Riemann Integral and Sum of Infinite Series

The use of the Riemann integral in summing infinite series is based on the ability to recognise that the partial sum of the infinite series in question is a Riemann sum of a function which has nice easily computable antiderivative. There remains the question that if a function is Riemann integrable, then is it the case that any sequence of Riemann sum with respect to a sequence of partition with norm converging to zero has limit equal to the Riemann integral? The answer is yes and this gives legitimacy to our use of the Riemann integral in evaluating infinite series.

Theorem. If $f : [a, b] \to \mathbf{R}$ is Riemann integrable on $[a, b]$ and P_n is a sequence of Partition of $[a, b]$ with norm $\|P_n\|$ converging to 0 as n tends to infinity, then for any sequence $\{S_n\}$ where each S_n is a Riemann sum respect to P_n, the limit of the sequence $\{S_n\}$ as n tends to infinity is equal to the Riemann integral of f on $[a, b]$.

Proof. Since f is Riemann integrable, i.e. there exists a number L such that given any $\varepsilon > 0$, we can find a $\delta > 0$ such that for any partition P for $[a, b]$ with norm $\| P \| < \delta$, and for any Riemann sum S with respect to P, $| S - L | < \varepsilon$. Since $\|P_n\|$ converges to 0, we can find an integer N such that for any $n > N$, $\|P_n\| < \delta$. Therefore, because each S_n is a Riemann sum with respect to P_n, $|S_n - L| < \varepsilon$ for all $n > N$. This means the sequence $\{S_n\}$ converges to L as n tends to infinity and L here is the Riemann integral of f on $[a, b]$. This completes the proof.

Remark. In most application of the above theorem, the partition P_n above has $\|P_n\| = (b - a)/n$ as is the case when P_n is a regular partition and the Riemann sum S_n is usually one with respect to P_n and the point in each sub interval in the partition are chosen either to be the end point or beginning point of the subintervals. In more detail, the application is stated below.

Corollary. Suppose we are given a limit of the form $\lim_{n\to\infty} \sum_{i=1}^{n} g(i)$. If we can write $g(i) = f(x_i)\frac{b-a}{n}$, where either $x_i = a + i\frac{b-a}{n}$ or $x_i = a + (i-1)\frac{b-a}{n}$, then $\lim_{n\to\infty} \sum_{i=1}^{n} g(i) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\frac{b-a}{n} = \int_a^b f(x)dx$ if f is Riemann integrable on $[a, b]$. Therefore, if we can find an antiderivative F of f, then $\lim_{n\to\infty} \sum_{i=1}^{n} g(i) = F(b) - F(a)$ by the Fundamental Theorem of Calculus.