Real Numbers?

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by

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Preface

When we study calculus, we often take the real numbers for granted. It is especially difficult for the novice to trace the development of the real numbers through the more mathematically natural but conceptually difficult path. The properties of the real numbers are well known but often assumed as a matter of fact. We shall recreate the real numbers through Dedekind's cuts, starting from the rational numbers and then prove all the properties of the real numbers that we often assumed. One last step in this approach is to show that there is only one real number system, a complete totally ordered field up to isomorphism. By this isomorphism, we understand and translate all the properties that the real numbers possess as Dedekind's cuts, to our usual concept of the real numbers as 'extension' of the rational numbers. There are other models for the real numbers, but the Dedekind's cuts offer the clearest but by no means easy, logical development.

This book aims to provide the novice first year undergraduates or a beginner thirsting for the meaning of the real number system, a step by step approach to finding out about the real numbers. The need for calculus students, who we can say study properties of function from the real numbers to the real numbers or from Cartesian products of the real numbers to Cartesian products of the real numbers, to know about the object that they are studying is apparent. More precisely, if we are studying the properties of the functions from the real numbers to the real numbers, we ought to know or at least be familiar with the real numbers themselves.

The creativity evident in the construction of the Dedekind's cuts and the deep result that there is one and only one complete totally ordered field up to isomorphism can only be appreciated with some effort and concentration. The book is written to allow for appreciation in steps rather than all at one go. If one wants to know just the real numbers as cuts, start with chapter two. Then if one wants to know about addition on the real numbers, one can proceed next to chapter three. If one wants to know about the overall picture, then start with chapter one, the least technical of all the chapters.

This book has come to be written for the following reason. The author often has to clarify the fine points and the understanding of the real numbers, in particular, their properties, with students, who often regard the real numbers as little more than abstract algebra with inequalities. It is the hope that this book will make some difference towards an appreciation of the definitions of limits, continuity and differentiability and theorems such as the intermediate value theorem, extreme value theorem and the mean value theorem. In writing this book, the author has considered among other things, how anyone would begin this journey to know the real numbers better, the possible pitfalls, the explanation and the technical details from the point of view of someone doing it for the first time. This book that has gone through many revisions is also a labour of love. Many people have contributed to it in one way or another. They include the students, for whom it is mainly intended, and my family without whose support and love I will not be able to find the reason for writing it. Last but not least, thanks are due to Ian Shelly who has prompted me to write and my spirit for wanting to give.

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Contents

CHAPTER ONE

Real Numbers! What are they?

Numbers have different origins, but mostly in the abstract. There are many ways of thinking and representing integers. We speak of 'real' numbers. What do we mean by a number that is real? A question, hardly understood and least to answer. Let us start by counting.

ONE, TWO, THREE, FOUR, FIVE, SIX, ...

The counting numbers, as the name suggests, has to do with counting. Starting with one, we get its successor, two, followed by its successor, three, and so on. If we were to ask: "What this number (which is represented by a symbol) is?", there is a number of answers we can give. We can start by recalling the sequence of successor followed by successor, starting from one, until we reach our desired number. We are accustomed to see numbers abbreviated by universally understood symbols. We would subsequently simplify or modify this to a system that takes into account certain specified numbers together with the operation of addition (and sometimes multiplication) that gives a representation of a number by a shorthand that will let the beholder know and reconstruct the number by addition of certain basic chosen numbers. For example, 1024 would mean one THOUSAND, zero HUNDRED, two TEN and four ONE. How useful or effective is this representation? It would depend on the chosen specified blocks of numbers and how you would represent these blocks.

A symbol for each block would be cumbersome. For one, we would run out of symbols. For second, these symbols may not be universally understood. Nevertheless, these symbols are unavoidable if we want to make ourselves understood. For instance, 1024 could equally mean one $\underline{6}^3$, zero $\underline{6}^2$, two $\underline{6}$ and four $\underline{1}$. There would also be the need for special symbols to represent certain numbers. These would be the first few numbers needed to exchange certain number of blocks with another block. This is useful and desirable for commerce and trade. For our standard base TEN representation, we would need the symbols for (zero), one, two, ..., nine. A symbol for zero is not essential for this purpose. Ten ONE is TEN or one block of TEN, ten blocks of TEN is HUNDRED and so on. For the base 2 representation we would need only zero and one. So, two ONE is $\underline{2}$, two blocks of $\underline{2}$ is $\underline{2}^2$ and so on. The representation ought to be unique otherwise we would be open to dispute. Because a symbol for each block is cumbersome, although there is a place for this, place value becomes a device to circumvent the need for these symbols or to reduce the number of symbols required. But then there is still the problem of which way do we assume the place value to begin, from the right to the left or from the left to the right? A symbol for zero here would be essential because we cannot disregard the place value even though there is zero block there. 1024 and 4201 could mean the same thing unless we all agree on which way the place value is written down. We shall adopt the base 10 representation from right to left starting from the smallest to the largest blocks.

The Integers

To set the rudiments of arithmetic, we introduce the notion of negative numbers. Arithmetic is of course desirable for trading purposes. Mathematicians view negative numbers as the solution to the linear equation,

$$x + n = 0,$$

for each counting number n. For this approach, of course, the notion of zero would have to be introduced. There is not a satisfactory way of defining or saying what zero means at least mathematically or what the equation means, unless we begin from the meaning of the counting numbers and its construction, the meaning of addition, multiplication and the ordering of the counting numbers with the meaning of zero as satisfying the following:

$$m + 0 = m \text{ for all counting numbers } m \text{ and } 0 + 0 = 0.$$

So, we must then assume the existence of such a number zero and rightly this is a very important number and we should indeed construct the counting numbers together with zero. We shall not venture into the technical detail and justification of the existence of the negative numbers. The construction of the negative numbers at this point would cloud our view and make us lose sight of the overall picture. For now, we view the existence of the negative numbers as guaranteed by construction. Thus, the integers consist of the counting numbers, their negatives and zero. Ordering comes naturally with the counting numbers together with zero. Successor comes after each number, starting from zero. It is not clear that the ordering is naturally extended to the negative numbers. If we view negative numbers as deficits, that are similar to money owed or quantity owed, then a natural ordering would be the same as that of counting numbers with zero at the start. One would have -1 comes after 0, -2 comes after -1, -3 comes after -2 and so on. Then 0 would be in a unique position as the starting number for both the counting numbers and their negatives. This would be fine until we view these numbers as distances. A deficit distance would assume an ordering as -1 comes before 0 and 1 comes after 0, ..., basically dictated by direction. So, for the ordering on the negative to be consistent with direction, we would have -1 comes after -2 and -2 comes after -3 and so on. The number zero does play a unique role in this ordering and an even greater part in fact is played by the counting numbers, if we were to define meaning of ordering (in an algebraic context).

The Rational Numbers

A ratio of 1: n gives a fraction $\frac{1}{n}$ for each counting number n. Then addition of m of this $\frac{1}{n}$ means $\frac{m}{n}$. This representation is not unique and we have to allow for cancellation, as for instance, $\frac{1}{2}$ means the same as $\frac{2}{4}$. There is the question of deciding when two representations are the same without having to know the common factor of m and n. We say $\frac{a}{b}$ and $\frac{c}{d}$ are the same if, and only if, $a\,d = b\,c$. We write $\frac{a}{b} = \frac{c}{d}$ when this happens. We can define negative fractions as similar to how we define integers as the solution to $x + r = 0$ for each fraction r. The collection of the fractions, their negatives and zero constitute the rational numbers $\mathbb{Q}$. There is another way of constructing the rational numbers from the integers by set theoretic consideration, a more mathematical way. We are keeping to our theme of extending a system to include the solution of all linear equations.

Ordering does not come easily this time with the rational numbers. The set of "positive" rational numbers has the following properties. It should, of course, contain the counting numbers. For any two "positive" rational numbers p and q,

A. $p + q$ is again a "positive" rational number *and*

B. $p\,q$ is again a "positive" rational number.

Notice that the counting numbers satisfy these two properties.

This meaning of "positive" is artificial and unnatural. We know it ought to apply to the counting numbers and it does as we shall see later. At least it is consistent with our perception of what positive would mean. Property **B** involves multiplication. It is easily seen that multiplication of a counting number by -1 gives a negative integer and so multiplication of a fraction by -1 gives us a negative rational number.

We now additionally insist that this set of "positive" rational numbers, together with its negative, that is the set consisting of the result of multiplying each "positive" rational numbers by -1, and zero form the entire set of the rational numbers. Can we do this? That is, do we have a candidate for this set of "positive" rational numbers? Yes. Our construction of the rational numbers involves the following ingredients: the fractions, their negatives and zero. Plainly, the negative fractions are the result of multiplication of the fractions by -1. Obviously, the fractions satisfy Properties **A** and **B**. Is our insistence misplaced? Is there a set of different "positive" rational numbers than the fractions? The answer is no. Our insistence does pay off. This definition of "positive" would capture the essence of the meaning of positive. Indeed, it gives it new meaning. Note that 1 is "positive", a notion we would accept readily. But with the new meaning, it would require some thought. We would use a contradiction argument to show this. If 1 is not "positive", then it's negative -1 would be "positive" and by Property **B**, $(-1)(-1) = 1$ would be "positive", contradicting our assumption that 1 is not "positive". Then by Property **A**, all the counting numbers, being defined successively by adding 1, are "positive". Also note that for any counting number n, $\frac{1}{n}$ is "positive". This can be verified as follows. If $\frac{1}{n}$ is not "positive", then $\frac{1}{n}$ is the multiplication of a "positive" number by -1 because it is not 0. Thus, $-\frac{1}{n}$ is "positive" and so, since n is "positive", $-1 = -\frac{1}{n}\cdot n$ would be "positive" by Property **B**, contradicting that -1 is not "positive". Therefore, we conclude that $\frac{1}{n}$ is "positive". It then follows from Property **B** that any fraction $\frac{m}{n}$ is "positive" for any counting numbers m and n, since $\frac{m}{n} = m\cdot\frac{1}{n}$. Thus, our fractions are "positive". Then the "positive" rational numbers are precisely the fractions. This is because if there is a "positive" number p not a fraction, then since $p \neq 0$, $-p$ is a fraction and so $-p$ is "positive" (since any fraction is "positive") and so p is not "positive", contradicting that p is "positive". We call this subset of "positive" rational numbers, a *positive cone*. It is precisely the set of fractions.

The positive cone or the fractions would serve as a kind of reference for the ordering. It is a natural division of the rational numbers into two parts, a special part that decides the "direction" of an ordering and another. There is always a division of the rational numbers at any point into the 'left' and 'right'; what we needed is a reference point, zero, and a translation operation to give meaning to 'left' and 'right'.

For any two rational numbers a and b, we say a is greater than b ($a > b$) if, and only if, $a - b$ is "positive", i.e., $a - b$ belongs to this special part.

This ordering is consistent with the ordering on the counting numbers. This is seen as follows. For any counting number n, $n + 1 > n$ because $(n + 1) - n = 1$ is "positive". Since $n + 1$ is the successor of n, this ordering is consistent with the previous ordering determined by the sequence of successor followed by successor. In particular for any fraction r, $r > 0$ because $r - 0 = r$ is "positive". If we now define any rational number r to be positive when $r > 0$, then positive would mean the same as "positive".

In here is buried the notion of "reality", a reference to the continuous nature of time, past, present and future. We cannot in any way pinpoint what the next instant of time is, but we can say the future is what comes after the present instant. The next instant does not exist until it becomes the present instant. What lies beyond the present is the future. The difficulty in describing time is the same as the difficulty in describing reality.

What We Would Like the Real Numbers to Possess

If we can view the real numbers as the extension of the rational numbers, then we would want the properties that the rational numbers possessed that are so useful, to carry over to this extension. We shall describe in abstract terms these properties.

The rational number system is an example of an object, which mathematician calls a *field*. It is a set F that comes with two binary operations called addition (+) and multiplication (×), two unique elements called respectively 0 and 1, two unary operations, one on F denoted by $-: F \to F$ and the other on $F - \{0\}$, denoted by $*: F - \{0\} \to F - \{0\}$, satisfying the following 9 properties.

For all a in F,

1. $a + 0 = 0 + a = a$;
2. $a + (-a) = (-a) + a = 0$.

For all a, b and c in F,

3. $a + (b + c) = (a + b) + c$; (Associativity)
4. $a + b = b + a$; (Commutativity)
5. $a(b + c) = ab + ac$. (Distributivity)

For all a in $F - \{0\}$,

6. $1 \times a = a \times 1 = a$;
7. $a \times (*a) = (*a) \times a = 1$.

For all a, b and c in $F - \{0\}$,

8. $a \times (b \times c) = (a \times b) \times c$; (Associativity)
9. $a \times b = b \times a$. (Commutativity)

The unary operation * for the rational numbers $\mathbb{Q}$ corresponds to taking reciprocals on non-zero rational numbers. The other operations are suggestive of the symbols.

A *totally ordered field F* is a field F together with a positive cone P such that 0 does not belong to P, the union of P, its reflection, $-P = \{-a: a \text{ belongs to } P\}$, and $\{0\}$ is equal to F and P satisfies the following two properties, that for all a and b in P,

(A) $a + b$ belongs to P and

(B) $a \times b$ belongs to P.

The ordering, '>', called a total ordering on F, is defined by, $b > a$ if, and only if, $b - a$ belongs to P. Thus, for any x in F, $x > 0$ if, and only if x belongs to P.

The rational numbers $\mathbb{Q}$ satisfy all the 9 properties with the usual operations of addition, multiplication, taking negatives and reciprocals and has the total ordering described earlier with the (positive) fractions as the positive cone. To investigate the desirable property the real number system should possess, we have to reinvent the whole system of representing numbers. In the theory of mensuration, there are numbers or quantities that are not commensurable. Take for instance, $\sqrt{2}$, the square root of 2. Is this a number? Geometric intuition says it is. One thing is sure as we are taught - we can "approximate" $\sqrt{2}$ by fractions. There are rational numbers as close to $\sqrt{2}$ as we like 'before' and 'after' $\sqrt{2}$. We cannot pin down $\sqrt{2}$ as a rational number. It is not a symbol readily understood as -2 or $\frac{1}{2}$. But what we can say is this: If there is such a number, then its square would give us the integer 2. We can be bold. We can extend, in any sense as we would, our rational numbers to some system containing the solution of the equation, $x^2 = 2$. But then we would just open up a Pandora's box. What about $\sqrt{3}, \sqrt{5}, \sqrt{7}, \cdots, \sqrt{p}$, p a prime and so on? It is well-known none of these numbers is rational. What about cube root of 2? These are solutions to polynomial equation of the form,

$$x^n = p, \text{ where } p \text{ and } n \text{ are counting numbers.}$$

What about solution to all polynomial equations? It then becomes an impossible task to describe all these "numbers". In particular, there are "numbers", such as the Euler constant e and π, that are not the solution of a polynomial equation. So, extending our rational numbers in this way would not include these numbers. But what is plausible is that, no matter what their origins may be, there are rational numbers as close to these numbers as we like 'before' and 'after' these numbers. This forms the basic concept of the *cut* of Dedekind. We have to think of numbers differently as if there is a hierarchy of numbers. We may not know what $\sqrt{2}$ is but we know there are rational numbers as close as we like on the 'left' of it or less than it, if we can give an ordering on our set of real numbers. This gives enough information about $\sqrt{2}$ for all practical purposes. Indeed, it is a collection of rational numbers that can give us all the information we required about $\sqrt{2}$. In this way, we need a collection of rational numbers to describe a number. We shall come back to this later when we embark on making this more precise.

We say the totally ordered field F has the *Archimedean Property* if for all $x > 0$ in F, and for all $y > 0$ in F, there exists a counting number n such that

$$(n1)\times x > y.$$

We can rewrite the last inequality as $\frac{1}{n1}\times y < x$ or $\left(*(n1)\right)\times y < x$. What this says is that given any x and $y > 0$ in F, no matter how small x is, we can find a counting number n such that $\frac{1}{n}\times y < x$, if we identify $\frac{1}{n}$ with $\frac{1}{n1}$. Obviously, the rational numbers $\mathbb{Q}$ has the Archimedean Property. This is a property that we would wish the real numbers to have.

We shall need to add a new property that says that the set of real numbers does exist in a different sense. We know there are rational numbers arbitrarily bigger than $\sqrt{2}$. We can collect all the rational numbers below or less than $\sqrt{2}$. In a way, $\sqrt{2}$ would be the largest such number if it exists, bigger than all the rational numbers below $\sqrt{2}$. The existence of such a number would have guaranteed the meaning of $\sqrt{2}$. But of course, we would have to think of rational numbers in a different way. To describe more precisely what we mean, we make the following definition.

Consider a subset A of F. A is said to be *bounded above* if there exists x in F such that for all a in A, $a \le x$. Here, $a \le x$ means $a < x$ or $a = x$. We say A is *bounded below* if there exists y in F such that for all a in A, $y \le a$. The number x is called an *upper bound* for A and y a *lower bound* for A. We say A is *bounded* if it is both bounded above and bounded below.

If A is bounded above, then it has an upper bound. It is natural to ask if it has the smallest such upper bound. That means, if M is the smallest such upper bound, then, of course, M is in F and for any x in F with $x < M$, x cannot be an upper bound for A and so consequently, there exists a a_0 in A such that $x < a_0$.

Definition 1. M is the *least upper bound* or *supremum* (*sup*) of a subset A of F if for all a in A, $a \le M$ and for any $x < M$, there exists b in A such that $x < b$.

A more descriptive way of describing M is this: For any number x less than M, we can always find an element b in A such that $x < b \le M$.

Similarly, we can define the greatest lower bound or infimum of A.

Definition 2. m is the *greatest lower bound* or *infimum* (*inf*) of a subset A of F if for all a in A, $m \le a$ and for any $x > m$, there exists b in A such that $b < x$.

We can thus characterize m by saying that for any $x > m$, *we can always find an element b in A such that* $x > b \ge m$.

The notion of supremum or infimum would be in vain if they do not exist. We would like them to be included in our consideration. A totally ordered field in which every bounded subset has an infimum and a supremum is special in that the "boundaries" of the bounded subsets are elements in the field. This prompts the next important definition.

Definition 3. A totally ordered field F is *complete* if every non-empty bounded above subset of F has a supremum.

The significance of this definition is that the supremum is a member of F. That means any bounded above subset has its 'upper' boundary residing in F and there is no room for a gap to exist in F.

The term 'complete' has several meanings. The present meaning is sometimes referred to as *order complete*.

This property is new. It is desirable for $\sqrt{2}$ to have a meaning. The rational numbers $\mathbb{Q}$ is not complete. Take for example the subset $A = \{x \text{ in } \mathbb{Q} : x > 0 \text{ and } x^2 < 2\}$ of $\mathbb{Q}$. It does not have a supremum (in $\mathbb{Q}$). Note that we cannot as yet write $\sqrt{2}$ as its existence has not been established. We can only talk about it hypothetically. Even if we assume that its existence has been established, we still have to know how the usual ordering on the rational numbers will place $\sqrt{2}$ by some extension of the ordering. That $\sqrt{2}$ is not a rational number can easily be shown as follows. Suppose $\sqrt{2}$ is rational, say $\sqrt{2} = \dfrac{p}{q}$, where p and q are counting numbers with no common factors other than 1. Then $p = \sqrt{2}q$ and so $p^2 = 2\,q^2$. Thus, p^2 is even. Since product of odd numbers yields odd number, p is even and so $p = 2\,k$. Thus $2\,q^2 = p^2 = 4\,k^2$ and we get $q^2 = 2\,k^2$. A similar argument yields q is even. Hence, 2 is a common factor of p and q, contradicting that p and q have no common factors other than 1. Thus, $\sqrt{2}$ will belong to a different scheme of things. A is plainly bounded above for we see that for any a in A, $a < 2$. We shall now show that A has no supremum.

Suppose A has a supremum M. Then $M \geq a$ for all a in A and that if $k < M$, then there exists b in A such that $k < b$. By definition of A, $a^2 < 2$ for all a in A. Note that $a^2 \leq M^2$ for all a in A. We can compare M^2 and 2 to produce a contradiction. Suppose $M^2 > 2$. Then $\dfrac{M^2-2}{M+2} > 0$ as plainly, $M+2 > 0$. Let $k = M - \dfrac{M^2-2}{M+2} = \dfrac{2M+2}{M+2}$. Then $k < M$. Note that $k^2 - 2 = \dfrac{4(M+1)^2 - 2(M+2)^2}{(M+2)^2} = \dfrac{2(M^2-2)}{(M+2)^2} > 0$ since $M^2 > 2$. Thus, $k^2 > 2$. But since $k < M$, there exists b in A such that $k < b$. Thus $k^2 < b^2 < 2$. This contradicts $k^2 > 2$. Therefore, $M^2 \leq 2$. Since $M^2 \neq 2$ because M is a rational number, $M^2 < 2$. We shall now derive another contradiction. So, we now assume $2 - M^2 > 0$. Let now $k = M + \dfrac{2-M^2}{M+2} = \dfrac{2M+2}{M+2}$. Then $k > M$. Observe that $k^2 - 2 = \dfrac{2(M^2-2)}{(M+2)^2} < 0$ since $M^2 < 2$. Therefore, $k^2 < 2$. Hence k belongs to A and so $k \leq M$. This contradicts $k > M$. Consequently, these two contradictions imply that $M = \sup A$ does not exist.

