Riemann Integral and Bounded function.

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In a generalisation of area under a graph of a function, it is normally assumed that the function under consideration be bounded. For bounded function, the range of the function is bounded and hence any subset of the range is also bounded. A consequence of this and also of the completeness of the real numbers is that supremum and infimum exist for any subset of the range. This means upper Riemann sum and lower Riemann sum with respect to any partition can be defined. Another consequence of the boundedness of the function is that the set of all upper Riemann sum is bounded below and so we have then the existence of the infimum of that set and this is the ***Upper Riemann Integral*** of the function. It is also a consequence of the boundedness of the function that the set of all lower Riemann sum is bounded above and so we have the existence of the supremum of this set and this is the ***Lower Riemann Integral***. This is the mechanism behind the meaning of the Riemann integral. The (bounded) function is then said to be Riemann integrable, if and only if, the lower and upper Riemann integrals are the same. Then the definition of Riemann integral in terms of the usual Riemann sum follows, though not so easily, as a consequence of the above. The question is then asked: What about unbounded function? Do we have a sensible generalisation of the area in this case? We cannot use the completeness property of the real numbers.

Because subset not bounded above does not have supremum and subset not bounded below does not have infimum, for unbounded functions lower and upper Riemann sums cannot be defined. Even if the domain is a closed and bounded interval and the function is defined on this interval but not bounded above say, Riemann sum for any partition can be arbitrarily large. We can observe this as follows.

Take any partition P: $a = x_0 < x_1 < ... < x_n = b$ with arbitrary norm $\|P\|$. Let $A = \min\{x_i - x_{i-1} : i = 1, ..., n\}$ Take a Riemann sum $\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) = K$, where $\xi_i \in [x_{i-1}, x_i]$. Then let $F = \max\{f(\xi_i) : i = 1, ..., n\}$. We shall show that for any Large $N > K$, if f is not bounded above, we can find a Riemann sum S with respect to the same partition P such that $S > N$. Since f is not bounded above, there exists c in $[a, b]$ such that $f(c) > \frac{N-K}{A} + F$. Then c is in $[x_{j-1}, x_j]$ for some j, $1 \le j \le n$. Then the Riemann sum

$$\begin{aligned} S &= \sum_{i \neq j} f(\xi_i)(x_i - x_{i-1}) + f(c)(x_j - x_{j-1}) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) + (f(c) - f(\xi_j))(x_j - x_{j-1}) \\ &> K + \left(\frac{N-K}{A} + F - f(\xi_j)\right)(x_j - x_{j-1}) \ge K + \left(\frac{N-K}{A}\right)(x_j - x_{j-1}) \ge K + N - K = N \end{aligned}$$

Similarly, if f is not bounded below we can find arbitrarily negatively large Riemann sum with respect to the same partition P. No matter how small the norm of the partition, if the function is unbounded, the corresponding set of Riemann sum is unbounded. Therefore, there does not exists a number L such that given any $\varepsilon > 0$, we can find a $\delta > 0$ such that for any partition P with norm $\| P \| < \delta$, and for any Riemann sum S with respect to P, $| S - L | < \varepsilon$. This consideration shows affirmatively that any function that is not bounded CANNOT be Riemann integrable. But some unbounded function do have a finite area. It is difficult to generalise here. But for functions with one or two discontinuity where the left or right limits is either $+\infty$ or $-\infty$, we can talk about the existence of improper (Riemann) integral. For instance, for the function $f(x) = \begin{cases} 1/\sqrt{x}, x > 0 \\ 0, x = 0 \end{cases}$, the area under the graph between $x = 0$ and $x = 1$ is given by $\lim_{t\to 0^+} \int_t^1 f(x)dx = \lim_{t\to 0^+} \left[2\sqrt{x}\,\right]_t^1 = 2$. We can similarly consider functions with a finite number of such discontinuities.

Riemann Integrability

Theorem 1. For a bounded function $f: [a, b] \to \mathbf{R}$ the following statements are equivalent.

