(\chi Sq^{4m-4}Sq^2Sq^1\right)Sq^2 = 0$ and we can choose a representative for the operation $\chi\Lambda$ to be $\chi Sq^{4m-4}\phi_{0,2}$, where $\phi_{0,2}$ is the Adams operation of degree 4. Hence, $\chi\Lambda(u^{2(2^{2^{s+1}+s-1}+g2^{2^{s+1}+s})}) = \chi Sq^{4m-4}\phi_{0,2}(u^{2(2^{2^{s+1}+s-1}+g2^{2^{s+1}+s})})$ modulo zero indeterminacy.

But $\phi_{0,2}(u^{2(2^{2^{s+1}+s-1}+g2^{2^{s+1}+s})}) = (2^{2^{s+1}+s-1} + g2^{2^{s+1}+s})u^{2(2^{2^{s+1}+s-1}+g2^{2^{s+1}+s}-1)}\phi_{0,2}(u^2) = 0$. It follows that $\Gamma(U(T(\nu))) = U(T(\nu))\cdot(k_3^2(\nu) + w_4(\nu)\cdot w_{4m-4}(\nu)) = 0$ modulo zero indeterminacy. This means $k_3^2(\nu) + w_4(\nu)\cdot w_{4m-4}(\nu) = 0$. The Stiefel-Whitney class of the normal bundle of $QP(2^s)$ is given by $w_{4i}(\nu) = \binom{m+i}{i}u^i = \binom{2^s+i}{i}u^i$, where $u \in H^4(QP(m);\mathbb{Z}_2)$ is the generator. Therefore, $w_4(\nu) = u$ and $w_{4m-4}(\nu) = w_{4(2^s-1)}(\nu) = u^{m-1}$. Hence, $k_3^2(\nu) = u^m \neq 0$.

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