The following is a variation or equivalent definition for completeness. First, note that for any non-empty bounded below subset A of F, $-A = \{-a : a \text{ belongs to } A\}$ is bounded above. In particular, $\inf A = -\sup(-A)$. It follows that if the supremum exists for any non-empty bounded above subset of F, then the infimum too exists for any non-empty bounded below subset of F.

Definition 3'. A totally ordered field F is *complete* if every non-empty bounded below subset of F has an infimum.

It is clear from the above that Definition 3 implies Definition 3'. It can be similarly observed that for any bounded above subset A of F, $-A = \{-a: a$ belongs to $A\}$ is bounded below and $\sup A = -\inf(-A)$. This will supply the argument for proving that Definition 3' implies Definition 3.

Assuming that we have constructed the real numbers, then the following tells us just what it is.

Theorem 4. The real numbers $\mathbb{R}$ is a complete totally ordered field.

There is essentially one such complete ordered field. This is not to say that there is exactly one such complete totally ordered field but that any two are *isomorphic*. We interpret this to mean that for all intent and purposes they are the same although they may be constructs of a different nature.

Proposition 5. The real numbers $\mathbb{R}$ has the Archimedean Property. That is to say,

for any $x, y > 0$ in $\mathbb{R}$, there is a counting number n such that $n\,x > y$. --------- (*)

(Here we are using the notation inherited from the rational numbers.)

When a totally ordered field has the Archimedean Property, we say it is *Archimedean*. Thus, $\mathbb{R}$ is Archimedean.

Proof of Proposition 5. We shall prove Proposition 5 by contradiction. Suppose $\mathbb{R}$ is not Archimedean. Then by negating the statement (*), we get,

there exists $x, y > 0$ such that for all counting number n, $n\,x \le y$. ------ (**)

Take the set $K = \{n\,x: n$ a counting number$\}$. Then by (**), K is bounded above by y and is non-empty. Because $\mathbb{R}$ is complete, the supremum M of K exists. Hence, for any counting number n, $n\,x \le M$ since $n\,x$ belongs to K. Now, $(n + 1)x$ belongs to K too. Therefore, $(n + 1)x \le M$. It follows that for any counting number n,

$n\,x \le M - x < M$. ------------------- (***)

Thus, $M - x$ is an upper bound for K. Because $M - x < M$ and that M is the supremum of K, there is an element $n_0\,x$ in K, for some counting number n_0, such that $M - x < n_0\,x$. But by (***), $n_0\,x \le M - x$. This contradicts that $M - x < n_0\,x$. Therefore, $\mathbb{R}$ is Archimedean.

Real numbers are hard to conceptualize, particularly so, when we are so accustomed to the arithmetic of the rational numbers to expect that they are a natural extension of the rational numbers. When we say, take a small number $\varepsilon > 0$, we like to think of ε as a rational number, since we are more comfortable with the rational numbers. For all practical purposes, this is what we need to think of ε. We may indeed just say, take a small rational number $\varepsilon > 0$ instead. The following justifies this.

Corollary 6. For any $\varepsilon > 0$, there is a counting number n such that $\frac{1}{n} < \varepsilon$.

Proof. By the Archimedean Property of $\mathbb{R}$, there exists a counting number n such that $n\varepsilon > 1$. Therefore, $\frac{1}{n} < \varepsilon$.

Note that $\frac{1}{n}$ is a rational number. So, the above corollary says that for any $\varepsilon > 0$, no matter how small ε is, we can find a rational number $\frac{1}{n}$ such that $0 < \frac{1}{n} < \varepsilon$. So, for all practical purposes, in place of ε, we can use $\frac{1}{n}$ instead.

Now that we have presented a class of numbers, which consists of numbers that are not rational numbers and accepted that there is an ordering that applies to the whole of $\mathbb{R}$, we might ask ourselves the question, "How often can we find a rational number or for that matter, irrational number?". "Very often" is the answer. In mathematical terms, we mean the rational numbers or the irrational numbers are *dense*. The following Corollary gives meaning to the term 'density of the rational numbers'.

Corollary 7. For any x and y in $\mathbb{R}$ and $x < y$, there exists an integer n and a counting number m such that $x < \frac{n}{m} < y$.

A descriptive way of stating Corollary 7 will be "between any two real numbers, there is a rational number".

Proof of Corollary 7. The proof goes like this. Take two real numbers x and y such that $x < y$. Then $y - x > 0$. It follows by Corollary 6 that there is a counting number m such that $\frac{1}{m} < y - x$. The rest of the proof will be divided into 3 cases. The easiest case will be when $x = 0$. Then we have $0 < \frac{1}{m} < y$ and so the required rational number is $\frac{1}{m}$ for this case. The second case is when $x > 0$. We now invoke the Archimedean Property of $\mathbb{R}$. By this property, there is a counting number n, such that $n\frac{1}{m} > x$. Having established the existence of such an integer n, we can then by successively taking one away from this number n to obtain the least integer n such that $n\frac{1}{m} > x$. That means $(n-1)\frac{1}{m} \le x$. Therefore,

$$y = (y-x) + x > \frac{1}{m} + (n-1)\frac{1}{m} = \frac{n}{m} > x .$$

For this case, the required rational number is $\frac{n}{m}$. The remaining case is when $x < 0$. That is, $-x > 0$. Then by the Archimedean property, there is a counting number n such that $n\frac{1}{m} > -x$. As before, we choose the least integer n such that $n\frac{1}{m} \ge -x$. Then we have $(n-1)\frac{1}{m} < -x$. Therefore,

$$y = (y-x) + x > \frac{1}{m} - \frac{n}{m} = (1-n)\frac{1}{m} > x .$$

For this case, the required rational number is $\frac{1-n}{m}$. This completes the proof.

Having established the density of the rational numbers, we expect that the irrational numbers are also dense in $\mathbb{R}$. More is true here. The irrational numbers are more numerous than the rational numbers. This statement will make sense only when we have some means of "measuring" subsets of the real numbers. Indeed, the "measure" of the set of rational numbers is zero but not so for the set of irrational numbers. We would need a theory of measure to establish this. The fact that $\mathbb{R}$ is uncountable and the rational numbers $\mathbb{Q}$ is countable gives us some idea of the difference in "size" of the set of irrational numbers and the set of rational numbers. A set is said to be *countable* if we can match its elements one to one with elements of the counting numbers. It can be shown that $\mathbb{Q}$ is countable though not finite. But it is much harder to show that $\mathbb{R}$ is not countable. One can do this by showing that the real numbers between 0 and 1 is not countable. This can be shown by way of contradiction. First, by assuming that we have a matching function from the counting numbers to the real numbers between 0 and 1 and thus we can write them as a sequence. Then assuming that each term of this sequence can be written as an infinite decimal and with this sequence of infinite decimals to produce a number different from any term of this sequence and thus showing that we can never have a matching function. This approach will need some criterion to distinguish infinite decimals converging to different limits. For now, we are content with the following.

Corollary 8. For any two rational numbers a and b with $a < b$, there is an irrational number α such that $a < \alpha < b$.

Proof. The proof of this Corollary is by actually producing the required irrational number α by making use of a known irrational number. Take an irrational number $k > 0$. For instance, we can take $k = \sqrt{2}$. By the Archimedean Property of $\mathbb{R}$, there is a counting number n such that $n\,(b - a) > k$. That is,

$$a + \frac{k}{n} < b\,.$$

Then $a < a + \frac{k}{n} < b$. Since k is irrational, $a + \frac{k}{n}$ is also irrational because a is rational and $\frac{k}{n}$ is irrational. Take $\alpha = a + \frac{k}{n}$ to be our required irrational number. This establishes the truth of this corollary.

Now, here is a curious observation. Each integer is separated from the nearest integer by one. We can expect to find an integer between x and $x + 1$ for any real number x. This is stated more precisely as follows.

Corollary 9. For any real number x, there is an integer n_0 such that $x < n_0 \leq x + 1$.

Proof. If x is an integer, then we only have to take n_0 to be $x + 1$. Now, assume x is not an integer. Suppose $x > 0$. Then by the Archimedean Property of **R**, there is a counting number n such that $n = n\times 1 > x$. Take the least such integer N with $N > x$. Then $N - 1 \leq x$. Therefore, $x < N \leq x + 1$. So, take $n_0 = N$. Observe that we actually have $x < N < x + 1$, since $x + 1$ is not an integer. Now, for the case x is not an integer and $x < 0$. Then, as shown before, there exists an integer M such that $-x < M < -x + 1$. Thus, $x - 1 < -M < x$ and so $x < -M + 1 < x + 1$. For this case, take $n_0 = 1 - M$. This completes the proof.

In the next few chapters, we shall construct the real numbers and show that it is a complete totally ordered field.

CHAPTER TWO

A Cut, An Instant

Because I know that time is always time
And place is always and only place
And what is actual is actual only for one time
And only for one place
.... T.S. Eliot

Before we begin the construction of the real numbers, we take another look at the object we shall use, namely the rational numbers in a different way. As we shall see, in an analogy with time, when we stand at the threshold of time dividing the past and the future, there is an urge to move and yet not moved for time's ceaseless motion like an invisible hand divides the past and future at any instant now and forever. Pondering upon the *total ordering* on the rational numbers, we have a situation that mirrors the ever-present time. At any point P (either to be made precisely or a rational number), there is according to the total ordering a division of the rational numbers into two classes of numbers, the class whose numbers are all greater than P mimicking the future of P and the class whose numbers are all less than P mimicking the past of P, *a cut in time*, in analogy as an instant in time. The nature of the point P or the *cut* is the subject to be discussed. This cut in time is infallibly associated with an instant, the present instant. The untethered instant has order amidst the chaos. We cannot tell when it is and yet we know. The set of rational numbers then has a new meaning and in this new meaning is born the seed of the real numbers.

A Cut in Time

We cannot quite yet describe the present but we can for practical reason describe what comes close to being the present. A measure of time, the second, the humanly imaginable unit of time is our yardstick. As we speak, the present becomes in a fraction of the second the past and the future seems in a fraction of the second like the present. Time never stands still. We can in our humanly possible way describe what comes close to an instant as a cut in time. With this philosophical musing, we define what we call a *cut* of the rational numbers.

Definition 1. A *cut* of $\mathbb{Q}$ is an ordered pair of subsets of the rational numbers (L, R) satisfying the following three properties:

(1) $L \neq \emptyset, R \neq \emptyset$.
(2) $L \cap R = \emptyset, L \cup R = \mathbb{Q}$.
(3) If x is in L and y is in R, then $x < y$.

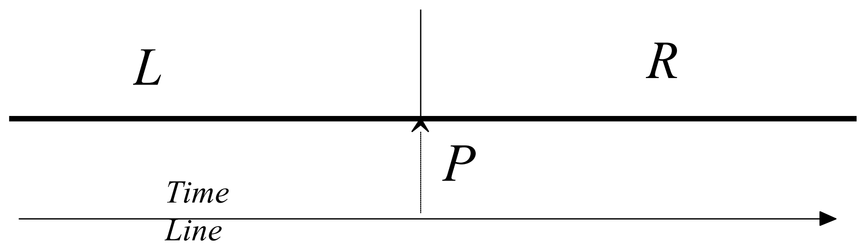
The subset L is called the left set of the cut and R the right set of the cut. By Property (2), every cut is determined by its left and right sets, each of which determines the other. Therefore, we shall identify a cut with its left set.

Thus, each 'point' or each instant is given by a cut. Each 'point', as time, has its past and future. If we are given a reference point, then we can begin to think of each rational number as a cut. This then gives new meaning to the rational numbers.

Definition 2. A subset ξ of $\mathbb{Q}$ is called a *real number* if it satisfies the following two properties.

(A) ξ is the left set of some cut; and

(B) ξ has no greatest element with respect to the total ordering on $\mathbb{Q}$.



The present instant, the fleeting moment, once you think you have it in your grasp, it has already escaped you; yet we know the present instant. A point, an instant, has taken on a new meaning as a cut in time. A 'real' number is not a number yet but part of a continuum of time in analogy. The abstract notions of addition and multiplication have yet to be invented. Property **B** models the fleeting nature of an instant.

Technically not all cuts are real numbers. Take for example the following cuts.

(1) $(L, R) = \{ x \in \mathbb{Q} : x^2 < 2\} \cup \{ x \in \mathbb{Q} : x^2 > 2\}$.

(2) $(L, R) = \{ x \in \mathbb{Q} : x < 2\} \cup \{ x \in \mathbb{Q} : x \geq 2\}$.

(3) $(L, R) = \{ x \in \mathbb{Q} : x \leq 2\} \cup \{ x \in \mathbb{Q} : x > 2\}$.

Cuts (1) and (2) are real numbers but cut (3) is not a real number. The cuts (2) and (3) correspond in some sense the integer 2, but we do not want both. We identify a real number with the left cut that has no greatest element. Like the essence of an instant, knowing but unattainable, the real number is represented by the left cut that has no greatest element. Out of these special partitions of rational numbers, identified as the left-hand sets of the cuts and hence out of the basic building blocks in set theory, we shall realise the real numbers. We shall have to redefine what seems so natural in the rational numbers. The identification of the cuts with their left-hand sets provides us with the ease to redefine operations of addition and multiplication and others in terms of the basic operations in set theory. The following is a way of deciding when a subset of the rational numbers is a real number under our identification.

Lemma 3. A subset ξ of $\mathbb{Q}$ is a real number if, and only if, the following four conditions are satisfied.

1. $\xi \neq \varnothing$.
2. $\xi \neq \mathbb{Q}$.
3. For any x in ξ, if y is a rational number such that $y < x$, then y is also in ξ.
4. ξ has no greatest number.

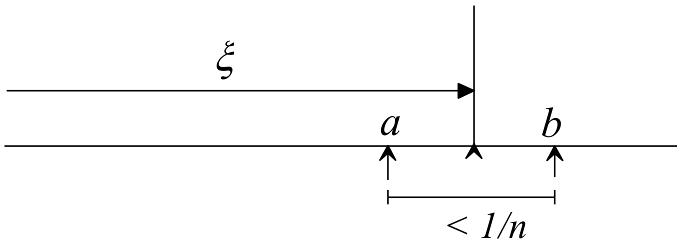
Proof. Let $L = \xi$ and $R = \mathbb{Q} - \xi$. Then conditions (1) and (2) imply that $L \neq \varnothing$ and $R \neq \varnothing$. For any x in L and any y in R, $x < y$. This is because if on the contrary that $x \geq y$, then by condition (3) above, y is in ξ. But since y is in $R = \mathbb{Q} - \xi$, y is not in ξ. This contradicts that y is in ξ. Therefore, by Definition 1, (L, R) is a cut. Condition (4) then says that L has no greatest number. Therefore, ξ is a real number. Conversely if ξ is a real number, then the conditions (1) to (4) are automatically satisfied.

The essence of a cut is captured in the following lemma. It says that there are points on the left and right of a cut that are as close to one another as one wishes.

Lemma 4. 1. If ξ is a real number and n is a counting number greater or equal to 1, then there exists (rational numbers) a in ξ and b not in ξ such that $b - a < \frac{1}{n}$.

2. For any rational number a in a real number ξ, there exists a counting number m such that $a + \frac{1}{m} \in \xi$.

Proof. Part 1. The real line will help us to visualise this lemma. Look at the following picture carefully.

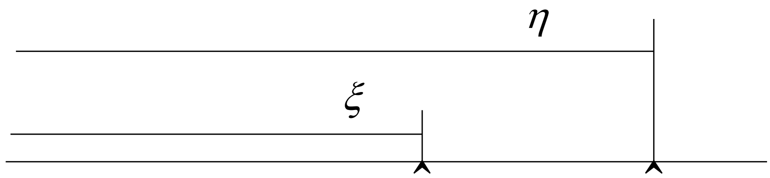


Since ξ is a real number, we can take an integer c not in ξ and an integer d in ξ. Then starting from d, we shall add lengths of $\frac{1}{n^2}$ to give rational numbers that are consecutively $\frac{1}{n^2}$ apart. Define $x_i = d + \frac{i}{n^2}$. Then $x_0 = d, x_1, x_2, x_3, \ldots, x_{n^2(c-d)} = c$ is a sequence of rational numbers such that $x_0 = d < x_1 < x_2 < x_3 < \ldots < x_{n^2(c-d)} = c$. Therefore, for some integer j such that $0 \le j < n^2(c - d)$, x_j is in ξ but x_{j+1} is not in ξ. Take $b = x_{j+1}$ and $a = x_j$. Then for $n > 1$, $b - a = \frac{1}{n^2} < \frac{1}{n}$. For $n = 1$, use $n + 1$ instead as above to obtain b and a such that $b - a < \frac{1}{n+1} < \frac{1}{n}$.

Proof of Part 2. Take any rational number a in the real number ξ. Suppose on the contrary that we cannot find such a number m with $a + \frac{1}{m} \in \xi$. Then for all counting number n, $a + \frac{1}{n} \notin \xi$. That is, for all counting number n, $x < a + \frac{1}{n}$ for all x in ξ. This implies that for each x in ξ, $x \le a$. This is shown by way of contradiction. Suppose on the contrary that $x > a$. Then by the Archimedean property of $\mathbb{Q}$, there exists a counting number p such that $p \times (x - a) > 1$. That is, $\frac{1}{p} < x - a$. Hence, $x > a + \frac{1}{p}$. This contradicts $x < a + \frac{1}{p}$. Therefore, for each x in ξ, $x \le a$. Since a is in ξ, a is a maximum element in ξ, contradicting that ξ does not have a maximum element because it is a real number. Thus, for each a in ξ, there exists a counting number m such that $a + \frac{1}{m} \in \xi$.

A cut has a natural ordering arising out of a subset inclusion. Like the marching of time extending into the future, this analogy aptly describes the ordering. We make the definition below.

Definition 5. For any two real numbers ξ and η, we say $\xi \le \eta$ if, and only if, $\xi \subseteq \eta$.



Let the set of real numbers be denoted by $\mathbb{R}$. Clearly this ordering '$\le$' is a *reflexive*, *transitive* and *antisymmetric* relation on the set $\mathbb{R}$. What does this mean? '$\le$' is *reflexive* means that for all ξ in $\mathbb{R}$, $\xi \le \xi$. It is obviously true since $\xi \subseteq \xi$ for any set ξ. Transitivity means if $\xi \le \eta$ and $\eta \le \kappa$, then $\xi \le \kappa$. Clearly if $\xi \le \eta$ and $\eta \le \kappa$, then $\xi \subseteq \eta$ and $\eta \subseteq \kappa$ so that $\xi \subseteq \kappa$, which means that $\xi \le \kappa$. '$\le$' is antisymmetric means that if $\xi \le \eta$ and $\eta \le \xi$, then $\xi = \eta$. This is plainly true, since $\xi \subseteq \eta$ and $\eta \subseteq \xi$ imply that $\xi = \eta$.

Lemma 6. This ordering '$\le$' on $\mathbb{R}$ is a total ordering.

Proof. We have shown above that '$\le$' is a partial ordering on $\mathbb{R}$. We need to show that any two real numbers ξ and η are comparable. That is, either $\xi \le \eta$ or $\eta \le \xi$. Since both ξ and η are subsets of $\mathbb{Q}$, either $\xi \subseteq \eta$ or $\xi \not\subseteq \eta$. If $\xi \subseteq \eta$, then $\xi \le \eta$ and we have nothing to prove. It remains to show that if $\xi \not\subseteq \eta$, then $\eta \le \xi$. Now, if $\xi \not\subseteq \eta$, then there exists a rational number x in ξ such that x is not in η ($x \notin \eta$). Then for any y in η, $y < x$ by Property 3 of Definition 1 because η is a cut. Therefore, by Property 3 of Lemma 3, for any y in η, y is in ξ since x is in ξ. Thus, $\eta \subseteq \xi$. Therefore, $\eta \le \xi$.

Then $\mathbb{R}$ is complete in the sense of order. We shall show this below. First, let us examine what we have not done so far. We have not yet defined the operations of addition and multiplication on $\mathbb{R}$. The only structure we have on $\mathbb{R}$ at the moment is a total ordering.

Lemma 7. Let S be a subset of $\mathbb{R}$ which is bounded above. Then S has a least upper bound or supremum in $\mathbb{R}$.

Set theoretically the supremum is easily found. But we do need to show that we can actually obtain a real number this way. Much of the proof of Lemma 7 goes in showing this. Remember each element in S is a subset of the rational numbers.