1. The upper and lower Riemann integrals are the same.
2. Given $\varepsilon > 0$, there exists a partition P for the interval $[a, b]$ such that the difference $U(P) - L(P) < \varepsilon$, where $U(P)$ is the upper Riemann sum and $L(P)$ is the lower Riemann sum with respect to P for f.
3. f is Riemann integrable, i.e., there exists a number L such that given any $\varepsilon > 0$, we can find a $\delta > 0$ such that for any partition P for $[a, b]$ with norm $\|P\| < \delta$, and for any Riemann sum S with respect to P, $|S - L| < \varepsilon$.

Proof. (1) if and only if (2) is easy.

(3) $\Rightarrow$(2). Assume f is Riemann integrable. Given $\varepsilon > 0$, then for any partition P for $[a, b]$ with norm $\|P\| < \delta$, and for any Riemann sum S with respect to P, $|S - L| < \varepsilon/4$. Let $T = \{$ Riemann sum S: S has the same partition $P\}$.
Then for any S in T, $L - \varepsilon/4 < S < L + \varepsilon/4$.
Suppose P: $a = x_0 < x_1 < \ldots < x_n = b$ *with* $\|P\| < \delta$. Let $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ for $i = 1,\ldots, n$. Then for each i, $1 \le i \le n$, there exists c_i in $[x_{i-1}, x_i]$ such that $f(c_i) > M_i - \frac{\varepsilon}{4(b-a)}$. Then

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) > \sum_{i=1}^{n}\left(M_i - \frac{\varepsilon}{4(b-a)}\right)(x_i - x_{i-1}) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) - \frac{\varepsilon}{4} = U(P) - \frac{\varepsilon}{4},$$

where $U(P)$ is the upper Riemann sum with respect to the partition P.
Therefore, $U(P) < \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) + \frac{\varepsilon}{4} < L + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = L + \frac{\varepsilon}{2}$. Now let $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ for $i = 1,\ldots, n$. Then for each i, $1 \le i \le n$, there exists d_i in $[x_{i-1}, x_i]$ such that $f(d_i) < m_i + \frac{\varepsilon}{4(b-a)}$. *Then*

$$\sum_{i=1}^{n} f(d_i)(x_i - x_{i-1}) < \sum_{i=1}^{n}\left(m_i + \frac{\varepsilon}{4(b-a)}\right)(x_i - x_{i-1}) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) + \frac{\varepsilon}{4} = L(P) + \frac{\varepsilon}{4},$$

where $L(P)$ is the lower Riemann sum with respect to the partition P.

Thus, $L(P) > \sum_{i=1}^{n} f(d_i)(x_i - x_{i-1}) - \frac{\varepsilon}{4} > L - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = L - \frac{\varepsilon}{2}$.
Therefore, $U(P) - L(P) < \varepsilon$. Hence (2) follows.

(1) $\Leftrightarrow$(2). Suppose the lower integral $\underline{\int} f$ is the same as the upper integral $\overline{\int} f$.
Then by the definition of the lower integral, given $\varepsilon > 0$, there exists a partition P such that the lower sum $L(P, f) > \underline{\int} f - \frac{\varepsilon}{2}$. Similarly by the definition of the upper integral, there exists a partition Q such that the upper Riemann sum $U(Q, f) < \overline{\int} f + \frac{\varepsilon}{2}$. Let R be the partition $P \cup Q$, then

$$\underline{\int} f - \frac{\varepsilon}{2} < L(P, f) \le L(P \cup Q, f) \le U(P \cup Q, f) \le U(Q, f) < \overline{\int} f + \frac{\varepsilon}{2}.$$

Thus, $U(P \cup Q, f) - L(P \cup Q, f) < \overline{\int} f + \frac{\varepsilon}{2} - \underline{\int} f + \frac{\varepsilon}{2} = \varepsilon$ since $\overline{\int} f = \underline{\int} f$. So R is the required partition.

Conversely, if for any $\varepsilon > 0$, there exists a partition R such that $U(R, f) - L(R, f) < \varepsilon$, then $\overline{\int} f \le U(R, f) < L(R, f) + \varepsilon \le \underline{\int} f + \varepsilon$. Therefore, $\overline{\int} f < \underline{\int} f + \varepsilon$ for any $\varepsilon > 0$. Thus, $\overline{\int} f \le \underline{\int} f$. But since for any partition Q, $L(P, f) \le U(Q, f)$, $L(P, f) \le \overline{\int} f$ because $\overline{\int} f$ is the infimum of all the upper Riemann sums. Then $\underline{\int} f \le \overline{\int} f$ since $\underline{\int} f$ is the supremum of all the lower Riemann sums. Therefore, $\overline{\int} f = \underline{\int} f$.

(1) *and* (2) $\Rightarrow$ (3)

Given $\varepsilon > 0$, there exists a partition P : $a = x_0 < x_1 < \ldots < x_L = b$ with $L > 1$ such that $U(P, f) - L(P, f) < \frac{\varepsilon}{2}$. Let $K = \min \{x_i - x_{i-1} : i = 1, \ldots, L\}$. Now since f is bounded, there exists $M > 0$ such that $|f(x)| < M$. Let R: $a = y_0 < y_1 < \ldots < y_N = b$ be any partition such that $\|R\| < \delta$, where $\delta = \min(K, \frac{\varepsilon}{2(2ML+1)})$. Then because $\|R\| < K$, $N > L$ and for each $i = 1,2,.., L-1$ there exist $1 \le j_i \le N-1$ such that $y_{j_i-1} \le x_i < y_{j_i}$. Let $I = \{j_i : i = 1, \ldots, L-1\}$ .Then for any Riemann sum with respect to the partition R,

$$\begin{aligned}
&\sum_{i=1}^{N} f(\xi_i)(y_i - y_{i-1}), \text{ where } \xi_i \in [y_{i-1}, y_i] ,\\
&= \sum_{i \notin I} f(\xi_i)(y_i - y_i) + \sum_{i=1}^{L-1} f(\xi_{j_i})(y_{j_i} - y_{j_i-1})\\
&= \sum_{i \notin I} f(\xi_i)(y_i - y_i) + \sum_{i=1}^{L-1} f(x_i)(y_{j_i} - y_{j_i-1}) + \sum_{i=1}^{L-1} (f(\xi_{j_i}) - f(x_i))(y_{j_i} - y_{j_i-1})\\
&= \left(\sum_{i \notin I} f(\xi_i)(y_i - y_i) + \sum_{i=1}^{L-1} f(x_i)(y_{j_i} - x_i) + \sum_{i=1}^{L-1} f(x_i)(x_i - y_{j_i-1}) \right)\\
&\quad + \sum_{i=1}^{L-1} (f(\xi_{j_i}) - f(x_i))(y_{j_i} - y_{j_i-1}).
\end{aligned}$$

Note that the bracketed term is a Riemann sum S for the partition $R \cup P$. Thus

$$\begin{aligned}
&\sum_{i=1}^{N} f(\xi_i)(y_i - y_{i-1}) = S + \sum_{i=1}^{L-1} (f(\xi_{j_i}) - f(x_i))(y_{j_i} - y_{j_i-1})\\
&< U(R \cup P, f) + \sum_{i=1}^{L-1} 2M(y_{j_i} - y_{j_i-1}) \le U(P, f) + \sum_{i=1}^{L-1} 2M(y_{j_i} - y_{j_i-1})\\
&< L(P, f) + \frac{\varepsilon}{2} + 2\sum_{i=1}^{L-1} M\|R\| < \underline{\int} f + \frac{\varepsilon}{2} + 2ML\|R\|\\
&< \underline{\int} f + \frac{\varepsilon}{2} + 2ML\frac{\varepsilon}{2(2ML+1)} \le \underline{\int} f + \varepsilon \text{ since } \|R\| < \delta \le \frac{\varepsilon}{2(2ML+1)}.
\end{aligned}$$

Also

$$\begin{aligned}
&\sum_{i=1}^{N} f(\xi_i)(y_i - y_{i-1}) = S + \sum_{i=1}^{L-1} (f(\xi_{j_i}) - f(x_i))(y_{j_i} - y_{j_i-1})\\
&\ge L(R \cup P, f) + \sum_{i=1}^{L-1} (f(\xi_{j_i}) - f(x_i))(y_{j_i} - y_{j_i-1})\\
&\ge L(P, f) + \sum_{i=1}^{L-1} (f(\xi_{j_i}) - f(x_i))(y_{j_i} - y_{j_i-1})\\
&> U(P, f) - \frac{\varepsilon}{2} - \sum_{i=1}^{L-1} 2M(y_{j_i} - y_{j_i-1}) > U(P, f) - \frac{\varepsilon}{2} - 2\sum_{i=1}^{L-1} M\|R\|\\
&> \overline{\int} f - \frac{\varepsilon}{2} - 2ML\|R\|\\
&> \overline{\int} f - \frac{\varepsilon}{2} - 2ML\frac{\varepsilon}{2(2ML+1)} = \overline{\int} f - \varepsilon.
\end{aligned}$$