Proof of Lemma 7. Define $\eta = \bigcup\{\xi : \xi \in S\}$. Then $\xi \subseteq \eta$ for all ξ in S. Remember that each ξ in S is a cut and so is a subset of the rational numbers. Therefore, η is a subset of the rational numbers $\mathbb{Q}$. We need to show that η is a real number and that it is the supremum of S. We are given that S is bounded above. Therefore, there exists a real number κ such that $\xi \le \kappa$ for all ξ in S. That means $\xi \subseteq \kappa$ for all ξ in S. Therefore, $\eta = \bigcup\{\xi : \xi \in S\} \subseteq \kappa$. If we can show that η is a real number, then $\eta \le \kappa$. We shall now use Lemma 3 to show that η is a real number. Since $\xi \ne \varnothing$ for each ξ in S, we have then that $\eta \ne \varnothing$. Also $\eta \ne \mathbb{Q}$ because $\eta \subseteq \kappa$ and $\kappa \ne \mathbb{Q}$. Take now any x in η. Then $x \in \xi$ for some ξ in S. Thus, for any y in $\mathbb{Q}$ with $y < x$, y is in ξ by Property 3 of Lemma 3, since ξ is a real number. Hence, for any y in $\mathbb{Q}$ with $y < x$, $y \in \eta$. It now remains to show property 4 of Lemma 3 for η, that is, η has no maximum element. We shall show this by contradiction. Suppose on the contrary that η has a maximum element l. Then l is in some ξ in S. Also, for all $y \in \eta$, $y \le l$. This is also true of all y in this particular ξ, since $\xi \subseteq \eta$. Thus, l would be the greatest element in ξ. This contradicts that ξ has no greatest element since it is a real number. This completes the proof that η is a real number. Since for any ξ in S, $\xi \subseteq \bigcup\{\zeta : \zeta \in S\} = \eta$, $\xi \le \eta$ for any ξ in S. Therefore, η is an upper bound for

S. Now, let Ψ be any other upper bound of S. Then $\xi \le \Psi$ for all ξ in S. That means $\xi \subseteq \Psi$ for all ξ in S. Therefore, $\eta = \bigcup\{\xi : \xi \in S\} \subseteq \Psi$. Hence $\eta \le \Psi$. This shows that η is the least upper bound or the supremum of S.

We have thus proved the following.

Theorem 8. $\mathbb{R}$ has a complete total ordering.

The construction of the real numbers makes it stand apart from the rational numbers. Firstly, it is constructed out of subsets of the rational numbers. Secondly, the set of rational numbers is definitely not a subset of it, at least not in a natural sense. To make sense of the rational numbers in this new, as yet to be defined, system of real numbers, we shall embed our rational numbers into $\mathbb{R}$ in such a way that it is compatible with the ordering on the rational numbers.

The Embedding of the Rational Numbers in The Real Numbers

Define an embedding, $\varphi : \mathbb{Q} \to \mathbb{R}$, by $\varphi(a) = \{x \in \mathbb{Q} : x < a\}$ for any rational number a. This is the only natural way of embedding $\mathbb{Q}$ and gives the new meaning of $\mathbb{Q}$ that was mentioned earlier. This is well defined, for $\varphi(a)$ is a real number. Why? Obviously, $\varphi(a)$ is a cut that does not have a maximum element and so by definition, $\varphi(a)$ is a real number. The image of $\mathbb{Q}$ under φ is truly a copy of $\mathbb{Q}$. Until addition and multiplication are defined on the set of real numbers, we cannot show that the image behaves just like the rational number system. This will be done in the later chapter. We now show that φ is injective and is compatible with the ordering on the rational numbers. Mathematically, this is summarised as follows.

Lemma 9. φ: $\mathbb{Q} \to \mathbb{R}$ is injective. Furthermore, for any rational numbers a and b $a \le b \Leftrightarrow \varphi(a) \le \varphi(b)$.

Proof. We shall show that φ is *injective*. More precisely, we shall show that whenever $\varphi(a) = \varphi(b)$, then $a = b$. $\varphi(a) = \varphi(b)$ implies that $\{x \in \mathbb{Q} : x < a\} = \{x \in \mathbb{Q} : x < b\}$. If $a < b$, then $a \in \{x \in \mathbb{Q} : x < b\}$. Take $c = \dfrac{a+b}{2}$. We have then $a < c < b$, $c \in \{x \in \mathbb{Q} : x < b\}$ and $c \notin \{x \in \mathbb{Q} : x < a\}$. Therefore, $\{x \in \mathbb{Q} : x < b\} \ne \{x \in \mathbb{Q} : x < a\}$, contradicting $\{x \in \mathbb{Q} : x < b\} = \{x \in \mathbb{Q} : x < a\}$. Thus, a must be greater or equal to b. We can show similarly that a cannot be greater than b. Thus $a = b$. Hence φ is injective.

If $a \le b$, then $\varphi(a) = \{x \in \mathbb{Q} : x < a\} \subseteq \{x \in \mathbb{Q} : x < b\} = \varphi(b)$. Therefore, $\varphi(a) \le \varphi(b)$. Conversely, if $\varphi(a) \le \varphi(b)$, then $\{x \in \mathbb{Q} : x < a\} \subseteq \{x \in \mathbb{Q} : x < b\}$. It follows that $a \le b$, for otherwise, $a > b$ would imply that $\{x \in \mathbb{Q} : x < a\} \not\subseteq \{ x \in \mathbb{Q} : x < b\}$, since we would have $c = \dfrac{a+b}{2} \in \{x \in \mathbb{Q} : x < a\}$ and $c \notin \{x \in \mathbb{Q} : x < b\}$. This completes the proof of Lemma 9.

We now have a new model of the rational numbers embedded in the real numbers. We have yet to define multiplication and addition on $\mathbb{R}$ and yet to show that the embedding respects addition and multiplication. We shall come back to this in the later chapter. Having known that the embedding φ respects ordering, Lemma 4 will then have new interpretation as given below.

Corollary 10. If ξ is a real number, then for any counting number n, there exist rational numbers a and b such that

$$\varphi(a) < \xi \le \varphi(b) \text{ and } \varphi(b - a) < \varphi\left(\frac{1}{n}\right),$$

where the strict ordering '< ' is defined by $\xi < \eta$ if, and only if, $\xi \le \eta$ and $\xi \ne \eta$.

Proof. By Lemma 4, for any counting number n, there exist rational numbers a in ξ and $b \notin \xi$ such that $b - a < \dfrac{1}{n}$. Since $a \in \xi$, by Property 3 of Lemma 3, any rational number $x < a$ belongs to ξ. Therefore, $\{x \in \mathbb{Q} : x < a\} \subseteq \xi$. That means $\varphi(a) \subseteq \xi$, therefore $\varphi(a) \le \xi$. Because $a \notin \varphi(a)$ and $a \in \xi$, $\varphi(a) \ne \xi$. Thus $\varphi(a) < \xi$. Since $b \notin \xi$, for any x in ξ, $x < b$ because b is in the right set of the cut ξ. Hence $\xi \subseteq \{x \in \mathbb{Q} : x < b\} = \varphi(b)$. Therefore, $\xi \le \varphi(b)$. By Lemma 9, $\varphi(b - a) \le \varphi\left(\dfrac{1}{n}\right)$. As $b - a < \dfrac{1}{n}$, $b - a \in \varphi\left(\dfrac{1}{n}\right)$. Now that we also have $b - a \notin \varphi(b - a)$, $\varphi(b - a) \ne \varphi\left(\dfrac{1}{n}\right)$. Therefore, $\varphi(b-a) < \varphi\left(\dfrac{1}{n}\right)$. This completes the proof.

Corollary 10 says that for any real number ξ, there are rational numbers (the embedded kind) before and after ξ that are arbitrary close to one another. In the next chapter we shall define addition on $\mathbb{R}$.

CHAPTER THREE

Relearning Addition

Like a child who has forgotten how to walk
Like a bird who has forgotten how to fly
We shall learn, step by step,
Carefully, gingerly, how to add
Not by natural designs, but by constructs
That spring from our forgotten past.

Addition on the real numbers comes naturally but subtly as we shall observe in our definition.

Definition of Addition.

Now that a real number is a collection of rational numbers, given two real numbers, a natural consideration for the addition of these two real numbers is to form a collection of rational numbers by taking the sum of a rational number from one real number and a rational number from the other.

For any ξ and η in $\mathbb{R}$, define $\xi + \eta = \{x + y: x \in \xi, y \in \eta\}$.

This looks natural but we need to know if it gives us a real number. It would not make any sense if it is not a real number. Note that $\xi + \eta \neq \varnothing$. We shall show that $\xi + \eta \neq \mathbb{Q}$. Since ξ is a real number, there exists a rational number $c \notin \xi$ such that for all x in ξ, $x < c$. Likewise, there exists a rational number $d \notin \eta$ such that for all y in η, $y < d$. Then for any x in ξ and any y in η, $x + y < c + d$. Therefore, $c + d \notin \xi + \eta$. Hence, $\xi + \eta \neq \mathbb{Q}$. Now, take any x in ξ and any y in η. For any rational number $z < x + y$, $z - x < y$. Hence, $z - x \in \eta$, by Property 3 of Lemma 3, Chapter 2, since η is a real number. Then $z = x + (z - x) \in \xi + \eta$. It follows that $\xi + \eta$ satisfies Property 3 of Lemma 3, Chapter 2. It remains to show that $\xi + \eta$ has no maximum element. We show this by way of contradiction. Suppose $\xi + \eta$ has a maximum element k in $\xi + \eta$. That is, for all r in $\xi + \eta$, $r \leq k$ and there exists x in ξ and y in η with $k = x + y$. Therefore, $r \leq k = x + y$ and so $r - x \leq y$ for any r in $\xi + \eta$. Now, take any l in η, $x + l \in \xi + \eta$. Thus, by the above argument, for any l in η, $(x + l) - x = l \leq y$ (taking $r = x + l$). Hence, since y is in η, y is the maximum element of η, contradicting that η has no maximum element (because η is a real number).

This contradiction tells us that $\xi + \eta$ has no maximum element. We can thus conclude, by Lemma 3 of Chapter 2, that $\xi + \eta$ is a real number.

We have thus proved the following.

Lemma 1. For any ξ and η in $\mathbb{R}$, $\xi + \eta$ is a real number.

Hence, the usual properties of addition will be the rules to be verified. They are simply not obvious. We need to check these rules by different means than we are accustomed to.

Lemma 2. Let ξ, η and ζ be any real numbers in $\mathbb{R}$ and a, b any rational numbers in $\mathbb{Q}$. Then the following holds.

1. $\xi + \eta = \eta + \xi$. (Commutativity)
2. $(\xi + \eta) + \zeta = \xi + (\eta + \zeta)$. (Associativity)
3. $\zeta + \varphi(0) = \zeta$.
4. $\eta \le \xi \Rightarrow \eta + \zeta \le \xi + \zeta$.
5. $\varphi(a + b) = \varphi(a) + \varphi(b)$.

Before we embark on a proof of this lemma, let us examine what it says. Property 1 and Property 2 are the usual properties that would be expected from any definition of addition. Property 3 says that $\varphi(0)$ is the zero for this addition. Notice the difference from the old zero. It is a subset of the rational numbers. Property 4 says that addition respects the ordering on the real numbers $\mathbb{R}$. Finally, Property 5 says that the embedding φ of the rational numbers $\mathbb{Q}$ into $\mathbb{R}$ respects both addition on $\mathbb{Q}$ as well as on $\mathbb{R}$.

Proof of Properties 1 and 2. Obviously, $\xi + \eta = \{x + y : x \in \xi, y \in \eta\} = \{y + x : x \in \xi, y \in \eta\} = \eta + \xi$, as $x + y = y + x$ for any rational numbers x and y. Now,

$(\xi + \eta) + \zeta = \{k + z : k \in \xi + \eta, z \in \zeta\} = \{(x + y) + z : x \in \xi, y \in \eta, z \in \zeta\}$

$= \{x + (y + z) : x \in \xi, y \in \eta, z \in \zeta\}$,

since for any rational numbers x, y and z, $(x + y) + z = x + (y + z)$,

$= \{x + l : x \in \xi, l \in \eta + \zeta\}$

$= \xi + (\eta + \zeta)$.

Proof of Property 3. Recall that $\varphi(0) = \{x \in \mathbb{Q} : x < 0\}$. Then

$$\zeta + \varphi(0) = \{x + y : x \in \zeta, y \in \varphi(0)\}.$$

Now, for any y in $\varphi(0)$, $y < 0$, therefore, for any x in ζ and y in $\varphi(0)$, $x + y < x$. Since ζ is a real number and x is in ζ, by Property 3 of Lemma 3, Chapter 2, $x + y$ is in ζ. It follows that for any x in ζ and any y in $\varphi(0)$, $x + y$ is in ζ. Therefore, $\zeta + \varphi(0) \subseteq \zeta$. Take any rational number a in ζ. By Lemma 4, Part 2 of Chapter 2, there exists a counting number n such that $a + \frac{1}{n} \in \zeta$. Then

$$a = \left(a + \frac{1}{n}\right) + \left(-\frac{1}{n}\right) \in \zeta + \varphi(0),$$

since $-\frac{1}{n}$ is in $\varphi(0)$ because $-\frac{1}{n} < 0$. Thus, $\zeta \subseteq \zeta + \varphi(0)$. Hence, $\zeta \le \zeta + \varphi(0)$. Therefore, $\zeta = \zeta + \varphi(0)$.

Proof of Property 4. $\eta \le \xi$ if, and only if, $\eta \subseteq \xi$. This means any element x in η is also in ξ. Therefore,

$$\eta + \zeta = \{x + y : x \in \eta, y \in \zeta\} \subseteq \{x + y : x \in \xi, y \in \zeta\} = \xi + \zeta.$$

Hence, $\eta + \zeta \le \xi + \zeta$.

Proof of Property 5. Let us write down the three subsets that are involved here.

$\varphi(a) = \{x \in \mathbb{Q} : x < a\}$; $\varphi(b) = \{x \in \mathbb{Q} : x < b\}$ and $\varphi(a + b) = \{x \in \mathbb{Q} : x < a + b\}$. Then

$$\varphi(a) + \varphi(b) = \{x + y \in \mathbb{Q} : x \in \varphi(a), y \in \varphi(b)\}$$
$$= \{x + y \in \mathbb{Q} : x < a, y < b\} \subseteq \{z \in \mathbb{Q} : z < a + b\} = \varphi(a + b).$$

Therefore, $\varphi(a) + \varphi(b) \le \varphi(a + b)$. Next, we shall show that $\varphi(a + b) \le \varphi(a) + \varphi(b)$.

Take any z in $\varphi(a + b)$. Then $z < a + b$ and so $z - a < b$. Hence, $z - a \in \varphi(b)$. By Lemma 4, Part 2 of Chapter 2, there exists a counting number m such that $z - a + \frac{1}{m} \in \varphi(b)$.

Therefore, $z = \left(a - \frac{1}{m}\right) + \left(z - a + \frac{1}{m}\right) \in \varphi(a) + \varphi(b)$, since $a - \frac{1}{m} < a$. This is true for every z in $\varphi(a + b)$ and so $\varphi(a + b) \subseteq \varphi(a) + \varphi(b)$. Hence, $\varphi(a + b) = \varphi(a) + \varphi(b)$.

Lemma 2, Property 3 says that the zero for the real number is $\varphi(0)$. Notice that the rational number 0 is not in $\varphi(0)$. We also need to show that $\varphi(0)$ has the property that the rational number 0 enjoys, namely that each element ξ in $\mathbb{R}$ has an additive inverse $-\xi$ such that their sum gives $\varphi(0)$. We want to define a unary operation $\mathbb{R} \to \mathbb{R}$ that assigns to each ξ in $\mathbb{R}$, its additive inverse, $-\xi$.

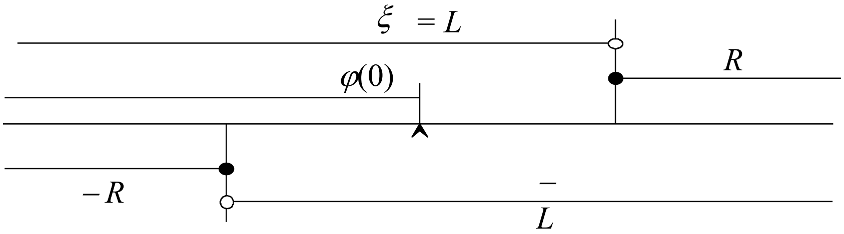
Definition 3. For any ξ in $\mathbb{R}$, define

$$-\xi = \{x \in \mathbb{Q} : \ x < -y \text{ for some } y \notin \xi\}.$$

There are two things we need to do. We need to show that $-\xi$ is a real number and that indeed $\xi + (-\xi) = \varphi(0)$.

Lemma 4. For any real number ξ, $-\xi$ is a real number.

Before we embark on the proof of this lemma, let us examine the reason and consideration behind the definition. The concept of additive inverse for a rational number is an extension of the definition of negative integers conceptually as discussed in Chapter 1. Both in concept and construction, the additive inverse for an integer is strongly associated with a direction and the notion of distance. Envisaged in another way, the additive inverse of an integer is the reflection of the integer about the point 0. Reflection is multiplication by -1. It has the effect of reversing the ordering of any two integers. Each real number carries with it a whole baggage of the rational numbers but without the structures. The additive inverse seems so unnatural to define. For distance, reflection is easily understood. What about a cut? A real number is represented by the left hand cut of a special cut. If we apply reflection to a CUT, the left-hand set becomes the right-hand set and the right-hand set becomes the left-hand set. Suppose the real number, ξ, is the left-hand set of the cut (L, R). Then the reflection of this cut gives a cut $(-R, -L)$, where $-R$ is the set $\{-r: r \in R\}$ and $-L = \{-x: x \in L\}$.



This may not give you a real number, for $-R$ may contain the greatest number. To obtain a real number, we must exclude the greatest number in $-R$ whenever it exists. There is a way to do this. Take any rational number y in R and consider its negative $-y$ in $-R$. Then take all rational numbers x such that $x < -y$. In this way, if $-y$ is the greatest element in $-R$, then the set $\{x \in \mathbb{Q}; x < -y\}$ will not have the greatest element. This explains the definition:

$$-\xi = \{x \in \mathbb{Q} : \ x < -y \text{ for some } y \notin \xi\}.$$

From the point of view of a cut, it is easy to see that $-\xi$ is the left-hand set of some cut, which has no maximal element and so it is a real number. All the same, we shall check if it is indeed a real number.

Proof of Lemma 4. First of all, $-\xi \neq \emptyset$. This is deduced as follows. Since $\mathbb{Q} - \xi \neq \emptyset$, we can choose a rational number $y \notin \xi$. Take any rational number $a > y$. Then $-a \in -\xi$, since $-a < -y$. Thus, $-\xi \neq \emptyset$. Let $K = \{-y\colon y \in \mathbb{Q}$ and $y \notin \xi\}$. Then $K \neq \mathbb{Q}$ because $\{y \in \mathbb{Q} : y \notin \xi\} \neq \mathbb{Q}$, since $\xi \neq \emptyset$. Then $-\xi \subseteq K$. This is because for any x in $-\xi$, there exists $y \notin \xi$, such that $x < -y$. Therefore, $-x > y$ and so $-x \notin \xi$ and that means x belongs to K. This is true for all x in $-\xi$ and so gives us the stated inclusion. Therefore, $-\xi \neq \mathbb{Q}$. Now, take any x in $-\xi$. Then for any rational number $b < x$, b is in $-\xi$. This is because there exists a rational number $y \notin \xi$ such that $x < -y$ and so $b < -y$ and that means b belongs to $-\xi$. Finally, we shall show that $-\xi$ has no maximal element. Suppose it has a maximal element M. That is, for all x in $-\xi$, $x \leq M$. Since M is in $-\xi$, there exists a rational number y_0 not in ξ such that $M < -y_0$. Then by taking $z = \dfrac{M - y_0}{2}$, we see that there is a rational number z such that $M < z < -y_0$. Obviously, $z \in -\xi$ and so $z \leq M$. This contradicts $M < z$ and so $-\xi$ has no maximal element. Therefore, we have shown that $-\xi$ satisfies Properties 1 to 4 of Lemma 3 of Chapter 2 and so $-\xi$ is a real number.

Next, we shall show that $-\xi$ is indeed the additive inverse of $-\xi$, i.e., it lives up to the meaning that the minus sign in front of it would suggest. We shall also show that the embedding φ of the rational numbers into the real numbers takes the additive inverse of a rational number, a in $\mathbb{Q}$, to the inverse of its image, $\varphi(a)$, in $\mathbb{R}$.

Lemma 5. 1. If ξ is a real number, then $\xi + (-\xi) = \varphi(0)$.

2. For any rational number a, $\varphi(-a) = -\varphi(a)$.

3. For any real numbers, ξ and η, $-(\xi + \eta) = (-\xi) + (-\eta)$

Proof.

Part 1. Firstly, we shall show that $\xi + (-\xi) \subseteq \varphi(0)$. Recall that

$$\xi + (-\xi) = \{x + y\colon x \in \xi, y \in -\xi\}.$$

For any y in $-\xi$, there exists a rational number y_0 not in ξ such that $y < -y_0$. Now, for any x in ξ, $x < y_0$, since y_0 is not in ξ. Therefore, $x + y < y_0 + (-y_0) = 0$. Hence, $x + y$ belongs to $\varphi(0)$. This is true for any $x + y$ in $\xi + (-\xi)$ and so $\xi + (-\xi) \subseteq \varphi(0)$ and $\xi + (-\xi) \leq \varphi(0)$.

Next, we shall show that any negative rational number is in $\xi + (-\xi)$. We begin with the following observation. By Lemma 4 of Chapter 2, for any counting number n, there exist rational numbers $b \notin \xi$ and $c \in \xi$ such that $0 < b - c < \frac{1}{2n}$. Therefore, $(2b - c) - c < \frac{1}{n}$ and so $-\frac{1}{n} < c + (c - 2b)$. Now, $c - 2b$ is in $-\xi$ because $c - 2b < -b$. Hence, $c + (c - 2b) \in \xi + (-\xi)$. Therefore, for any counting number n, $-\frac{1}{n}$ belongs to $\xi + (-\xi)$. For any rational number $q < 0$, $-q > 0$. Therefore, by the Archimedean Property of the rational numbers, there exists a counting number n_0, such that $n_0(-q) > 1$. Therefore, $-q > \frac{1}{n_0}$ and that means $q < -\frac{1}{n_0}$. Since we have just proven that $-\frac{1}{n_0}$ belongs to $\xi + (-\xi)$ and because $\xi + (-\xi)$ is a real number, q belongs to $\xi + (-\xi)$. Since this is true for any rational number $q < 0$, $\varphi(0) \subseteq \xi + (-\xi)$ and so $\varphi(0) \leq \xi + (-\xi)$. Hence, $\xi + (-\xi) = \varphi(0)$.