Since $\overline{\int} f = \underline{\int} f = L$, $\left|\sum_{i=1}^{N} f(\xi_i)(y_i - y_{i-1}) - L\right| < \varepsilon$. Thus for any partition R with norm $\|R\| < \delta$ and for any Riemann sum S with respect to R, $|S - L| < \varepsilon$.

Consequence of Theorem 1.

Theorem 2. Suppose $f : [a, b] \to \mathbf{R}$ is a real valued function. If f is Riemann integrable on $[a, b]$, then for any c in $[a, b]$, f is Riemann integrable on $[a, c]$ and on $[c, b]$.

Proof. Fix an $\varepsilon > 0$. Then since f is Riemann integrable on $[a, b]$, by Theorem 1, there exists a partition $P: \ a = x_0 < x_1 < \ldots < x_n = b$ such that the difference of the upper and lower Riemann sum,

$$U(P) - L(P) < \varepsilon, \quad \text{---------------------} \quad (1)$$

where $U(P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$, $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$,

$$L(P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

Then since for any refinement Q of P, that is any partition Q of $[a, b]$ such that $P \subseteq Q$, $U(Q) \le U(P)$ and $L(P) \le L(Q)$, we have that the difference $U(Q) - L(Q) < U(P) - L(P) < \varepsilon$. Now if $c = a$ *or* b we have nothing to prove. We now assume $a < c < b$. If c is not a point of P, then add c to P and replace P by this new partition and still have the inequality (1) satisfied. So we may, without loss of generality, assume that c is in P *that is* $c = x_k$ for some $0 < k < n$. Suppose now that P is given by

$$P: \ a = x_0 < x_1 < \ldots < x_k = c < x_{k+1} < \ldots < x_n = b$$

Then Δ: $a = x_0 < x_1 < \ldots < x_k = c$ is a partition for $[a, c]$ and Δ' : $x_k = c < x_{k+1} < \ldots < x_n = b$ is a partition for $[c, b]$. Thus the difference of the upper and lower Riemann sum over $[a, c]$ is

$$U(\Delta) - L(\Delta) = \sum_{i=1}^{k}(M_i - m_i)(x_i - x_{i-1}) \le \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) = U(P) - L(P) < \varepsilon .$$

Therefore, by Theorem 1, f is Riemann integrable over $[a, c]$. Similarly,

$$U(\Delta') - L(\Delta') = \sum_{i=k+1}^{n}(M_i - m_i)(x_i - x_{i-1}) \le \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) = U(P) - L(P) < \varepsilon .$$

Once more, by Theorem 1, f is Riemann integrable over $[c, b]$. This proves the theorem.

Corollary 3. Suppose $f : [a, b] \to \mathbf{R}$ is a real valued function. If f is Riemann integrable on $[a, b]$, then for any c and d in $[a, b]$ such that $a < c < d < b$, f is Riemann integrable on $[c, d]$.

Proof. By Theorem 2, f is Riemann integrable on $[a, d]$ and another application of Theorem 2 says that f is Riemann integrable on $[c, d]$.

Theorem 4. Suppose $f : [a, b] \to \mathbf{R}$ is a real valued function. If f is Riemann integrable on $[a, b]$, then $|f|$ is also Riemann integrable on $[a, b]$ and

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx.$$

Proof. This is just manipulation. We need to work with the properties of supremum and infimum. Notice that for any bounded subset A of $\mathbf{R}$, $\inf A = -\sup(-A)$. As f is Riemann integrable, there exists a partition, $P:\ a = x_0 < x_1 < \ldots < x_n = b$ of $[a, b]$, such that the difference of the upper and lower Riemann sum with respect to P for f,

$$U(P,f) - L(P,f) = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) < \varepsilon,$$

where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$, and $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} = -\sup\{-f(x) : x \in [x_{i-1}, x_i]\}$. Therefore,

$$\begin{aligned} M_i - m_i &= \sup\{f(x) : x \in [x_{i-1}, x_i]\} + \sup\{-f(x) : x \in [x_{i-1}, x_i]\} \\ &= \sup\{f(x) - f(y) :\ x, y \in [x_{i-1}, x_i]\}. \end{aligned}$$

Also the difference of the upper and lower Riemann sum with respect to P for $|f|$ *is*

$$U(P,|f|) - L(P,|f|) = \sum_{i=1}^{n}(M_i' - m_i')(x_i - x_{i-1}) < \varepsilon,$$

where $M_i' = \sup\{|f(x)| : x \in [x_{i-1}, x_i]\}$, and $m_i' = \inf\{|f(x)| : x \in [x_{i-1}, x_i]\} = -\sup\{-|f(x)| : x \in [x_{i-1}, x_i]\}$. Therefore,

$$\begin{aligned} M_i' - m_i' &= \sup\{|f(x)| : x \in [x_{i-1}, x_i]\} + \sup\{-|f(x)| : x \in [x_{i-1}, x_i]\} \\ &= \sup\{|f(x)| - |f(y)| :\ x, y \in [x_{i-1} - x_i]\}. \end{aligned}$$

In fact, since $a \in \{f(x) - f(y) :\ x, y \in [x_{i-1}, x_i]\}$, if and only if, $-a \in \{f(x) - f(y) :\ x, y \in [x_{i-1}, x_i]\}$, $M_i - m_i = \sup\{|f(x) - f(y)| :\ x, y \in [x_{i-1}, x_i]\}$. Similarly, $M_i' - m_i' = \sup\{||f(x)| - |f(y)|| :\ x, y \in [x_{i-1}, x_i]\}$.

Now since we have the following inequality that for any a, b in $\mathbf{R}$, $||a| - |b|| \le |a - b|$,

$$M_i' - m_i' \le M_i - m_i \quad \text{for } i = 1, \ldots, n.$$

It follows that

$$\begin{aligned} U(P,|f|) - L(P,|f|) &= \sum_{i=1}^{n}(M_i' - m_i')(x_i - x_{i-1}) \\ &\le \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) \le U(P,f) - L(P,f) < \varepsilon. \end{aligned}$$

Therefore, the Riemann condition for $|f|$ is fulfilled. Hence by Theorem 1, $|f|$ is Riemann integrable.

Now since $-|f| \le f \le |f|$ and $-|f|$ is also Riemann integrable we have then that

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|\,dx$$

which implies that $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx$. This completes the proof of Theorem 4.

We can use the above argument to prove the following.

Theorem 5. Suppose $f : [a, b] \to \mathbf{R}$ is a real valued function. If f is Riemann integrable on $[a, b]$, then $f^2 = f \times f$ is also Riemann integrable on $[a, b]$.

Proof. Let $K = \sup\{|f(x)| : x \in [a, b]\}$. If $K = 0$, we have nothing to prove since f would be the zero constant function. Assume $K > 0$. As in the proof of the last theorem, there exists a partition $P:\ a = x_0 < x_1 < \ldots < x_n = b$ of $[a, b]$ such that the difference of the upper and lower Riemann sum with respect to P for f,

$$U(P,f) - L(P,f) = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) < \varepsilon/(2K),$$

where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$, and $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$.
For $i = 1, \ldots, n$, let $M_i' = \sup\{f^2(x) : x \in [x_{i-1}, x_i]\}$, and $m_i' = \inf\{f^2(x) : x \in [x_{i-1}, x_i]\} = -\sup\{-f^2(x) : x \in [x_{i-1}, x_i]\}$. Then for $i = 1, \ldots, n$,

$$M_i' - m_i' = \sup\{f^2(x) : x \in [x_{i-1}, x_i]\} + \sup\{-f^2(x) : x \in [x_{i-1}, x_i]\}$$
$$= \sup\{f^2(x) - f^2(y) : x, y \in [x_{i-1}, x_i]\}$$
$$= \sup\{(f(x) - f(y))(f(x) + f(y)) : x, y \in [x_{i-1}, x_i]\}$$
$$\leq 2K \sup\{(f(x) - f(y)) : x, y \in [x_{i-1}, x_i]\}$$
$$= 2K(M_i - m_i).$$

Therefore, the difference of the upper Riemann sum and the lower Riemann sum with respect to P for f^2, is

$$U(P, f^2) - L(P, f^2) = \sum_{i=1}^{n}(M_i' - m_i')(x_i - x_{i-1}) \leq 2K\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1})$$
$$\leq 2K\,\varepsilon/(2K) = \varepsilon.$$

Therefore, by Theorem 1, f^2 is Riemann integrable on $[a, b]$.