Part 2. $\varphi(0) = \varphi(a + (-a)) = \varphi(a) + \varphi(-a)$, --------------------(*)

by Lemma 2 (5).

Thus, adding $-\varphi(a)$ to both sides of the equation (*) gives

$$-\varphi(a) + \varphi(0) = -\varphi(a) + (\varphi(a) + \varphi(-a)).$$

Therefore, by Lemma 2, part 3 and the associativity of addition (Lemma 2, part 2), $-\varphi(a) = (-\varphi(a) + \varphi(a)) + \varphi(-a)$. Hence, $-\varphi(a) = \varphi(0) + \varphi(-a) = \varphi(-a)$, by part 1 and Lemma 2, part 5. This completes the proof.

Part 3.

$((-\xi) + (-\eta)) + (\xi + \eta)$

$= (((-\xi) + (-\eta)) + \xi) + \eta$, by the associativity of addition (Lemma 2, part 2),

$= (\xi + ((-\xi) + (-\eta))) + \eta$, by the commutativity of addition (Lemma 2, part 1),

$= ((\xi + (-\xi)) + (-\eta)) + \eta$, by the associativity of addition (Lemma 2, part 2),

$= (\varphi(0) + (-\eta)) + \eta$, by part 1,

$= (-\eta) + \eta$, by Lemma 2, part 1 and part 3,

$= \varphi(0)$, by part 1.

Therefore,

$-(\xi + \eta) = \varphi(0) + (-(\xi + \eta))$, by Lemma 2, part 1 and part 3,

$= (((-\xi) + (-\eta)) + (\xi + \eta)) + (-(\xi + \eta))$

$= ((-\xi) + (-\eta)) + ((\xi + \eta) + (-(\xi + \eta)))$, by Lemma 2 part 2,

$= ((-\xi) + (-\eta)) + \varphi(0)$

$= (-\xi) + (-\eta)$, by Lemma 2 part 3.

Lemma 2, parts (1), (2), (3) and Lemma 5 imply that the set of real numbers with the operation of addition is an abelian group with identity element given by $\varphi(0)$. Hence, we have proved the following.

Theorem 6. $\mathbb{R}$ with the operation of addition is an abelian group with $\varphi(0)$ as its identity element.

The embedding φ gives then a copy of the rational numbers in $\mathbb{R}$, additively. This is summed up in the following theorem.

Theorem 7. The embedding, $\varphi: \mathbb{Q} \to \mathbb{R}$ of Chapter 2, is a group monomorphism of $(\mathbb{Q}, +)$ into $(\mathbb{R}, +)$.

Proof. This theorem is a consequence of Lemma 2, Part 5, Lemma 5, Part 2 and Theorem 6.

We have now succeeded in copying the additive structure of the rational numbers onto a subgroup of $\mathbb{R}$, namely the image of the embedding φ. The next task is to see how the multiplicative structure can be defined on $\mathbb{R}$ and how our embedding will also copy the multiplicative structure of the rational numbers $\mathbb{Q}$ onto $\varphi(\mathbb{Q})$. The task seems complicated and we shall do this in stages in the next chapter.

CHAPTER FOUR

Rethinking Multiplication

Multiplication for the rational numbers is no longer an obvious generalisation of addition. Its rule has become embedded in the definition of the rational numbers. For the real numbers, it seems we have to learn the meaning of multiplication by using the ubiquitous multiplication on the rational numbers. What is even harder is to recover the old meaning of multiplication this way. The only thing we can say about the definition below is that it seems to be the only thing we can try. We shall define multiplication in stages, first on real numbers $\geq \varphi(0)$ and then extend it to the whole of the real numbers.

We shall begin by considering the subset of the rational numbers constructed out of two real numbers $\geq \varphi(0)$ described in the lemma below. This is going to be the subset that gives the multiplication of these two real numbers. Of course, we shall need it to be a real number. The following lemma confirms it to be the case.

Lemma 1. For any two real numbers $\xi \geq \varphi(0)$ and $\eta \geq \varphi(0)$, define the subset of the rational numbers, L, by

$$L = \varphi(0) \cup \{x \in \mathbb{Q} : x = ab \text{ for some rational number } a \geq 0 \text{ in } \xi \text{ and some rational number } b \geq 0 \text{ in } \eta\}.$$

Then L is a real number.

The subset L seems unnatural as a candidate for the multiplication of ξ and η. But the "obvious" subset, $\{x \in \mathbb{Q} : x = ab$ for some a in ξ and some b in $\eta\}$, cannot be a candidate for the multiplication, simply because it can be too large or bounded below by 0 and so cannot be a left cut. Take for example $\varphi(0)$ and $\varphi(1)$. Then we have that the subset κ defined by $\kappa = \{x \in \mathbb{Q} : x = ab$ for some a in $\varphi(0)$ and some b in $\varphi(1)\}$ is the same as the set of rational numbers $\mathbb{Q}$. This is seen as follows. Obviously, $0 \in \varphi(1)$ implies that $0 \in \kappa$. For any $x > 0$, $-x \in \varphi(0)$. Since $-1 \in \varphi(1)$, for any $x > 0$, $x = (-x)(-1)$ is in κ. Also, for any $x < 0$, $2x \in \varphi(0)$ and so since $1/2 \in \varphi(1)$, $x = (2x)(1/2)$ is in κ. Thus, $\kappa = \mathbb{Q}$ and it cannot be a cut and so not a real number. The additional condition in the definition of L is to ensure that we obtain a real number again.

Before we embark on the proof of Lemma 1, we state the following characterisation of when a real number ξ satisfies $\xi > \varphi(0)$. We write $\xi > \varphi(0)$ if, and only if, $\xi \geq \varphi(0)$ and $\xi \neq \varphi(0)$.

Lemma 2. For any real number ξ, $\xi > \varphi(0)$ if, and only if, $0 \in \xi$. In particular, if $\xi > \varphi(0)$, then there exists a rational number $a > 0$ in ξ.

Proof. $\xi > \varphi(0)$ implies that $\varphi(0)$ is a proper subset of ξ. This means there exists a rational number x in ξ such that $x \notin \varphi(0)$. This means that $x \geq 0$. Therefore, $0 \in \xi$, since ξ is a real number. Conversely, if $0 \in \xi$, then for any rational number $q < 0$, $q \in \xi$ again, since ξ is a real number. This means that $\varphi(0) \subseteq \xi$ and so $\varphi(0) \leq \xi$. Since $0 \notin \varphi(0)$, $\xi > \varphi(0)$. This completes the proof.

Proof of Lemma 1.

If any one of the real numbers ξ or η is equal to $\varphi(0)$, then L is equal to $\varphi(0)$ because the set $\{x \in \mathbb{Q} : x = ab$ for some rational number $a \geq 0$ in ξ and some rational number $b \geq 0$ in $\eta\}$ is empty. Therefore, L is a real number.

We now assume that $\xi, \eta > \varphi(0)$

Obviously, $L \neq \varnothing$, since $\varphi(0) \subseteq L$. Next, we shall show that $L \neq \mathbb{Q}$. For any $\xi, \eta \geq \varphi(0)$, either $\xi \geq \eta$ or $\eta \geq \xi$, since the ordering $\geq$ is a total ordering. Without loss of generality, we assume that $\eta \leq \xi$, i.e., $\eta \subseteq \xi$. Since ξ is a real number and $\xi > \varphi(0)$, there exists a rational number $x_0 > 0$ not in ξ such that for all x in ξ, $x < x_0$. Similarly, there exists a rational number $y_0 > 0$ not in η such that for all y in η, $y < y_0$. Then for any rational number $a \geq 0$ in ξ and any rational number $b \geq 0$ in η, we have that $a < x_0$ and $b < y_0$ and $a\,b < x_0\, y_0$. It follows that $x_0\, y_0 \notin \{x \in \mathbb{Q} : x = ab$ for some rational number $a \geq 0$ in ξ and some rational number $b \geq 0$ in $\eta\}$ and so since $x_0\, y_0 > 0$, $x_0\, y_0 \notin L$. Hence, $L \neq \mathbb{Q}$.

Now, take any rational number z in L. We shall next show that for any rational number $x < z$, $x \in L$. Note that z is in L implies either $z \in \varphi(0)$ or $z \in \{x \in \mathbb{Q} : x = ab$ for some rational number $a \geq 0$ in ξ and some rational number $b \geq 0$ in $\eta\}$. If $z \in \varphi(0)$ or $z = 0$ (that is, $z \leq 0$), then we have nothing to prove, since any $x < z$ implies that $x \in \varphi(0) \subseteq L$. If $z > 0$, then $z \in \{x \in \mathbb{Q} : x = ab$ for some rational number $a \geq 0$ in ξ and some rational number $b \geq 0$ in $\eta\}$ and there exist rational number $a_0 > 0$ in ξ and rational number $b_0 > 0$ in η such that $z = a_0\, b_0$. Now, take any $x < z$, if $x < 0$, then $x \in \varphi(0)$ and so $x \in L$. We are thus left with showing that for any rational number x such that $0 \leq x < z$, x is in L. For any of these rational numbers x, we have $0 \leq x < a_0\, b_0$. Therefore, $x/a_0 < b_0$ and so $x/a_0 \in \eta$, since η is a real number and b_0 is in η. Hence, $x = a_0\,(x/a_0)$, with a_0 in ξ and x/a_0 in η, belongs to L.

Next, we shall show that L has no maximal element. Suppose next that L has a maximal element M. We shall derive a contradiction. By Lemma 2, since $\xi, \eta > \varphi(0)$, there exist rational numbers $a > 0$ in ξ and $b > 0$ in η. Therefore, $a\,b > 0$ and is an element of L. Because M is the maximal number of L, $M \geq a\,b$ and so $M > 0$. Consequently, $M \notin \varphi(0)$. It follows that $M \in \{x \in \mathbb{Q} : x = ab$ for some rational number $a \geq 0$ in ξ and some rational number $b \geq 0$ in $\eta\}$. Hence, there exist rational number $a_0 > 0$ in ξ and rational number $b_0 > 0$ in η such that $M = a_0\,b_0$. Then consider $a_0 = M / b_0$. Because $a_0 \geq 0$, for any a in ξ such that $a \leq 0$, $a \leq a_0$. Now, for any a in ξ such that $a > 0$, $a\,b_0 \leq M$ so that $a \leq M / b_0 = a_0$. Thus, for any rational number a in ξ, $a \leq a_0$. That means a_0 is a maximal number for ξ, contradicting that ξ is a real number. Therefore, the maximal number M for L does not exist. Hence, it follows from Chapter 2, Lemma 3 that L is a real number.

Definition 3. For any real numbers ξ and $\eta \geq \varphi(0)$, we define the product $\xi\,\eta$ by

$$\xi\,\eta = \varphi(0) \cup \{x \in \mathbb{Q} : x = ab \text{ for some rational number } a \geq 0 \text{ in } \xi \text{ and some rational number } b \geq 0 \text{ in } \eta\}.$$

By Lemma 1, the product $\xi\,\eta$ is a real number. It remains to check if this product defines a multiplication. There are some immediate consequences of the definition. Firstly, $\xi\,\eta \geq \varphi(0)$ since $\varphi(0) \subseteq \xi\,\eta$. Secondly, if both ξ and $\eta > \varphi(0)$, then $\xi\,\eta > \varphi(0)$, since $0 \in \xi\,\eta$ because 0 belongs to both ξ and η.

Proposition 4. Let ξ and η be any real numbers $\geq \varphi(0)$.

1. $\xi\,\eta = \eta\,\xi$. (Commutativity)
2. If any one of ξ or η is equal to $\varphi(0)$, then $\xi\,\eta = \varphi(0)$.

Proof. Simply examine the definition of the product.

$\xi\,\eta = \varphi(0) \cup \{x \in \mathbb{Q} : x = ab$ for some rational number $a \geq 0$ in ξ and some rational number $b \geq 0$ in $\eta\}$

$= \varphi(0) \cup \{x \in \mathbb{Q} : x = ba$ for some rational number $b \geq 0$ in η and some rational number $a \geq 0$ in $\xi\}$

$= \eta\,\xi$.

Part 2 is a consequence of the fact that if any one of ξ or $\eta = \varphi(0)$, then $\{x \in \mathbb{Q} : x = ab$ for some rational number $a \geq 0$ in ξ and some rational number $b \geq 0$ in $\eta\} = \varnothing$.

Definition 3 is the first of a series of definitions to define the multiplication on the real numbers $\mathbb{R}$. Before we extend the definition to the whole of $\mathbb{R}$, the next thing we shall do is to define the multiplicative inverse of a real number $\xi > \varphi(0)$. Later, we shall extend this definition of the inverse to all of $\mathbb{R} - \{\varphi(0)\}$.

Consider the next construct obtained from any real number $\xi > \varphi(0)$. Define

$$\mathcal{M} = \{x \in \mathbb{Q} : x < 1/y \text{ for some } y \notin \xi\}.$$

Let us examine this set. First of all, the inequality in the definition makes sense because for $y \notin \xi$, $y > 0$, since by Lemma 2, $0 \in \xi$ and ξ is a (left) cut. Since for any $y \notin \xi$, $1/y > 0$, this set consists of all rational numbers less than or equal to 0 together with the reciprocals of all rational numbers not in ξ with the exception that if the complement of ξ has a least number (as in $\varphi(1)$), we do not get the reciprocal of this number. This set looks naturally like the multiplicative inverse of ξ. The usual multiplicative inverse of a non-zero rational number is its reciprocal. Here, ξ is a set, so we shall have to take for its multiplicative inverse a collection of the reciprocals cleverly and to include at least sufficient rational numbers to give us a real number. Then the next thing to do is to show that it really is the multiplicative inverse.

Proposition 5. If $\xi > \varphi(0)$, then the subset $\mathcal{M}$ of the rational number is a real number.

Proof. 1. $\mathcal{M}$ is obviously not empty since $0 \in \mathcal{M}$.

2. $\mathcal{M} \neq \mathbb{Q}$. This is seen as follows. Since $\xi > \varphi(0)$, by Lemma 2, there exists a rational number $x_0 > 0$ in ξ. Then for any rational $y \notin \xi$, because ξ is a real number, $y > x_0 > 0$. Thus, $1/x_0 > 1/y$ for any $y \notin \xi$. For any x in $\mathcal{M}$, there exists some $y \notin \xi$ such that $x < 1/y$. Therefore, for any x in $\mathcal{M}$, $x < 1/x_0$. Thus $1/x_0 \notin \mathcal{M}$ and so $\mathcal{M} \neq \mathbb{Q}$.

3. Take any rational number z in $\mathcal{M}$. Then there exists some $y \notin \xi$ such that $z < 1/y$. Hence, for any rational number $x < z$, $x < 1/y$ and so $x \in \mathcal{M}$.

4. $\mathcal{M}$ does not have a maximal number. Suppose on the contrary $\mathcal{M}$ has a maximal number M. Then $M \geq x$ for all x in $\mathcal{M}$. Note that $M \in \mathcal{M}$ implies that there exists some $y \notin \xi$ such that $M < 1/y$. Take the rational number $q = (M + 1/y)/2$ satisfying $M < q < 1/y$. Then $q \in \mathcal{M}$ and so $M \geq q$, contradicting $M < q$. Therefore, M does not exist.

We are now ready to make our definition.

Definition 6. For any real number $\xi > \varphi(0)$, define

$$\frac{1}{\xi} = \left\{ x \in \mathbb{Q} : x < \frac{1}{y} \text{ for some } y \notin \xi \right\}.$$

By Proposition 5, $\frac{1}{\xi}$ is a real number. An immediate consequence of the definition is that $0 \in \frac{1}{\xi}$ and so $\frac{1}{\xi} > \varphi(0)$.

As the notation suggests, $\frac{1}{\xi}$ is the multiplicative inverse of ξ. This is stated in the following lemma.

Lemma 7. If ξ is a real number and $\xi > \varphi(0)$, then $\xi \frac{1}{\xi} = \varphi(1)$.

Before we prove this lemma, we shall record a technical consequence of Lemma 4 of Chapter 2 that we shall be using.

Lemma 8. If ξ is a real number, then for any counting number n, there exist rational numbers a and b, $b \notin \xi$, $a \in \xi$ such that $b - a < \frac{1}{n}$ and there exists a rational number $c \notin \xi$ such that $c < b$.

This lemma simply says that we can always choose the rational number b *there* not equal to the minimum of the complement of ξ in $\mathbb{Q}$, if this minimum exists.

Proof of Lemma 8. By Lemma 4 of Chapter 2, for any counting number n, there exist rational numbers a and c, $a \in \xi$, $c \notin \xi$ such that $0 < c - a < \frac{1}{2n}$. Take $b = 2c - a$. This choice of b satisfies $b > c$ and so $b \notin \xi$ and $b - a = 2c - 2a < \frac{1}{n}$.

Proof of Lemma 7. Recall that the product

$\xi \frac{1}{\xi} = \varphi(0) \cup \{x \in \mathbb{Q} : x = ab$ for some rational number $a \geq 0$ in ξ and some rational number $b \geq 0$ in $\frac{1}{\xi}\}$.

Firstly, we shall show that $\xi \frac{1}{\xi} \leq \varphi(1)$. Since $\varphi(0) \subseteq \varphi(1)$ and $0 \in \varphi(1)$, we only need to show that for any rational number $a > 0$ in ξ and any rational number $b > 0$ in $\frac{1}{\xi}$, $a\,b \in$

$\varphi(1)$. By the definition of $\frac{1}{\xi}$, $b > 0$ in $\frac{1}{\xi}$ implies that there exists rational number $y \notin \xi$ such that $(0 <)\ b < 1/y$. Since $a \in \xi$ and ξ is a real number, $a < y$. Therefore, $a\,b < y\,(1/y) = 1$ and so $a\,b \in \varphi(1)$. Thus, $\xi\frac{1}{\xi} \subseteq \varphi(1)$ and that means $\xi\frac{1}{\xi} \le \varphi(1)$.

Next, we need to show that $\varphi(1) \le \xi\frac{1}{\xi}$. Now, for any rational number $x \le 0$, $x \in \xi\frac{1}{\xi}$, since $\varphi(0)\cup\{0\} \subseteq \xi\frac{1}{\xi}$ because of the fact that 0 belongs to both ξ and $\frac{1}{\xi}$ and that $\varphi(0) \subseteq \xi\frac{1}{\xi}$. Then it remains for us to show that for any rational x such that $0 < x < 1$, $x \in \xi\frac{1}{\xi}$. Choose a rational number $y_0 \notin \frac{1}{\xi}$. Then for all rational number $z \notin \xi$,

$$y_0 \ge 1/z\ (> 0). \quad \text{--} \quad (1)$$

Next, since $\mathbb{Q}$ is Archimedean, there exists a counting number n such that $n\,(1 - x) > y_0$, that is to say,

$$1 - \frac{1}{n}y_0 > x. \quad \text{--} \quad (2)$$

By Lemma 8, there exist rational numbers x_0 and z_0, where $x_0 \in \xi$ and $z_0 \notin \xi$, such that

$$0 < z_0 - x_0 < \frac{1}{n} \quad \text{---}(3)$$

and there is a rational number $z_1 \notin \xi$ such that $(0 <)\ z_1 < z_0$. Then $1/z_0 < 1/z_1$ and so $1/z_0 \in \frac{1}{\xi}$. Dividing (3) by z_0, we have $0 < 1 - \frac{x_0}{z_0} < \frac{1}{nz_0}$. Therefore,

$$\frac{x_0}{z_0} > 1 - \frac{1}{nz_0}. \quad \text{---}(4)$$

Note that $\frac{x_0}{z_0} = x_0\frac{1}{z_0} \in \xi\frac{1}{\xi}$. By (1), $y_0 \ge \frac{1}{z_0}$ since $z_0 \notin \xi$. Thus, $-\frac{1}{nz_0} \ge -\frac{y_0}{n}$. Therefore, by (4) and (2), we have $\frac{x_0}{z_0} > 1 - \frac{1}{n}y_0 > x$. Thus, $x \in \xi\frac{1}{\xi}$. This is true for all $x < 1$. Therefore, $\varphi(1) \subseteq \xi\frac{1}{\xi}$ and so $\varphi(1) \le \xi\frac{1}{\xi}$. We have thus proved that $\xi\frac{1}{\xi} = \varphi(1)$.

Next, we shall verify that our multiplication on the real numbers $\mathbb{R}$ has the desired properties, at least for the moment, for the real numbers $\geq \varphi(0)$.

Lemma 9. (Properties of Multiplication)

Let ξ, η and ζ be any real numbers $\geq \varphi(0)$. Let a and b be any rational numbers ≥ 0.

Then

1. $\xi\,\eta \geq \varphi(0)$; if ξ and $\eta > \varphi(0)$, then $\xi\,\eta > \varphi(0)$.
2. $\xi\,\eta = \eta\,\xi$. (Commutativity)
3. $(\xi\,\eta)\,\zeta = \xi\,(\eta\,\zeta)$. (Associativity)
4. $\xi\,(\eta + \zeta) = \xi\,\eta + \xi\,\zeta$. (Distributivity)
5. $\xi\,\varphi(1) = \xi$.
6. $\varphi(a\,b) = \varphi(a)\,\varphi(b)$.

Property 1 is an immediate consequence of the definition as explained just after the Definition 3. Property 2 is Proposition 4, part 1 and Property 3 is an easy consequence of the definition of the multiplication and the associativity of multiplication on the rational numbers. We shall prove the remaining three properties.

Proof of Property 5 of Lemma 9. Observe that $\xi\,\varphi(1) = \varphi(0) \cup \{x \in \mathbb{Q} : x = ab$ for some rational number $a \geq 0$ in ξ and some rational number $b \geq 0$ in $\varphi(1)\}$. Since $\xi \geq \varphi(0)$, $\varphi(0) \subseteq \xi$. Now, for any $a \geq 0$ in ξ and any rational number $b \geq 0$ in $\varphi(1)$, $a\,b \leq a$, since $b < 1$. It follows that $a\,b \in \xi$, since $a \in \xi$ and ξ is a real number. This implies that $\{x \in \mathbb{Q} : x = ab$ for some rational number $a \geq 0$ in ξ and some rational number $b \geq 0$ in $\varphi(1)\} \subseteq \xi$. Hence, $\xi\,\varphi(1) = \varphi(0) \cup \{x \in \mathbb{Q} : x = ab$ for some rational number $a \geq 0$ in ξ and some rational number $b \geq 0$ in $\varphi(1)\} \subseteq \xi$. That means $\xi\,\varphi(1) \leq \xi$. Next, we shall show that $\xi \subseteq \xi\,\varphi(1)$. Let x be a rational number in ξ. If $x < 0$, then $x \in \varphi(0) \subseteq \xi\,\varphi(1)$. If $x \geq 0$, then by Lemma 4, Part 2 of Chapter 2, there exists a counting number n such that $x + \frac{1}{n} \in \xi$. Obviously, $x + \frac{1}{n} > x \geq 0$. Thus $0 \leq \frac{x}{x + \frac{1}{n}} < 1$ and so $\frac{x}{x + \frac{1}{n}} \in \varphi(1)$. Hence, $x = \left(x + \frac{1}{n}\right)\left(\frac{x}{x + \frac{1}{n}}\right) \in \xi\,\varphi(1)$. It follows that $\xi \subseteq \xi\,\varphi(1)$. We can now conclude that $\xi = \xi\,\varphi(1)$.

Proof of Property 4 of Lemma 9. First observe that η and $\zeta \geq \varphi(0)$ implies that $\eta + \zeta \geq \eta + \varphi(0) = \varphi(0) + \eta \geq \varphi(0) + \varphi(0) = \varphi(0)$ by Lemma 2, Part 4, Part 1 and Part 5 of Chapter 3. Similarly, since by Part 1, $\xi\,\eta$, $\xi\,\zeta \geq \varphi(0)$, we conclude that $\xi\,\eta + \xi\,\zeta \geq \varphi(0)$. We shall show that $\xi\,(\eta + \zeta) \subseteq \xi\,\eta + \xi\,\zeta$. Take a rational number x in $\xi\,(\eta + \zeta)$. If $x < 0$ then $x \in \varphi(0) \subseteq \xi\,\eta + \xi\,\zeta$. We now assume that $x \geq 0$. Then there exists rational number $a \geq 0$ in ξ and rational number $b \geq 0$ in $(\eta + \zeta)$ such that $x = a\,b$. Since $b \in (\eta + \zeta)$, there exist rational numbers c in η and d in ζ such that $b = c + d$.

Therefore, $x = a\,b = a\,(c + d) = a\,c + a\,d$. Since $c + d = b \geq 0$, either $c \geq 0$ or $d \geq 0$.

If both c and d are ≥ 0, then $a\,c \in \xi\,\eta$ and $a\,d \in \xi\,\zeta$ and so $x = a\,c + a\,d \in \xi\,\eta + \xi\,\zeta$.

If $c \geq 0$ and $d < 0$, then $c + d < c$ and so $b = c + d \in \eta$, since c is in η. Thus, $x = a\,b \in \xi\,\eta$. Now, $\xi\,\eta = \xi\,\eta + \varphi(0) \leq \xi\,\eta + \xi\,\zeta$, by Lemma 2, Part 4 of Chapter 3, since $\xi\,\zeta \geq \varphi(0)$. Therefore, $x \in \xi\,\eta + \xi\,\zeta$. Likewise, if $c < 0$ and $d \geq 0$, then $c + d < d$ and so $b = c + d \in \zeta$, since d is in ζ. Hence, as before, we have $x = a\,b \in \xi\,\zeta \leq \xi\,\eta + \xi\,\zeta$. Once again, we get $x \in \xi\,\eta + \xi\,\zeta$. Thus, we can conclude that $\xi\,(\eta + \zeta) \subseteq \xi\,\eta + \xi\,\zeta$.

Next, we show that $\xi\,\eta + \xi\,\zeta \subseteq \xi\,(\eta + \zeta)$. By Lemma 2, Part 4 of Chapter 3, $\eta + \zeta \geq \eta$, ζ, since η and $\zeta \geq \varphi(0)$. We have thus $\eta \subseteq \eta + \zeta$ and $\zeta \subseteq \eta + \zeta$. Then $\xi\,\eta \subseteq \xi\,(\eta + \zeta)$ and $\xi\,\zeta \subseteq \xi\,(\eta + \zeta)$. This is seen as follows. Take any x in $\xi\,\eta$. If $x < 0$, then $x \in \xi\,(\eta + \zeta)$. If $x \geq 0$, then there exist rational number $a \geq 0$ in ξ and rational number b in η such that $x = a\,b$. Since $\eta \subseteq \eta + \zeta$, $b \in \eta + \zeta$. Therefore, $x = a\,b \in \xi\,(\eta + \zeta)$. Hence, $\xi\,\eta \subseteq \xi\,(\eta + \zeta)$. It follows similarly that $\xi\,\zeta \subseteq \xi\,(\eta + \zeta)$. Now, take any rational number x in $\xi\,\eta$ and any rational number y in $\xi\,\zeta$. If $x < 0$, then $x + y < y$ and so $x + y \in \xi\,\zeta \subseteq \xi\,(\eta + \zeta)$, since $\xi\,\zeta$ is a real number. Likewise, if $y < 0$, then $x + y < x$ and so $x + y \in \xi\,\eta \subseteq \xi\,(\eta + \zeta)$. If both x and y are ≥ 0, then there exist rational numbers a, $a' \geq 0$ in ξ, $b \geq 0$ in η and $b' \geq 0$ in ζ such that $x = a\,b$ and $y = a'\,b'$. If $a' \leq a$, then $x + y = a\,b + a'\,b' \leq a\,b + a\,b' = a(b + b') \in \xi(\eta + \zeta)$. Hence, $x + y \in \xi(\eta + \zeta)$ because $\xi(\eta + \zeta)$ is a real number. Finally, and similarly, if $a \leq a'$, then $x + y \leq a'\,(b + b') \in \xi(\eta + \zeta)$ and so we deduce in the same manner that $x + y \in \xi(\eta + \zeta)$. Since this is true for any x in $\xi\,\eta$ and any y in $\xi\,\zeta$, we conclude that $\xi\,\eta + \xi\,\zeta \subseteq \xi\,(\eta + \zeta)$. Therefore, $\xi\,\eta + \xi\,\zeta = \xi\,(\eta + \zeta)$.

Proof of Property 6 of Lemma 9. If either a or b is 0, then $\varphi(a\,b) = \varphi(0)$ and $\varphi(a)\varphi(b) = \varphi(0)$ by Proposition 4, Part 2. In this case the equality is trivial. We now assume a and b are both > 0. Let y be any rational number in $\varphi(a)\varphi(b)$. If $y < 0$, then $y < a\,b$ and so $y \in \varphi(a\,b)$. We now assume $y \geq 0$. Then there exist rational numbers c and d such that $0 \leq c < a$ and $0 \leq d < b$ and $y = c\,d$. Therefore, $y = c\,d < a\,b$ and so $y \in \varphi(a\,b)$. Hence, any rational number y in $\varphi(a)\varphi(b)$ is also in $\varphi(a\,b)$. Thus, $\varphi(a)\varphi(b) \subseteq \varphi(a\,b)$. Conversely, take any rational number x in $\varphi(a\,b)$. If $x \leq 0$, then $x \in \varphi(a)\varphi(b)$ since $\varphi(a)\varphi(b) > \varphi(0)$. We now assume $0 < x < a\,b$. Then $0 < \frac{x}{a} < b$ since $a > 0$. Therefore, $b - \frac{x}{a} > 0$. Then by the

Archimedean property of $\mathbb{Q}$, there exists a counting number n such that $n\left(b-\frac{x}{a}\right)>\frac{x}{a}$.
Hence, $\left(b-\frac{x}{a}\right)>\frac{1}{n}\frac{x}{a}$. Thus,

$$b>\left(1+\frac{1}{n}\right)\frac{x}{a}.$$

Therefore, $\left(1+\frac{1}{n}\right)\frac{x}{a}\in\varphi(b)$. Also, since $\frac{a}{1+\frac{1}{n}}<a$, $\frac{a}{1+\frac{1}{n}}\in\varphi(a)$. Hence,

$x=\left(\frac{a}{1+\frac{1}{n}}\right)\left(\left(1+\frac{1}{n}\right)\frac{x}{a}\right)\in\varphi(a)\varphi(b)$). We can now conclude that $\varphi(a\,b)\subseteq\varphi(a)\varphi(b)$.

Thus, $\varphi(a\,b)=\varphi(a)\varphi(b)$. This completes the proof.

Extension of Multiplication to All of the Real Numbers

Extending the multiplication to all of the real numbers $\mathbb{R}$ is a natural progression for our construction. We shall use the device: convert a pair of real numbers, if needed, to a pair of real numbers $\geq \varphi(0)$ by taking the appropriate additive inverse, then multiply the resulting pair according to the definition for real numbers $\geq \varphi(0)$, follow this by, if appropriate, taking the additive inverse of the real number so obtained. For real number $< \varphi(0)$, the operation of taking the additive inverse behaves like the usual reflection for the rational numbers. This is made precise in the following lemma. This lends support to the economy of visualisation of the real numbers as an infinite "line".

Lemma 10. For any real number ξ, $\xi<\varphi(0)$ if, and only if, $-\xi>\varphi(0)$.

Proof. Suppose $\xi<\varphi(0)$. Then $\xi\subseteq\varphi(0)$ and there exists a rational number $y<0$ such that $y\notin\xi$. Take any rational number z in $\varphi(0)$. Then $z<0<-y$. This means that z and 0 belong to $-\xi$. This implies that $\varphi(0)\subseteq-\xi$ and that means $\varphi(0)<-\xi$, since $0\in-\xi$.

Conversely, suppose $\varphi(0)<-\xi$. Then $\varphi(0)\subseteq-\xi$ and there exists a rational number y_0 in $-\xi$ such that $y_0\notin\varphi(0)$. Hence $y_0\geq 0$ and there exists some rational number $y_1\notin\xi$ such that $0\leq y_0<-y_1$. Therefore, $y_1<-y_0\leq 0$. Since $y_1\notin\xi$, for any rational number x in ξ, x

$< y_1 < 0$. Thus, $\xi \subseteq \varphi(0)$. Since $y_1 \in \varphi(0)$ and $y_1 \notin \xi$, $\xi < \varphi(0)$. This completes the proof.

Definition 11. Suppose ξ, η and ζ are real numbers such that $\xi < \varphi(0)$, $\eta < \varphi(0)$ and $\zeta \geq \varphi(0)$. The following defines the multiplication of ξ with any one of η or ζ in any order.

$\xi\,\zeta = -((-\xi)\,\zeta) = -(\zeta(-\xi)) = \zeta\,\xi$ and

$\xi\,\eta = (-\xi)(-\eta) = (-\eta)(-\xi) = \eta\,\xi.$

Observe that with Definition 11, multiplication of any two real numbers not both $\geq \varphi(0)$ is defined. This means that we now have a multiplication on the set of real numbers. A consequence of the definition is commutativity for multiplication. Another deduction from the definition is that for any real number ξ, $\varphi(0)\xi = \xi\,\varphi(0) = \varphi(0)$. With this definition, we have a multiplication map, $\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, that takes an ordered pair (ξ, η) and assigns to it, the product $\xi\,\eta$. It now remains to define the multiplicative inverse of any real number $\xi < \varphi(0)$.

Definition 12. For any real number $\xi < \varphi(0)$, define the multiplicative inverse by

$$\frac{1}{\xi} = -\left(\frac{1}{-\xi}\right).$$

We can easily check that this is really the multiplicative inverse of ξ. To facilitate the checking, we use the following Lemma.

Lemma 13. For any real number ξ, $-(-\xi) = \xi$.

Proof. $\xi = \xi + \varphi(0) = \xi + ((-\xi) + (-(-\xi)))$

$= (\xi + (-\xi)) + (-(-\xi))$, by the associativity of addition,

$= \varphi(0) + (-(-\xi))$

$= (-(-\xi)) + \varphi(0)$, by the commutativity of addition (Lemma 2(1) Chapter 3),

$= -(-\xi)$, by Lemma 2, part 3 of Chapter 3.

Thus, for $\xi < \varphi(0)$, $-\xi > \varphi(0)$ and so $\frac{1}{-\xi} > \varphi(0)$ as remarked in the paragraph following Definition 6 of the multiplicative inverse for real numbers > $\varphi(0)$. It follows then that $\frac{1}{\xi} = -\left(\frac{1}{-\xi}\right) < \varphi(0)$ by Lemma 10. Hence, by Definition 11,

$$\xi\frac{1}{\xi} = (-\xi)\left(-\frac{1}{\xi}\right) = (-\xi)\left(-\left(-\left(\frac{1}{-\xi}\right)\right)\right), \text{ by Definition 12,}$$

$$= (-\xi)\left(\frac{1}{-\xi}\right), \text{ by Lemma 13,}$$

$$= \varphi(1), \text{ by Lemma 7.}$$

This shows that the definition of the multiplicative inverse of the real number $\xi < \varphi(0)$ is the correct one.

Now, we collect all the properties that the set $\mathbb{R}$ of real numbers possesses with respect to multiplication and addition together with the properties in Lemma 2 of Chapter 3 and characterise $\mathbb{R}$ as a field.

Lemma 14. For any real numbers ξ, η and ζ and any rational numbers a and b, the following properties hold.

1. $\xi\,\eta = \eta\,\xi$. (Commutativity)

2. $(\xi\,\eta)\,\zeta = \xi\,(\eta\,\zeta)$. (Associativity)

3. $\xi\,(\eta + \zeta) = \xi\,\eta + \xi\,\zeta$. (Distributivity)

4. $\xi\,\varphi(1) = \xi$.

5. If $\xi \neq \varphi(0)$, then $\xi\frac{1}{\xi} = \varphi(1)$.

6. If $\xi > \varphi(0)$, then $\eta > \zeta \Rightarrow \xi\,\eta > \xi\,\zeta$.

7. $\varphi(a)\,\varphi(b) = \varphi(a\,b)$.

Property 1 of Lemma 14 is a consequence of Lemma 9, part 1 and Definition 11. Property 5 has already been shown. We shall prove the remaining properties.

Proof of Property 2 of Lemma 14. If any one of ξ, η or ζ is $\varphi(0)$, then trivially, by Proposition 4, part 2, both sides of the equation are the same as $\varphi(0)$ and so we have nothing to prove. If $\xi, \eta, \zeta > \varphi(0)$, then this is just Lemma 9, Part 2. So, we have the remaining seven cases to check out.

Case 1. $\xi, \eta > \varphi(0)$, $\zeta < \varphi(0)$.

$(\xi\,\eta)\,\zeta = -\,(\,(\xi\,\eta)(-\zeta))$, since $\xi\,\eta > \varphi(0)$ and $\zeta < \varphi(0)$,

$= -(\xi\,(\eta\,(-\zeta)))$, by Lemma 9 Property 3,

$= \quad \xi\,(-(\eta\,(-\zeta)))$, by Definition 11, since $\xi > \varphi(0)$ and $-(\eta\,(-\zeta)) < \varphi(0)$ and Lemma 13,

$= \quad \xi\,(\eta\,\zeta)$, by Definition 11, since $\eta > \varphi(0)$ and $\zeta < \varphi(0)$.

Case 2. $\xi, \zeta > \varphi(0)$, $\eta < \varphi(0)$.

$(\xi\,\eta)\,\zeta = (-(\xi(-\,\eta)))\zeta$, by Definition 11, since $\xi > \varphi(0)$ and $\eta < \varphi(0)$,

$= -\,(\,(-(-(\xi(-\,\eta))))\zeta)$, by Definition 11, since $-(\xi(-\,\eta)) < \varphi(0)$ and $\zeta > \varphi(0)$,

$= -\,(\,(\xi(-\,\eta))\zeta)$, by Lemma 13,

$= -\,(\,\xi((-\,\eta)\zeta)\,)$, by Lemma 9, Property 3, since $\xi, -\eta, \zeta > \varphi(0)$,

$= \xi(-\,((-\,\eta)\zeta))$, by Definition 11, since $\xi > \varphi(0)$ and $-\,((-\,\eta)\zeta) < \varphi(0)$ and Lemma 13,

$= \xi(\eta\,\zeta)$, by Definition 11, since $\eta < \varphi(0)$ and $\zeta > \varphi(0)$.

Case 3. $\xi > \varphi(0)$, $\eta, \zeta < \varphi(0)$.

$(\xi\,\eta)\,\zeta = (-(\xi(-\,\eta)))\zeta$, by Definition 11, since $\xi > \varphi(0)$ and $\eta < \varphi(0)$,

$= (-\,(-(\xi(-\,\eta))))(-\zeta)$, by Definition 11, since $\zeta < \varphi(0)$ and $-(\xi(-\,\eta)) < \varphi(0)$,

$= (\xi(-\,\eta))(-\zeta)$, by Lemma 13,

$= \xi((-\eta)(-\zeta))$, by Lemma 9, Property 3, since $\xi, -\eta, -\zeta > \varphi(0)$,

$= \xi(\eta\zeta)$, by Definition 11, since $\eta, \zeta < \varphi(0)$.

Case 4. $\xi < \varphi(0)$, $\eta, \zeta > \varphi(0)$.

$(\xi\eta)\zeta = (\eta\xi)\zeta$, by commutativity of multiplication,

$= \eta(\xi\zeta)$, by Case 2 above,

$= \eta(\zeta\xi)$, by commutativity of multiplication,

$= (\eta\zeta)\xi$, by Case 1 above,

$= \xi(\eta\zeta)$, by commutativity of multiplication.

Case 5. $\xi, \eta < \varphi(0)$, $\zeta > \varphi(0)$.

$(\xi\eta)\zeta = \zeta(\xi\eta)$, by commutativity of multiplication,

$= \zeta(\eta\xi)$, by commutativity of multiplication,

$= (\zeta\eta)\xi$, by Case 3 above,

$= \xi(\zeta\eta)$, by commutativity of multiplication,

$= \xi(\eta\zeta)$, by commutativity of multiplication.

Case 6. $\xi, \zeta < \varphi(0)$, $\eta > \varphi(0)$.

$(\xi\eta)\zeta = \zeta(\xi\eta)$ by commutativity of multiplication

$= (\zeta\xi)\eta$, by Case 5 above,

$= (\xi\zeta)\eta$, by commutativity of multiplication,

$= \xi(\zeta\eta)$, by Case 5 above,

$= \xi(\eta\zeta)$, by commutativity of multiplication.

Case 7. $\xi, \eta, \zeta < \varphi(0)$.

$(\xi\eta)\zeta = ((-\xi)(-\eta))\zeta$, by Definition 11, since $\xi, \eta < \varphi(0)$.

$= -(((-\xi)(-\eta))(-\zeta))$, by Definition 11, since $\zeta < \varphi(0)$ and $(-\xi)(-\eta) > \varphi(0)$,

$= -((-\xi)((-\eta)(-\zeta)))$, by Property 3 of Lemma 9, since $-\xi, -\eta, -\zeta > \varphi(0)$,

$= \xi((-\eta)(-\zeta))$, by Definition 11, since $\xi < \varphi(0)$ and $(-\eta)(-\zeta) > \varphi(0)$

$= \xi(\eta\,\zeta)$, by Definition 11, since $\eta, \zeta < \varphi(0)$.

This completes the proof of Property 2 of Lemma 14.

Proof of Property 4 of Lemma 14.

If $\xi \geq \varphi(0)$, then $\xi\,\varphi(1) = \xi$, by Property 5 of Lemma 9. For real number $\xi < \varphi(0)$,

$\xi\,\varphi(1) = -((-\xi)\varphi(1))$, by Definition 11,

$= -(-\xi)$, by Property 5 of Lemma 9, since $-\xi > \varphi(0)$,

$= \xi$, by Lemma 13.

The following is a useful observation of the operation of taking additive inverse with multiplication. It is an easy consequence of Definition 11. It is stated here for reference as we shall be using it implicitly.