An easy consequence of the above theorem is the following.

Corollary 6. Suppose f and g are Riemann integrable on $[a, b]$, then $fg = f \times g$ is also Riemann integrable on $[a, b]$,

Proof. Note that $fg = 1/2\,[(f + g)^2 - f^2 - g^2]$. The result then follows from Theorem 5.

Theorem 7. Suppose f is Riemann integrable on $[a, b]$. If there exists $K > 0$ such that for all x in $[a, b]$, $|f(x)| \geq K$, then $1/f$ is also Riemann integrable on $[a, b]$.

Proof. Fix an $\varepsilon > 0$. Then since f is Riemann integrable on $[a, b]$, by Theorem 1, there exists a partition $P: a = x_0 < x_1 < \ldots < x_n = b$ such that the difference of the upper and lower Riemann sum,

$$U(P, f) - L(P, f) < \varepsilon K^2, \qquad \text{-------------------------- (2)}$$

where $U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$, $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$,

$L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$, $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$.

Then $U(P, f) - L(P, f) = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1})$. -------------------- (3)

Let $M_i' = \sup\{1/f(x) : x \in [x_{i-1}, x_i]\}$, and $m_i' = \inf\{1/f(x) : x \in [x_{i-1}, x_i]\} = -\sup\{-1/f(x) : x \in [x_{i-1}, x_i]\}$. Then for $i = 1, \ldots, n$,

$$M_i' - m_i' = \sup\{1/f(x) : x \in [x_{i-1}, x_i]\} + \sup\{-1/f(x) : x \in [x_{i-1}, x_i]\}$$
$$= \sup\{1/f(x) - 1/f(y) : x, y \in [x_{i-1}, x_i]\}$$
$$= \sup\{|1/f(x) - 1/f(y)| : x, y \in [x_{i-1}, x_i]\}$$
$$= \sup\{|f(x) - f(y)|/|f(x)f(y)| : x, y \in [x_{i-1}, x_i]\}$$
$$\leq \sup\{|f(x) - f(y)|/K^2 : x, y \in [x_{i-1}, x_i]\},$$

because for any x in $[a, b]$, $1/|f(x)| \leq 1/K$,

$$= \sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\}/K^2$$
$$= (M_i - m_i)/K^2. \qquad \text{-------------------------- (4)}$$

Now $U(P, 1/f) = \sum_{i=1}^{n} M_i'(x_i - x_{i-1})$ and $L(P, 1/f) = \sum_{i=1}^{n} m_i'(x_i - x_{i-1})$, the difference of the upper and lower Riemann sum with respect to P for $1/f$,

$$U(P, 1/f) - L(P, 1/f) = \sum_{i=1}^{n}(M_i' - m_i')(x_i - x_{i-1}) \leq 1/K^2\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}), \text{ by (4)},$$
$$= (1/K^2)(U(P, f) - L(P, f)), \text{ by (3)},$$

$$< \ (1/K^2)\varepsilon K^{\ 2} = \varepsilon.$$

Hence, Riemann's condition for $1/f$ is satisfied on $[a, b]$ and so by Theorem 1, $1/f$ is Riemann integrable on $[a, b]$. This completes the proof.

The following results concerning supremum and infimum of bounded subsets are very useful and we have used it often here.

Let A and B be bounded subsets of $\mathbf{R}$.

1. If for every a in A, there exists a point b in B such that $a \le b$, then $\sup A \le \sup B$.
2. $\sup(A + B) = \sup A + \sup B$.
3. $\inf A = -\sup(-A)$.
4. If $k \ge 0$, then $\sup(kA) = k \sup A$.

Conclusion. Together with the additivity of Riemann integrable functions, Corollary 6 says that the set of all Riemann integrable functions on $[a, b]$ form a commutative ring.