Lemma 15. For any real numbers ξ and η,

$$-(\xi(-\eta)) = (-\xi)(-\eta) = -((-\xi)\eta) = \xi\,\eta.$$

Proof of Property 3 of Lemma 14.

If $\xi, \eta, \zeta \geq \varphi(0)$, then $\xi(\eta + \zeta) = \xi\,\eta + \xi\,\zeta$, by Property 4 of Lemma 9. If $\xi = \varphi(0)$, trivially, $\xi(\eta + \zeta) = \xi\,\eta + \xi\,\zeta = \varphi(0)$, by Proposition 4. If $\eta = \varphi(0)$, then $\xi(\eta + \zeta) = \xi(\varphi(0) + \zeta) = \xi\,\zeta = \varphi(0) + \xi\,\zeta = \xi\,\eta + \xi\,\zeta$. Similarly, if $\zeta = \varphi(0)$, then $\xi(\eta + \zeta) = \xi\,\eta + \xi\,\zeta$. The remainder of the proof is covered in the following six (not necessarily mutually exclusive) cases.

Case 1. $\xi > \varphi(0)$, $\eta \geq \varphi(0)$, $\eta + \zeta \geq \varphi(0)$,

If $\zeta \geq \varphi(0)$, then the distributivity is already shown in the preceding paragraph. We now assume $\zeta < \varphi(0)$.

$\xi(\eta + \zeta) + \xi(-\zeta) = \xi((\eta + \zeta) + (-\zeta))$, by Lemma 9, Property 4, since $\eta + \zeta, -\zeta \geq \varphi(0)$,

$= \xi((\eta + (\zeta + (-\zeta)))$, by associativity of addition, (Property 2 of Lemma 2, Chapter 3),

$= \xi(\eta + \varphi(0))$, by Property 3, Lemma 2 of Chapter 3,

$= \xi\eta$.

Adding $-(\xi(-\zeta))$ to both sides, we get

$(\xi(\eta+\zeta)+\xi(-\zeta))+(-(\xi(-\zeta)))=\xi\eta+(-(\xi(-\zeta)))$

$=\xi\eta+\xi\zeta$, by Lemma 15.

Thus, $\xi(\eta+\zeta)+(\xi(-\zeta))+(-(\xi(-\zeta))))=\xi\eta+\xi\zeta$, by the associativity of addition. Hence, the left-hand side is equal to $\xi(\eta+\zeta)+\varphi(0)=\xi(\eta+\zeta)$. This proves $\xi(\eta+\zeta)=\xi\eta+\xi\zeta$.

Case 2. $\xi>\varphi(0)$, $\zeta\geq\varphi(0)$, $\eta+\zeta\geq\varphi(0)$.

$\xi(\eta+\zeta)=\xi(\zeta+\eta)$, by the commutativity of addition (Lemma 2, Property 1 of Chapter 3),

$=\xi\zeta+\xi\eta$, by Case 1 above,

$=\xi\eta+\xi\zeta$, by the commutativity of addition.

Case 3. $\xi>\varphi(0)$, $\eta+\zeta<\varphi(0)$.

$\xi(\eta+\zeta)=-(\xi(-(\eta+\zeta)))$, by Lemma 15,

$=-(\xi((-\eta)+(-\zeta)))$, by Lemma 5, Part 3 of Chapter 3,

$=-(\xi(-\eta)+\xi(-\zeta))$, by Case 1, if $\eta\leq\varphi(0)$, or Case 2, if $\zeta\leq\varphi(0)$, since $\xi>\varphi(0)$, and $(-\eta)+(-\zeta)>\varphi(0)$,

$=-(\xi(-\eta))+(-(\xi(-\zeta)))$, by Lemma 5, Part 3 of Chapter 3,

$=\xi\eta+\xi\zeta$, by Lemma 15.

Case 4. $\xi<\varphi(0)$, $\eta\geq\varphi(0)$ and $\eta+\zeta\geq\varphi(0)$.

$\xi(\eta+\zeta)=-((-\xi)(\eta+\zeta))$, by Lemma 15,

$=-((-\xi)\eta+(-\xi)\zeta))$, by Case 1, since $-\xi>\varphi(0)$, $\eta\geq\varphi(0)$ and $\eta+\zeta\geq\varphi(0)$,

$=-((-\xi)\eta)+(-((-\xi)\zeta))$, by Lemma 4 Part 3 of Chapter 3,

$=\xi\eta+\xi\zeta$, by Lemma 15.

Case 5. $\xi<\varphi(0)$, $\zeta\geq\varphi(0)$ and $\eta+\zeta\geq\varphi(0)$.

$\xi(\eta+\zeta)=\xi(\zeta+\eta)$, by the commutativity of addition,

$= \xi\zeta + \xi\eta$, by Case 4 above,

$= \xi\eta + \xi\zeta$, by the commutativity of addition (Lemma 2, Property 1 of Chapter 3).

Case 6. $\xi < \varphi(0)$, $\eta + \zeta < \varphi(0)$.

$\xi(\eta + \zeta) = -((-\xi)(\eta + \zeta))$, by Lemma15,

$= -((-\xi)\eta + (-\xi)\zeta)$, by Case 3, since $-\xi > \varphi(0)$ and $\eta + \zeta < \varphi(0)$,

$= -((-\xi)\eta) + (-((-\xi)\zeta))$, by Lemma 5, Part 3 of Chapter 3,

$= \xi\eta + \xi\zeta$, by Lemma 15.

This completes the proof of Property 3.

Before we proceed to prove the remaining properties, we state the following equivalent result of Property 4, Lemma 2 of Chapter 3 for strict inequality.

Lemma 16. For any real numbers ξ, η and ζ,

$$\eta < \xi \Rightarrow \eta + \zeta < \xi + \zeta.$$

Proof. By Property 4, Lemma 2 of Chapter 3, $\eta < \xi \Rightarrow \eta + \zeta \le \xi + \zeta$. We need only show that there exists a rational number, d, in $\xi + \zeta$ but not in $\eta + \zeta$. Now, since $\eta < \xi$, there exists a rational number a in ξ, but $a \notin \eta$. Therefore, for all x in η, $x < a$.

By Lemma 4, Part 2 of Chapter 2, there exists a counting number n such that $a + 1/n \in \xi$. Also, by Lemma 4, Part 1 of Chapter 2, there exists rational numbers b in ζ and $c \notin \zeta$ such that $c - b < 1/n$. Therefore, for all rational number y in ζ, $y < c < b + 1/n$. Thus, for all rational number x in η and all rational number y in ζ, $x + y < a + b + 1/n$. It follows that $d = a + b + 1/n \notin \eta + \zeta$. Now, $d = (a + 1/n) + b \in \xi + \zeta$, since $a + 1/n \in \xi$ and $b \in \zeta$. We can thus conclude that $\eta + \zeta < \xi + \zeta$.

Proof of Property 6 of Lemma 14.

By Lemma 16, $\eta > \zeta$ implies that $\eta + (-\zeta) > \zeta + (-\zeta) = \varphi(0)$. Since $\xi > \varphi(0)$, by Property 1 of Lemma 9, $\xi(\eta + (-\zeta)) > \varphi(0)$. Therefore, by Property 3 of Lemma 14, we have $\xi\eta + \xi(-\zeta) = \xi(\eta + (-\zeta)) > \varphi(0)$. It then follows by Lemma 16 that

$$(\xi\eta + \xi(-\zeta)) + (-(\xi(-\zeta))) > \varphi(0) + (-(\xi(-\zeta))).$$

Now,

$\varphi(0) + (-(\xi(-\zeta))) = (-(\xi(-\zeta)))$, by Lemma 2 Property 1 and 3 of Chapter 3,

$= \xi\,\zeta$, by Lemma 15.

Also, we have,

$$(\xi\,\eta + \xi\,(-\zeta)) + (-(\xi\,(-\zeta))) = \xi\,\eta + (\xi\,(-\zeta) + (-(\xi\,(-\zeta)))),$$

by Property 2 Lemma 2 of Chapter 3

$$= \xi\,\eta + \varphi(0)$$

$= \xi\,\eta$, by Lemma 2, Property 3 of Chapter 3.

Hence $\xi\,\eta > \xi\,\zeta$.

Proof of Property 7 of Lemma 14.

If $a = 0$ or $b = 0$, then trivially $\varphi(a)\,\varphi(b) = \varphi(a\,b) = \varphi(0)$.

If $a, b > 0$, then $\varphi(a)\,\varphi(b) = \varphi(a\,b)$, by Property 6 of Lemma 9.

If $a > 0$ and $b < 0$, then $a\,b < 0$. Therefore, as $\varphi(b) < \varphi(0)$,

$\varphi(a)\,\varphi(b) = -\,(\varphi(a)\,(-\varphi(b)))$

$= -\,(\varphi(a)\,\varphi(-b))$, by Lemma 5, Part 2 of Chapter 3,

$= -\varphi(a\,(-b))$, by Lemma 9, Property 6, since $a, -b > 0$,

$= \varphi(-(a(-b)))$, by Lemma 5, Part 2 of Chapter 3,

$= \varphi(a\,b)$.

If $a < 0$ and $b > 0$, then $\varphi(a\,b) = \varphi(b\,a) = \varphi(b)\,\varphi(a)$, by the above argument.

Thus, $\varphi\,(a\,b) = \varphi(b)\,\varphi(a) = \varphi(a)\,\varphi(b)$.

If $a, b < 0$, $\varphi(a)\,\varphi(b) = (-\varphi(a))(-\,\varphi(b))$

$= \varphi(-a)\varphi(-b)$, by Lemma 5, Part 2 of Chapter 3,

$= \varphi((-a)(-b))$, by Lemma 9, Part 6, since $-a, -b > 0$,

$= \varphi(a\,b)$.

This concludes the proof of Lemma 14, Part 7.

We next extract some properties of the real numbers with respect to multiplication. We summarise this in the following theorem.

Theorem 17. $\mathbb{R}-\{\varphi(0)\}$ is an abelian group with respect to multiplication and its identity element is $\varphi(1)$. That is, multiplication on $\mathbb{R}$ is commutative and associative, has an identity element $\varphi(1)$ and a unary operation that assigns to each real number $\xi \neq \varphi(0)$, its multiplicative inverse $\frac{1}{\xi}$.

This is a consequence of Lemma 14, (1), (2) and (4). Lemma 14, Theorem 6 of Chapter 3 and Theorem 8 of Chapter 2 actually imply the following:

Theorem 18. $\mathbb{R}$ is a field with $\varphi(0)$ as the identity element for addition and $\varphi(1)$, the identity element for multiplication. The field $\mathbb{R}$ has a total ordering with respect to "$\leq$". $\mathbb{R}$ is a complete, totally ordered field.

Lemma 9 of Chapter 2, Lemma 2, Part 5 of Chapter 3 and Lemma 14, Part 7 imply that the embedding of the rational numbers into $\mathbb{R}$, $\varphi:\mathbb{Q}\to\mathbb{R}$, is a monomorphism of the field of rational numbers into $\mathbb{R}$ preserving the ordering on $\mathbb{Q}$. We have now constructed a field $\mathbb{R}$ with the rational numbers embedded in $\mathbb{R}$ as a subfield. To think of the real numbers $\mathbb{R}$ as an extension of the rational numbers $\mathbb{Q}$, we identify each rational number a in $\mathbb{Q}$ as its image $\varphi(a)$ in $\mathbb{R}$. In this way the rational numbers are thought of as the subfield $\varphi(\mathbb{Q})$ of $\mathbb{R}$. When we speak of the rational numbers, we think of the usual rational numbers, but when we want to prove any property of the real numbers we use the properties of a complete totally ordered field and think now of the rational numbers as the embedded subfield $\varphi(\mathbb{Q})$. It is hard to work with the model of the real numbers that we have constructed; most results about real numbers are results about complete totally ordered field and are proved using the properties of a complete totally ordered field. This is because there is essentially one such complete totally ordered field and we shall discuss this in the next chapter.

We conclude this chapter by showing that the remark stated just before Lemma 10 is explained as the fact that the additive inverse of any real numbers is precisely the multiplication of this real number by the (rational) number $\varphi(-1)$, the equivalent of -1 in $\mathbb{R}$.

Lemma 19. For any real number ξ, $-\xi = \varphi(-1)\xi$.

Proof. By Lemma 2, Part 5 of Chapter 3, $\varphi(0) = \varphi(1) + \varphi(-1)$.

Now,

$\varphi(0) = \varphi(0)\xi = (\varphi(1) + \varphi(-1))\xi$

$= \varphi(1)\xi + \varphi(-1)\xi$, by distributivity,

$= \xi + \varphi(-1)\xi$.

Therefore, adding the additive inverse $-\xi$ to both sides of the equation above yields

$-\xi + \varphi(0) = -\xi + (\xi + \varphi(-1)\xi)$

$= ((-\xi) + \xi) + \varphi(-1)\xi$, by the associativity of addition,

$= \varphi(0) + \varphi(-1)\xi$

$= \varphi(-1)\xi + \varphi(0)$, by the commutativity of addition.

Thus, by Lemma 2, Part 3 of Chapter 3, $-\xi = \varphi(-1)\xi$.

This completes the proof.

CHAPTER FIVE

Order! Order! There is only one order

The ordering on the real numbers $\mathbb{R}$ is given by subset inclusion. Let us now examine how we can think of this ordering as given essentially by a positive cone in $\mathbb{R}$ as described in Chapter 1.

Let $P = \{\xi \in \mathbb{R} : \xi > \varphi(0)\}$.

For any real number $\xi \neq \varphi(0)$, $\xi > \varphi(0)$ or $\xi < \varphi(0)$. By Lemma 10, Chapter 4, $\xi < \varphi(0)$ if, and only if, $-\xi > \varphi(0)$. Thus, the set

$$-P = \{-\xi \in \mathbb{R} : \xi > \varphi(0)\} = \{\xi \in \mathbb{R} : \xi < \varphi(0)\}.$$

Hence, $\mathbb{R} = P \cup -P \cup \{\varphi(0)\}$. In particular, we have the following result.

Theorem 1.

1. $\varphi(0) \notin P$.
2. $\xi, \eta \in P \Rightarrow \xi + \eta \in P$.
3. $\xi, \eta \in P \Rightarrow \xi\eta \in P$.
4. $\xi \neq \varphi(0) \Rightarrow \xi \in P$ or $\xi \in -P$.

Proof. Part 1 is obvious. For Part 2, $\xi, \eta \in P$ implies that $\xi, \eta > \varphi(0)$. It follows by Lemma 16 of Chapter 4 that $\xi + \eta > \varphi(0) + \eta = \eta > \varphi(0)$. Hence, $\xi + \eta \in P$. Part 3 is just a consequence of Lemma 9, Part 1 of Chapter 4. Part 4 follows from Lemma 10 of Chapter 4 and the fact that the ordering on $\mathbb{R}$ is a total ordering.

Thus, P is a positive cone for the real numbers. The ordering with respect to the positive cone is defined by ξ ">" η if, and only if, $\xi - \eta \in P$. Now, $\xi - \eta \in P$ if, and only if, $\xi - \eta > \varphi(0)$ if, and only if, $\xi > \eta + \varphi(0) = \eta$. So, the ordering defined by the positive cone is the same as that defined by set inclusion. Thus, if we define a real number ξ to be *positive* if $\xi \in P$, then the positive cone is none other than the *positive* real numbers and ξ is *positive* if $\xi > \varphi(0)$. So, we may, if we wish, *identify* $\varphi(0)$ with 0 but we must be careful to distinguish them. Notice then that a positive cone for the embedded rational number $\varphi(\mathbb{Q})$ is $P \cap \varphi(\mathbb{Q})$, which is the same as the set $\{\varphi(x): x$ a rational number and $x > 0\}$ and is the image of the positive cone of the rational numbers $\mathbb{Q}$. The uniqueness for the positive cone for the rational numbers $\mathbb{Q}$ has been established in

Chapter 1. It is none other than the set of positive rational numbers. However, the uniqueness of the positive cone for $\mathbb{R}$ and hence the uniqueness of the ordering on $\mathbb{R}$ is a little harder to show.

We now formally set the definition of a positive real number and a negative real number.

Definition 2. A real number ξ is said to be *positive* if $\xi > \varphi(0)$. ξ is said to be *negative* if $\xi < \varphi(0)$.

We shall need the following technical lemma regarding the existence of the n-th root of a positive real number. We require only the existence of the square root of a positive real number to show the uniqueness of the positive cone. Nonetheless, the same proof applies for n-th root.

Lemma 3. Let n be any counting number. If ξ is a positive real number, that is, $\xi > \varphi(0)$, then there exists a positive real number η such that $\eta^n = \xi$. Specifically, for $n = 2$, there exists a real number η such that $\eta^2 = \xi$.

We shall prove Lemma 3 only for $n = 2$. First, we make the following definition.

Definition 4. For any $\xi > \varphi(0)$, define $\eta = \{x \in \mathbb{Q} : x^2 < k \text{ for all } k \notin \xi\} \cup \varphi(0)$. Obviously, $0 \in \eta$.

The set η above is our candidate for the square root of ξ. Note that the construction of the square root has a practical and constructible nature in the set theoretic sense. When we talk about solution to $x^2 = p$, assumption is made about the existence of the square root of p without question. Symbol like $\sqrt{p}$ is employed to mean the square root of p. The meaning of $\sqrt{p}$ is not clear, particularly so when p is a prime number. The meaning and existence of $\sqrt{p}$ must be established less we shall be talking about an entity that does not exist. In this context, we shall use the embedded rational numbers $\varphi(\mathbb{Q})$. Use $\varphi(p)$ instead of the prime p. With $\xi = \varphi(p)$ and $n = 2$, the set η define above will be the square root of $\varphi(p)$. Undoubtedly, $\eta \notin \varphi(\mathbb{Q})$. We may deduce this as follows. Suppose $\eta \in \varphi(\mathbb{Q})$. Then there exists a rational number q such that $\eta = \varphi(q)$. Therefore, $\varphi(p) = \eta^2 = \varphi(q^2)$, by Chapter 4, Lemma 14, part 7. Since φ is injective, $p = q^2$. We may assume that q is positive. Now, as q is a rational number, $q = \dfrac{m}{n}$, where m and n are positive integers such that the highest common factor of m and n is 1. It follows then from $p = \dfrac{m^2}{n^2}$ that the prime p must divide m. Thus, $m = p\,k$ for some positive integer k. Then $n^2 p = m^2 = p^2 k^2$ implies that $n^2 = pk^2$ and so p must divide n. It follows that p must be a common factor of m and n other than 1. This contradicts that the highest common factor of m and n is 1. Hence, $\eta \notin \varphi(\mathbb{Q})$.

Of course, we need to show that η is a real number.

Lemma 5. For any real number $\xi > \varphi(0)$, the set η defined in Definition 4 is a real number.

Proof. 1. Since $\varphi(0) \subseteq \eta$, $\eta \neq \varnothing$.

2. Since ξ is a (left) cut, there exists a positive rational number $q > 1$ not in ξ. Then for any rational number x in η, if $x > 0$, then $x^2 < q^2$ and it follows that $x < q$. Therefore, $q \notin \eta$ and so $\eta \neq \mathbb{Q}$.

3. Let $l \in \eta$ and $x < l$. We shall show that $x \in \eta$. If $x \le 0$, then $x \in \eta$. It is sufficient to show that if $x, l > 0$ and $x < l$, then $x \in \eta$. Now, $0 < x < l$ implies that $x^2 < l^2$ and $l \in \eta$ implies that $l^2 < k$ for all k not in ξ. It follows that $x^2 < k$ for all k not in ξ and so $x \in \eta$, by Definition 4.

4. Next, we shall show that η has no maximal number. Suppose on the contrary that η has a maximal number M. Take a rational number $q > 0$ in ξ. This number exists by Lemma 2 of Chapter 4. Since $\mathbb{Q}$ is Archimedean, there is a counting number n such that $n\,q > 1$, that is, $0 < \frac{1}{n} < q$. In particular, $\frac{1}{n^2} < q$. Therefore, $\left(\frac{1}{n}\right)^2 < q < k$ for any rational number k not in ξ. Hence, $\frac{1}{n} \in \eta$ and $\frac{1}{n} > 0$. Thus, $M > 0$. Then $M^2 \in \xi$. This is because, if on the contrary that $M^2 \notin \xi$, then $M \in \eta$ would imply $M^2 < M^2$, which is absurd. Then by Lemma 4, part 2 of Chapter 2, there exists a counting number m such that $M^2 + \frac{1}{m} \in \xi$. Let $x_0 = M^2 + \frac{1}{m}$. Obviously, $x_0 > M^2$. By the Archimedean property of the rational numbers, there exists a counting number n_1 such that $n_1(x_0 - M^2) > M^2$. Thus $(x_0 - M^2) > \frac{1}{n_1}M^2$. Therefore, $x_0 > \left(1+\frac{1}{n_1}\right)M^2 > M^2$. Repeating this argument with $\left(1+\frac{1}{n_1}\right)M^2$ in place of M^2, we obtain another counting number n_2 such that

$$x_0 > \left(1+\frac{1}{n_2}\right)\left(1+\frac{1}{n_1}\right)M^2 > \left(1+\frac{1}{n_1}\right)M^2 > M^2 .$$

Now, let $k = \max\,\{n_1, n_2\}$. Then $M^2 < \left(1+\frac{1}{k}\right)^2 M^2 < \left(1+\frac{1}{n_2}\right)\left(1+\frac{1}{n_1}\right)M^2 < x_0$. This means that $\left(1+\frac{1}{k}\right)^2 M^2 < y$ for all y not in ξ, since $x_0 \in \xi$ and ξ is a (left) cut. Hence, by the definition of η, $\left(1+\frac{1}{k}\right)M \in \eta$. (The above proceeding works equally well with the exponent 2 replaced by any other counting number N, taking $k = \max(n_1, n_2, \dots, n_N)$, where the counting numbers n_j are obtained by repeating the above argument N times and we get $M^N < \left(1+\frac{1}{k}\right)^N M^N < x_0$.) Now, $M < \left(1+\frac{1}{k}\right)M$ but M is the maximal number in η and so $M \geq \left(1+\frac{1}{k}\right)M$ and we have a contradiction. Thus, the maximal number M does not exist. By Lemma 3 of Chapter 2, η is a real number. This completes the proof of Lemma 5.

Lemma 6. Let η be defined as in Definition 4 for any real number $\xi > \varphi(0)$. Then $\eta^2 = \xi$.

Proof. Recall $\eta^2 = \varphi(0) \cup \{x \in \mathbb{Q} : x = a\, b$ for some rational numbers $a, b \geq 0$ in $\eta\}$.

Firstly, we shall show that $\eta^2 \subseteq \xi$. Obviously, $\varphi(0) \subseteq \xi$. Since $0 \in \eta$, $0 \in \eta^2$. Observe that $0 \in \xi$ because $\xi > \varphi(0)$. It remains to show that for any rational number $a, b > 0$ in η, $a\, b$ is in ξ. Since $a, b \in \eta$, for all rational number $k \notin \xi$, $a^2, b^2 < k$. Therefore, for all rational number $k \notin \xi$, $(a\, b)^2 = a^2\, b^2 < k^2$. This means that $a\, b < k$ for all rational number $k \notin \xi$. Thus, $a\, b \in \xi$. We can then conclude that $\eta^2 \subseteq \xi$.

Next, we shall show that $\xi \subseteq \eta^2$. As before it is enough to show that for all rational number $y > 0$ in ξ, $y \in \eta^2$, since $\varphi(0) \cup \{0\} \subseteq \xi, \eta^2$. If, on the contrary $y \notin \eta^2$, then $y > a^2$ for all rational number $a > 0$ in η, otherwise $y \leq a^2$ for some $a > 0$ in η would imply that $y \in \eta^2$. Therefore, we have for all a in η,

$$a^2 < y < k \text{ for all } k \notin \xi. \qquad \text{---------------------------} \quad (1)$$

Now, by Lemma 4 of Chapter 2, for any counting number n, there exist rational numbers $a \in \eta$, $b \notin \eta$ such that $b - a < 1/n$. Since $\eta > \varphi(0)$, $b \notin \eta$ implies that $b > 0$ and for some rational number q not in ξ, $b^2 \geq q$. Hence by (1) we get

$$a^2 < y < q \leq b^2 < \left(a + \frac{1}{n}\right)^2. \qquad \text{---------------------------} \ (2)$$

We are going to prepare y for a contradiction. Since $y \in \xi$, there exists a rational number x_0 in ξ such that $y < x_0$. Choose a fixed rational number $k_0 \notin \eta$. Then $k_0 > 0$, since $\eta > \varphi(0)$. Since $x_0 - y > 0$, by the Archimedean property of $\mathbb{Q}$, there exists a counting number n_0 such that

$$n_0\left(\frac{x_0 - y}{2}\right) > 1 + 2k_0 \qquad \text{------------------------------} \quad (*)$$

and hence

$$\frac{1}{n_0}\left(1 + 2k_0\right) < \frac{x_0 - y}{2}. \qquad \text{-------------------------------} \quad (3)$$

With this counting number n_0 we obtain rational numbers a in η and b not in η satisfying (2). That is,

$$a^2 < y < b^2 < \left(a + \frac{1}{n_0}\right)^2 . \text{ ------------------------- (4)}$$

Now, $\left(a + \frac{1}{n_0}\right)^2 - a^2 = \frac{1}{n_0}\left(\frac{1}{n_0} + 2a\right) < \frac{1}{n_0}(1 + 2k_0)$ since $\frac{1}{n_o} < 1$ and $a < k_0$ because $a \in \eta$ and $k_0 \notin \eta$. Therefore, $\left(a + \frac{1}{n_0}\right)^2 < a^2 + \frac{1}{n_0}(1 + 2k_0) < a^2 + \frac{x_0 - y}{2}$ by (3). Hence, by (4), $\left(a + \frac{1}{n_0}\right)^2 < y + \frac{x_0 - y}{2} = \frac{x_0 + y}{2} < \frac{x_0 + x_0}{2} = x_0$, since $y < x_0$. Now, since $x_0 \in \xi$, for any rational number $k \notin \xi$, $x_0 < k$. Therefore, for all $k \notin \xi$, $\left(a + \frac{1}{n_0}\right)^2 < k$. This means $\left(a + \frac{1}{n_0}\right) \in \eta$. But $\left(a + \frac{1}{n_0}\right) \notin \eta$ because $\left(a + \frac{1}{n_0}\right) > b$, $b \notin \eta$ and η is a (left) cut. Thus, we have a contradiction and we conclude that $y \in \eta^2$. This is true for any rational number y in ξ and so $\xi \subseteq \eta^2$. Thus, $\eta^2 = \xi$. This completes the proof of Lemma 6.

For completeness, we remark that a candidate for the N-th root of a real number $\xi > \varphi(0)$ would be the following real number,

$$\eta_N = \varphi(0) \cup \{x \in \mathbb{Q} : x^N < k \text{ for all rational number } k \notin \xi\}.$$

The verification that η_N is a real number is similar to that for $N = 2$ as indicated in the proof therein. The proof that $(\eta_N)^N = \xi$ is the same as that for $N = 2$ except that in place of (*), we use the existence of a counting number n_0 such that

$$n_0\left(\frac{x_0 - y}{2}\right) > 1 + {}^N C_1\, k_0 + {}^N C_2\, k_0^2 + \ldots + {}^N C_{N-1}\, k_0^{N-1},$$

for some fixed rational number $k_0 \notin \eta$.

The uniqueness of the ordering on the real numbers is summarised as follows.

Theorem 7. The real number field $\mathbb{R}$ has exactly one total ordering. That is to say, if there is another positive cone P' for $\mathbb{R}$, then $P' = P$.

Proof. Suppose P' is another positive cone, satisfying $\mathbb{R} = P' \cup -P' \cup \{\varphi(0)\}$ and that for any real numbers $\xi, \eta \in P'$, $\xi + \eta, \xi\,\eta \in P'$. Take any real number $\xi \neq \varphi(0)$, either ξ

$\in P'$ or $\xi \in -P'$. If $\xi \in P'$, then $\xi^2 \in P'$. If $\xi \in -P'$, then $-\xi \in P'$ and so by Lemma 15, Chapter 4, $\xi^2 = (-\xi)^2 \in P'$. Therefore, for any $\xi \neq \varphi(0)$, $\xi^2 \in P'$. For any $\xi \in P$, $\xi > \varphi(0)$ and so by Lemma 6, there exists a real number $\eta > \varphi(0)$ such that $\xi = \eta^2$. Therefore, $\xi \in P'$. Hence $P \subseteq P'$. Thus, the complement $\mathbb{R} - P'$ of P' satisfies $\mathbb{R} - P' \subseteq \mathbb{R} - P$. *That is*, $-P' \cup \{\varphi(0)\} \subseteq -P \cup \{\varphi(0)\}$. Since $\varphi(0) \notin -P'$, $-P' \subseteq -P$. It follows then that $-(-P') \subseteq -(-P)$. Since for any subset of real numbers M, $-(-M) = M$, by Lemma 13 of Chapter 4, we conclude that $P' = -(-P') \subseteq -(-P) = P$. Thus, $P' = P$ and so the total ordering on $\mathbb{R}$ is unique.

For any complete totally ordered field $\mathcal{F}$, there is a subfield isomorphic to the rational numbers. This is seen as follows. Let 1 denote the identity element with respect to multiplication. Then $\mathbb{Q}$ is embedded as the subfield

$$\{\frac{m1}{n1} : m \text{ an integer and } n \text{ a counting number}\},$$

where we write $\frac{1}{n1}$ for the multiplicative inverse of $n1$. Let us see if this gives a new embedding of the rational number $\mathbb{Q}$ into our real numbers $\mathbb{R}$. The identity element for multiplication is $\varphi(1)$. The embedding of the rational number $\frac{m}{n}$ is given by $\frac{m\varphi(1)}{n\varphi(1)} = \frac{\varphi(m)}{\varphi(n)} = \varphi(m)\varphi\left(\frac{1}{n}\right) = \varphi\left(\frac{m}{n}\right)$, which is the same as the original embedding φ in Chapter 2. In the above we have used the fact that $\frac{1}{\varphi(n)} = \varphi\left(\frac{1}{n}\right)$. This is seen as follows. Since $\varphi(n)\varphi\left(\frac{1}{n}\right) = \varphi\left(n\frac{1}{n}\right) = \varphi(1)$, by Lemma 14 of Chapter 4,

$$\frac{1}{\varphi(n)} = \frac{1}{\varphi(n)}\varphi(1) = \frac{1}{\varphi(n)}\left(\varphi(n)\varphi\left(\frac{1}{n}\right)\right) = \left(\frac{1}{\varphi(n)}\varphi(n)\right)\varphi\left(\frac{1}{n}\right) = \varphi(1)\varphi\left(\frac{1}{n}\right) = \varphi\left(\frac{1}{n}\right).$$

The Archimedean property of a complete totally ordered field is a consequence of its completeness. The following is proven similarly as Proposition 5 of Chapter 1.

Theorem 8. Any complete totally ordered field is Archimedean.

As a consequence of the Archimedean property of a complete totally ordered field $\mathcal{F}$, the set of embedded rational numbers is *dense* in it. That is, for any β, γ in $\mathcal{F}$ and $\beta < \gamma$, there exists a rational number $\frac{m}{n}$ with m an integer and n a counting number such that $\beta < \frac{m1}{n1} < \gamma$. The proof is exactly the same as the proof of Corollary 7 of Chapter 1. We shall be using this fact again and again.

We have shown that the real numbers $\mathbb{R}$ is a complete totally ordered field. Does there exist a complete totally ordered field very different from $\mathbb{R}$? The answer is no, at least in the sense that though the objects or constructs may be different, nevertheless we can change from one object or construct to another for all practical purposes. This is what the next theorem says.

Theorem 9. For each complete totally ordered field $\mathcal{F}$, there is an order preserving isomorphism of $\mathcal{F}$ into $\mathbb{R}$ and this isomorphism is unique. That is to say, there is ONLY one such isomorphism.

We shall prove this theorem in stages. First, we shall define the isomorphism as stated in Theorem 9.

Lemma 10. For any β in a complete totally ordered field $\mathcal{F}$, the subset $M = \{x \in \mathbb{Q} : x1 < \beta\}$ is a real number in $\mathbb{R}$. (Here, we write for $x = \frac{m}{n}$, $x1 = \frac{m1}{n1}$.)

Proof. 1. M is not empty. This is seen as follows. For any β in $\mathcal{F}$, either $\beta > 0$ the identity element for addition, or $\beta < 0$ or $\beta = 0$. If $\beta > 0$, then $0 = (0)1 < \beta$ and so $0 \in M$ and $M \neq \varnothing$. If $\beta < 0$, then by the Archimedean property of $\mathcal{F}$, there exists a counting number n such that $n1 > (-\beta)$ and so $-n1 = (-n)1 < \beta$ and so $-n \in M$ and once more $M \neq \varnothing$. If $\beta = 0$, then any negative rational number is in M, since for any negative rational number q, $q1 < 0$. Obviously, for this case $M \neq \varnothing$.

2. $M \neq \mathbb{Q}$. If $\beta > 0$, then by the Archimedean property of $\mathcal{F}$, there exists a counting number n such that $n1 > \beta$. Thus, we have a rational number n not in M. If $\beta \leq 0$, then for any rational number $q > 0$, $\beta < q1$, since $q1 > 0$. That means $q \notin M$ and so $M \neq \mathbb{Q}$.

3. Take any y in M. Then $y1 < \beta$. Therefore, for any rational number $x < y$, we have then $x1 < y1 < \beta$ and so by definition of M, $x \in M$.

4. M has no maximal element. Suppose M has a maximal element J. Then for all x in M, $x \leq J$ and so $x1 \leq J1 < \beta$. Again by a consequence of the Archimedean property of $\mathcal{F}$ (density of the embedded rational field), there exists a rational number q such that $J1 < q1 < \beta$. (Refer to the proof of Corollary 7 of Chapter 1 for verification.) Thus, $J < q$ and $q \in M$. This contradicts that $q \leq J$. Therefore, M has no maximal number.

We now conclude by Lemma 3 of Chapter 2 that M is a real number.

We can now proceed to define the isomorphism.

Definition 11. For any complete totally ordered field $\mathcal{F}$, define a mapping $\psi: \mathcal{F} \to \mathbb{R}$, by $\psi(\beta) = \{x \in \mathbb{Q} : x1 < \beta\}$, where 1 denotes the identity element for multiplication in $\mathcal{F}$, for β in $\mathcal{F}$. By Lemma 10, this mapping is well defined.

Lemma 12. The mapping $\psi: \mathcal{F} \to \mathbb{R}$, is an order preserving bijection.

Proof. ψ is injective. This is seen as follows. Let β and γ be in $\mathcal{F}$ such that $\beta \neq \gamma$. Then either $\beta < \gamma$ or $\beta > \gamma$. Without loss of generality, we may assume $\beta < \gamma$. By Definition 11, $\psi(\beta) \subseteq \psi(\gamma)$. Now, by the density of the embedded rational field (see Corollary 7 of Chapter 1), there exists a rational number q such that $\beta < q1 < \gamma$. Then $\psi(\beta) < \psi(\gamma)$ since $q \in \psi(\gamma)$ but $q \notin \psi(\beta)$. This implies that the mapping ψ is order preserving. Thus $\psi(\beta) \neq \psi(\gamma)$ for $\beta \neq \gamma$ and so ψ is injective. Take any real number ξ in $\mathbb{R}$. Then consider the set $\kappa = \{x1 : x \in \xi\} \subseteq \mathcal{F}$. This set is bounded above. This is because there exists a rational number $q \notin \xi$ such that for all x in ξ, $x < q$, since ξ is a left-cut. Therefore, for all x in ξ, $x1 < q1$ and so κ is bounded above by $q1$ in $\mathcal{F}$. Since $\mathcal{F}$ is complete, κ has a least upper bound or supremum say δ in $\mathcal{F}$. Then $\psi(\delta) = \xi$. This is seen as follows. Obviously, for any rational number y in $\psi(\delta)$, $y1 < \delta$ and so $y1 \in \kappa$. This is deduced below. If on the contrary, $y1 \notin \kappa$, then $y > z$ for all z in ξ and so $y1 > z1$ and $y1$ would be an upper bound of κ. Thus, $y1 \geq \delta$, contradicting $y1 < \delta$. Hence, $y1 \in \kappa$ and so $y \in \xi$. This means $\psi(\delta) \subseteq \xi$. Obviously, $\xi \subseteq \psi(\delta)$, since the above argument can be reversed as follows. For any rational number x in ξ, $x1 \in \kappa$. Thus, $x1 \leq \delta$, since δ is the least upper bound of κ. We claim that $x1 \neq \delta$. This is because if $x1 = \delta$, then for all y in ξ, $y1 \leq x1$. Thus, for all y in ξ, $y \leq x$. Therefore, x would be a maximal number for ξ, contradicting that ξ has no maximal number. Thus, for all x in ξ, $x1 < \delta$ and so $x \in \psi(\delta)$. This implies that $\xi \subseteq \psi(\delta)$. Thus, $\psi(\delta) = \xi$. This shows that ψ is surjective and so is bijective.

Lemma 13. The mapping $\psi: \mathcal{F} \to \mathbb{R}$, is a field homomorphism. That is, for any β and γ in $\mathcal{F}$,

1. $\psi(\beta + \gamma) = \psi(\beta) + \psi(\gamma)$ and
2. $\psi(\beta\gamma) = \psi(\beta)\, \psi(\gamma)$.

Proof. 1. Firstly, we shall show that $\psi(\beta) + \psi(\gamma) \leq \psi(\beta + \gamma)$, that is $\psi(\beta) + \psi(\gamma) \subseteq \psi(\beta + \gamma)$. For any x in $\psi(\beta)$, $x1 < \beta$ and for any y in $\psi(\gamma)$, $y1 < \gamma$. Therefore, for any $x + y$ in $\psi(\beta) + \psi(\gamma)$ with $x \in \psi(\beta)$ and $y \in \psi(\gamma)$, $(x + y)1 = x1 + y1 < \beta + \gamma$. Hence, $x + y \in \psi(\beta + \gamma)$. Therefore, $\psi(\beta) + \psi(\gamma) \subseteq \psi(\beta + \gamma)$.

Now, take any z in $\psi(\beta + \gamma)$. Then $z1 < \beta + \gamma$ and so $z1 - \beta < \gamma$. Then by the density of the embedded rational field in $\mathcal{F}$, there exists a rational number q such that $z1 - \beta < q1 < \gamma$. Then we have $(z - q)1 = z1 - q1 < \beta$ and so $z - q \in \psi(\beta)$. Obviously, $q \in \psi(\gamma)$. Thus, $z = (z - q) + q \in \psi(\beta) + \psi(\gamma)$. This is true for any z in $\psi(\beta + \gamma)$ and so $\psi(\beta + \gamma) \subseteq \psi(\beta) + \psi(\gamma)$. Therefore, $\psi(\beta) + \psi(\gamma) = \psi(\beta + \gamma)$.

2. Firstly, we note that $\psi(0) = \varphi(0)$ the identity element for addition in $\mathbb{R}$. As a consequence of part (1) above, for any β in $\mathcal{F}$, $\psi(-\beta) = -\ \psi(\beta)$. (This can be deduced in a similar manner as in the proof of the corresponding fact for the embedding φ of the rational numbers into $\mathbb{R}$.) Note also that $\psi(1) = \varphi(1)$. If any one of β and γ is 0, then we have nothing to prove, since $\psi(\beta\,\gamma) = \psi(0) = \varphi(0)$ and since $\psi(\beta)\,\psi(\gamma)$ is either $\psi(\beta)\,\psi(0) = \psi(\beta)\varphi(0)$ or $\psi(0)\,\psi(\gamma) = \varphi(0)\,\psi(\gamma)$ and is always the same as $\varphi(0)$. It is sufficient to show Part 2 for β and γ not equal to 0 the identity element for addition in $\mathcal{F}$. We shall divide the remaining of the proof into two cases, firstly for β, $\gamma > 0$, then for either $\beta > 0$ and $\gamma < 0$ or $\gamma > 0$ and $\beta < 0$ or β, $\gamma < 0$.

Now, assume β, $\gamma > 0$. Note that for any $\delta > 0$ in $\mathcal{F}$, $\psi(\delta) > \varphi(0)$ and so by Lemma 2 of Chapter 4, $\{0\} \cup \varphi(0) \subseteq \psi(\delta)$. Note that we have $\psi(\beta)$, $\psi(\gamma)$, $\psi(\beta\,\gamma) > \varphi(0)$. Then we have $\psi(\beta)\,\psi(\gamma) > \varphi(0)$. Hence, $\{0\} \cup \varphi(0) \subseteq \psi(\beta)\,\psi(\gamma)$, $\psi(\beta\,\gamma)$. So, for any $x \leq 0$ in $\psi(\beta)\,\psi(\gamma)$, $x \in \psi(\beta\,\gamma)$. Therefore, we only need to show that for any $x > 0$ in $\psi(\beta)$ and any $y > 0$ in $\psi(\gamma)$, $x\,y \in \psi(\beta\,\gamma)$. Now, $x > 0$ in $\psi(\beta)$ implies that $x1 < \beta$ and likewise, $y > 0$ in $\psi(\gamma)$ implies that $y1 < \gamma$. Therefore, $(x\,y)1 = x1\ y1 < \beta\,\gamma$ and so $x\,y \in \psi(\beta\,\gamma)$. We can now conclude that $\psi(\beta)\,\psi(\gamma) \subseteq \psi(\beta\,\gamma)$. Next, we shall show that $\psi(\beta\,\gamma) \subseteq \psi(\beta)\,\psi(\gamma)$. Again, we need only show that for any $z > 0$ in $\psi(\beta\,\gamma)$, $z \in \psi(\beta)\,\psi(\gamma)$. (The proof of this part resembles that of the proof for Lemma 9, Part 6 of Chapter 4.) Now, since $z > 0$ and is in $\psi(\beta\,\gamma)$, $0 < z1 < \beta\,\gamma$. Therefore, multiplying the inequality by the multiplicative inverse of γ, we get $0 < z1\frac{1}{\gamma} < \beta$. By the density of the embedded rational field in $\mathcal{F}$, there exists a rational number a such that $0 < z1\frac{1}{\gamma} < a1 < \beta$. Then $a \in \psi(\beta)$ and $0 < z1 < a1\ \gamma < \beta\,\gamma$. Thus, $0 < \frac{z1}{a1} = \frac{z}{a}1 < \gamma$ and so $\frac{z}{a} \in \psi(\gamma)$. Therefore, $z = a\ \frac{z}{a} \in \psi(\beta)\,\psi(\gamma)$. This is true for any z in $\psi(\beta\,\gamma)$ and so $\psi(\beta\,\gamma) \subseteq \psi(\beta)\,\psi(\gamma)$ and the equality $\psi(\beta\,\gamma) = \psi(\beta)\,\psi(\gamma)$ follows.

Now, for $\beta > 0$ and $\gamma < 0$, $\psi(\beta) > \varphi(0)$ and $\psi(\gamma) < \varphi(0)$. Therefore, by the preceding remark, $\psi(\beta)\,\psi(\gamma) = -\,(\psi(\beta)(-\psi(\gamma)) = -((\psi(\beta)(\psi(-\gamma)) = -\ \psi(\beta(-\gamma))$ by the first case, since β, $(-\gamma) > 0$. Thus, $\psi(\beta)\,\psi(\gamma) = \ \psi(-(\beta(-\gamma))) = \psi(\beta\,\gamma)$. Similarly, if $\beta < 0$ and $\gamma > 0$, by interchanging β with γ, we can show that $\psi(\beta)\,\psi(\gamma) = \psi(\gamma)\,\psi(\beta) = \psi(\beta\,\gamma)$. Finally, when β, $\gamma < 0$, then $\psi(\beta)\,\psi(\gamma) = (-\psi(\beta))(-\psi(\gamma)) = \psi(-\beta)\,\psi(-\gamma) = \psi((-\beta)(-\gamma)) = \psi(\beta\,\gamma)$, by the first case. This completes the proof of Part 2 of Lemma 13.

Lemma 12 and Lemma 13 imply that there is only one complete totally ordered field upto isomorphism. The fields $\mathcal{F}$ and $\mathbb{R}$ may be constructs of different nature but they are all isomorphic in the sense that the mapping $\psi : \mathcal{F} \to \mathbb{R}$ is a *bijective* homomorphism, an isomorphism. More is true, that is, there is one and only one isomorphism of any complete totally ordered field $\mathcal{F}$ into $\mathbb{R}$. This is a consequence of the following theorem, which is of some interest on its own merit.

Theorem 14. Any automorphism Ψ: $\mathbb{R} \to \mathbb{R}$ (i.e. isomorphism of $\mathbb{R}$ into itself) is the identity homomorphism.

Proof. We observe that any automorphism Ψ would be order preserving. $\Psi(x^2) = \Psi(x)\Psi(x) = (\Psi(x))^2 > 0$ if $x \neq 0$. Here, we write 0 for $\varphi(0)$. Now, for any ξ in the positive cone P, $\xi = \eta^2$ for some η and so $\Psi(\xi) = \Psi(\eta^2) > 0$ so that $\Psi(\xi) \in P$. That means Ψ maps positive cone into positive cone. Hence, if $\xi, \eta \in \mathbb{R}$ and $\xi > \eta$, then $\xi - \eta \in P$ and so $\Psi(\xi) - \Psi(\eta) = \Psi(\xi - \eta) \in P$ and $\Psi(\xi) > \Psi(\eta)$. We shall write 1 for $\varphi(1)$ in $\mathbb{R}$. Then $n1 = n\varphi(1) = \varphi(n)$ for any integer n. For any counting number n, $\varphi\left(\frac{1}{n}\right) = \frac{1}{\varphi(n)} = \frac{1}{n\varphi(1)} = \frac{1}{n1}$ and so for any rational number $\frac{m}{n}$, $\varphi\left(\frac{m}{n}\right) = \varphi(m)\varphi\left(\frac{1}{n}\right) = \frac{m1}{n1}$. Thus, we can write $\varphi(\mathbb{Q}) = \{\frac{m1}{n1}$: m is an integer and n a counting number$\}$.

We shall show next that Ψ is the identity homomorphism on the embedded rational number fields $\varphi(\mathbb{Q})$. $\Psi(1) = 1$ since Ψ is a field homomorphism. Then it follows that $\Psi(-1) = -\Psi(1) = -1$. Therefore, for any integer n, $\Psi(n1) = n1$. This means that Ψ is the identity homomorphism on the embedded integers. Now, for any rational numbers, $\frac{m}{n}$, with m an integer and n a counting number,

$$\Psi\left(\frac{m1}{n1}\right) = \Psi(m1)\Psi\left(\frac{1}{n1}\right) = m1\frac{1}{n1} = \frac{m1}{n1},$$

assuming that $\Psi\left(\frac{1}{n1}\right) = \frac{1}{n1}$. We shall deduce this fact as follows.

$1 = \Psi(1) = \Psi\left(\frac{n1}{n1}\right) = \Psi(n1)\Psi\left(\frac{1}{n1}\right) = n1\Psi\left(\frac{1}{n1}\right)$. Therefore, multiplying by $\frac{1}{n1}$ on both sides, we get $\frac{1}{n1} = \Psi\left(\frac{1}{n1}\right)$. Thus, Ψ is the identity homomorphism on the embedded rational field.

Now, let ξ be a real number in $\mathbb{R}$ and $\xi \notin \varphi(\mathbb{Q})$. Suppose on the contrary that $\Psi(\xi) \neq \xi$, then either $\Psi(\xi) > \xi$ or $\Psi(\xi) < \xi$. Suppose $\Psi(\xi) > \xi$. Then by the Archimedean property of $\mathbb{R}$, there exists a counting number n such that

$$n(\Psi(\xi) - \xi) > 1.$$

Thus $\frac{1}{n1} < \Psi(\xi) - \xi$. Now, by Corollary 10 of Chapter 2, there exist rational numbers a and b such that $\varphi(a) < \xi < \varphi(b)$ and $\varphi(b - a) = \varphi(b) - \varphi(a) < \varphi\left(\frac{1}{n}\right)$. In our present notation, we have $a1 < \xi < b1$ and $b1 < a1 + \frac{1}{n1}$. Since Ψ is order preserving and is the identity homomorphism on the embedded rational subfield, it follows that $\Psi(\xi) < \Psi(b1) = b1 < a1 + \frac{1}{n1} < \xi + \frac{1}{n1}$. Thus, we get $\Psi(\xi) - \xi < \frac{1}{n1}$, contradicting $\Psi(\xi) - \xi > \frac{1}{n1}$. Similarly, if $\Psi(\xi) < \xi$, by the Archimedean property of $\mathbb{R}$, there exists a counting number m such that $m(\xi - \Psi(\xi)) > 1$ and so $\frac{1}{m1} < \xi - \Psi(\xi)$. With this counting number m, again by Corollary 12 of Chapter 2, there exist rational numbers a and b such that $a1 < \xi < b1$ and $b1 < a1 + \frac{1}{m1}$. Then we have $\xi < b1 < a1 + \frac{1}{m1}$. Applying the automorphism to the inequality $a1 < \xi$, we get $a1 = \Psi(a1) < \Psi(\xi)$. Thus, $\xi < a1 + \frac{1}{m1} < \Psi(\xi) + \frac{1}{m1}$ and so $\xi - \Psi(\xi) < \frac{1}{m1}$, contradicting $\xi - \Psi(\xi) > \frac{1}{m1}$. Therefore, $\Psi(\xi) = \xi$ and we conclude that Ψ is the identity homomorphism on $\mathbb{R}$. This completes the proof.

Theorem 15. There is exactly one field isomorphism from any complete totally ordered field $\mathcal{F}$ into $\mathbb{R}$.

Proof. Suppose there are two isomorphisms $\Psi, \Omega: \mathcal{F} \to \mathbb{R}$. Let $\Omega^{-1}: \mathbb{R} \to \mathcal{F}$ be the inverse function of Ω. Then Ω^{-1} is also an isomorphism. This is seen as follows. For any ξ, η in $\mathbb{R}$,

$$\Omega(\Omega^{-1}(\xi) + \Omega^{-1}(\eta)) = \Omega(\Omega^{-1}(\xi)) + \Omega(\Omega^{-1}(\eta))) = \xi + \eta = \Omega(\Omega^{-1}(\xi + \eta)).$$

Since Ω is injective, we have then $\Omega^{-1}(\xi) + \Omega^{-1}(\eta) = \Omega^{-1}(\xi + \eta)$. Similarly,

$$\Omega(\Omega^{-1}(\xi)\,\Omega^{-1}(\eta)) = \Omega(\Omega^{-1}(\xi))\,\Omega(\Omega^{-1}(\eta)) = \xi\,\eta = \Omega(\Omega^{-1}(\xi\,\eta)).$$

Again, since Ω is injective, we have then that $\Omega^{-1}(\xi)\,\Omega^{-1}(\eta) = \Omega^{-1}(\xi\,\eta)$. Therefore, $\Omega^{-1}: \mathbb{R} \to \mathcal{F}$ is a field isomorphism, since Ω^{-1} is also bijective. Then the composite isomorphism, $\Psi \circ \Omega^{-1}: \mathbb{R} \to \mathbb{R}$, is an automorphism. Therefore, by Theorem 14, $\Psi \circ \Omega^{-1} = 1_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is the identity homomorphism. Then $\Omega = 1_{\mathbb{R}} \circ \Omega = (\Psi \circ \Omega^{-1}) \circ \Omega = \Psi \circ (\Omega^{-1} \circ \Omega) = \Psi \circ 1_{\mathbb{R}} = \Psi$. This completes the proof of Theorem 15.

Proof of Theorem 9. This is now an easy consequence of Lemma 12, Lemma 13 and Theorem 15.

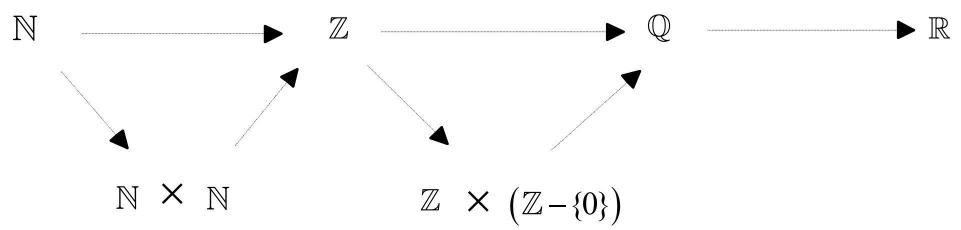
In the next chapter we shall trace the genealogy of the real numbers starting from the natural number system. We shall see how special and unique is the real number system.

CHAPTER SIX

Genealogy

Like the drifting seed in an ocean
Asking by whom has cast its existence
Whence we came, whimpering
Who are we? What are we?
Complicated, complex and perplexing questions
That has no answer in what or who we are
Only what we do
Even then only Silence shall answer

We shall now trace the genealogy of the real numbers starting from the indivisible atomic individual, the empty set. We shall observe the underlying theme, the first construct, the set of natural numbers (which now for this discussion includes the number 0) is unique in the sense that there is only one such natural number system up to isomorphism, a consequence of the *Dedekind's Recursion Theorem*, and that each embedding of the natural number system and subsequent constructs into the next is unique. The following chart depicts the lineage of the real numbers.



It is clear that the construct at each stage involves a new kind of object with the embedded starting object taking on a new and more complicated form. Let us start with VON NEUMANN's definition of the natural number system, $\mathbb{N}$. With the axiom of existence, we have the existence of an empty set. The number 0 is defined to be the empty set $\varnothing$, 1 is defined to be the set $\{\varnothing\}$, 2 is defined to be $\{\varnothing, \{\varnothing\}\}$ and, in general, $n + 1$ is defined to be $n \cup \{n\}$. It has two things going for it. Firstly, the number n has exactly n elements. Secondly, the ordering "<" is given by the relation "$\in$" ("is an element of"). For instance, 3 is defined by $2 \cup \{2\} = \{\varnothing, \{\varnothing\}\} \cup \{\{\varnothing, \{\varnothing\}\}\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$. Then $\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\} \in \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$. That is, 0, 1,2 < 3. And for $n > 0$, we have $0,1,\ldots, n-1 \in n$ and so $0,1,\ldots, n-1 < n$. Addition and multiplication can be defined by using the successor function, $S(n) = n \cup \{n\} = n + 1$ as follows. Addition can be defined recursively starting from $m + 0 = m$, by $m + S(n) = S(m+n)$ and multiplication, by $m \times 0 = 0$ and $m\times(n+1) = m\times n + m$. Then we can easily check that all the well-known rules of addition and multiplication are satisfied.

The next construct is the set of integers. To do this we shall need the notion of a Cartesian product and an equivalence relation as explained below.

Definition 6.1. Let A and B be two sets that may or may not be the same. An *ordered pair*, (a, b), where a is an element in A and b an element in B is defined to be the set $\{\{a\}, \{a, b\}\}$. That the element a comes first before b is determined by the set inclusion $\{a\} \subseteq \{a, b\}$. The *Cartesian product* of A and B, denoted by $A \times B$, is then the set consisting of all the ordered pairs, that is, $A \times B = \{(a, b)\colon a \in A, b \in B\}$.

In our usual sense of direction, for two natural numbers m and n, if $m > n$, then we would start from n and proceed to get to m by a sequence of application of the successor function S. The number of times that we apply S would give us a natural *number*. This number, as we can perceive, is $m-n$. We can think of this as applying the successor function S to the number 0 in $m + 0$ while keeping m fixed. We would have applied the successor function $m-n$ times. If we start from m and get to n then this would indicate an opposite direction. The direction is determined by the ordering $m > n$. So, going from m to n indicates a negative direction and hence gives rise to the "negative" of going from n to m. Hence, we would need an ordered pair (n, m) in order to take into account this sense of direction and, once we have this, we do not even need the ordering of m and n. So, we shall start with the Cartesian product of the set of natural numbers, $\mathbb{N}\times\mathbb{N}$. There is one complication. If $m > n$, then $m + a > n + a$. Thus, both pairs (n, m) and $(n + a, m + a)$ would represent the same *number*. We would want to use only one representative from the set $\{(n + a, m + a)\colon a \in \mathbb{N}\}$. So, we would need a way to distinguish different sets as representing different numbers. We would want to carve up the Cartesian product $\mathbb{N}\times\mathbb{N}$ into subsets, each of which represents a *number*. Elements in the set $\{(n+a, m+a)\colon a \in \mathbb{N}\}$ are *related* by the same difference. More precisely (a, b) and (c, d) are related if, and only if, $b - a = d - c$, *if we have defined subtraction*, but since we have not yet defined subtraction, (a, b) and (c, d) are related if, and only if, $a + d = b + c$. We shall need then the next concept in set theory, namely that of a relation and in particular, an equivalence relation.

Definition 6.2. Let A and B be two sets. A *relation* from A to B is a subset R of the Cartesian product $A \times B$. For an element a in A and an element b in B, we say a is *related* to b (and we write aRb) if, and only if, (a, b) is in R. If $A = B$, then we say the relation R is a relation on the set A.

Definition 6.3. Suppose R is a relation on a set A .

1. R is *reflexive* if for all a in A, aRa.

2. R is *symmetric* if for any a and b in A, aRb if, and only if, bRa.

3. R is *transitive* if for any a, b, c in A, aRb and bRc implies aRc.

We say R is an *equivalence relation* if R is reflexive, symmetric and transitive. Take an element a in A, the *equivalence class* of $[a]$ consists of all elements in A that are related to a. That is, $[a] = \{b \in A: bRa\}$. We can easily show that $[a] = [b]$ if, and only if, aRb. This would then carve up the set A into disjoint equivalence classes because $[a]\cap[b] = \varnothing$, if a is not related to b. A then becomes the disjoint union of these equivalence classes. We denote the set of all equivalence classes by A/R.

We now take A to be the Cartesian product $\mathbb{N}\times\mathbb{N}$. Define a relation $\sim$ on $\mathbb{N}\times\mathbb{N}$ by $(a, b) \sim (c, d)$ if and only if $a + d = b + c$. Thus, our relation is the subset of $(\mathbb{N}\times\mathbb{N})\times(\mathbb{N}\times\mathbb{N})$ given by $\{((a,b), (c,d))\in(\mathbb{N}\times\mathbb{N})\times(\mathbb{N}\times\mathbb{N}): a + d = b + c\}$. We then have a new way of looking at the natural numbers. We can easily show that $\sim$ is an equivalence relation. Denote the equivalence class of (a, b) by $[a, b]$. Then 0 would correspond to $[0,0] = [1,1]$, 1 would correspond to $[0,1] = [1,2]$ and in general any natural number n would correspond to $[0, n]$. The set of equivalence classes $\mathbb{N}\times\mathbb{N}/\sim$ is then defined to be the integers $\mathbb{Z}$. Addition and multiplication are defined by $[a, b] + [c, d] = [a + c, b + d]$ and $[a, b] \times [c, d] = [ad + bc, ac + b\,d]$. We can easily check that both addition and multiplication are meaningful, that is, if we take different representatives from each class of $[a, b]$ and $[c, d]$ and follow the definition, we should end up in the same equivalence class. Observe that both addition and multiplication are commutative and associative. The identity element for addition is $[0,0]$ and the identity element for multiplication is $[0,1]$. Then $\mathbb{Z}$ is an abelian group with respect to addition. The additive inverse $-[a, b]$ for any $[a, b]$ in $\mathbb{Z}$ is $[b, a]$ because $[a, b] + [b, a] = [a + b, b + a] = [0, 0]$. $\mathbb{Z}$ has no zero divisor because for any x and y in $\mathbb{Z}$, $x\,y = [0,0]$ if, and only if, $x = [0,0]$ or $y = [0,0]$ the zero number of $\mathbb{Z}$. We can now embed the natural numbers $\mathbb{N}$ into $\mathbb{Z}$, by g: $\mathbb{N} \to \mathbb{Z}$, defined by $g(n) = [0,n]$, for each n in $\mathbb{N}$. The order relation is then defined by the *positive cone* $P = \{[0, n]: n \in \mathbb{N} \text{ and } n \neq 0\}$. For any x and y in $\mathbb{Z}$, $x < y$ if, and only if, $y - x \in P$. Indeed, the embedding, g: $\mathbb{N} \to \mathbb{Z}$, respects addition, multiplication and the orderings on both $\mathbb{N}$ and $\mathbb{Z}$. If we now identify $[0, n]$ with n, then since the additive inverse $-[a, b] = [b, a]$, we have then the negative integer, $-n$ identified with $[n, 0]$. Then $\mathbb{Z}$ is none other than the embedded natural numbers and their negatives. The positive cone can then be written as $\{n: n \in g(\mathbb{N}) \text{ and } n \neq 0\}$.

The next construct is the system of rational numbers. Now, for the symbols $\frac{a}{b}$ and $\frac{c}{d}$, we can take them as ratios for the time being, representing the same rational number if $a\,d = b\,c$. If a is not equal to c or if b is not equal to d, then these are really distinctively looking ratios even when $a\,d = b\,c$ but we consider them as the same. This brings us to the use of an equivalence relation again. Since there are pairs involved and the denominator rightly should not be zero, we consider the Cartesian product, $\mathbb{Z}\times(\mathbb{Z}-\{0\})$, where we write 0 for [0,0]. For (a, b) and (c, d) in $\mathbb{Z}\times(\mathbb{Z}-\{0\})$, we define $(a, b) \sim (c, d)$ if, and only if, $a\,d = b\,c$. This relation is determined by the subset $\{((a,b), (c,d)) \in (\mathbb{Z}\times(\mathbb{Z}-\{0\}))\times(\mathbb{Z}\times(\mathbb{Z}-\{0\})): a\,d = b\,c\}$ of $(\mathbb{Z}\times(\mathbb{Z}-\{0\}))\times(\mathbb{Z}\times(\mathbb{Z}-\{0\}))$. We can easily show that this relation is an equivalence relation and we denote the equivalence class of (a, b) by $\left[\frac{a}{b}\right]$. Define the rational numbers to be the set of all equivalence classes, that is $\mathbb{Q} = \mathbb{Z}\times(\mathbb{Z}-\{0\})/\sim$. Addition is defined by $\left[\frac{a}{b}\right] + \left[\frac{c}{d}\right] = \left[\frac{ad+bc}{bd}\right]$ and multiplication by $\left[\frac{a}{b}\right] \times \left[\frac{c}{d}\right] = \left[\frac{ac}{bd}\right]$. Then the identity element for addition is $\left[\frac{0}{1}\right]$ and the identity element for multiplication is $\left[\frac{1}{1}\right]$. Addition and multiplication are easily shown to be commutative, associative and obey the distributive law. Obviously, the additive inverse of $\left[\frac{a}{b}\right]$ is $\left[\frac{-a}{b}\right]$ and the multiplicative inverse of $\left[\frac{a}{b}\right] \neq \left[\frac{0}{1}\right]$ is given by $\left[\frac{b}{a}\right]$. Then $\mathbb{Q}$ with these two operations of addition and multiplication is a field. The positive cone for $\mathbb{Q}$, as explained in Chapter 1, is $P = \left\{\left[\frac{a}{b}\right] : a, b > 0\right\}$ and gives rise to a total ordering on $\mathbb{Q}$, given by, for any x, y in $\mathbb{Q}$, $x > y$ if, and only if, $x - y \in P$. Thus, $\mathbb{Q}$ is a totally ordered field. The embedding, h: $\mathbb{Z} \to \mathbb{Q}$, given by, $h(x) = \left[\frac{x}{1}\right]$, embeds the integers $\mathbb{Z}$ in $\mathbb{Q}$. This embedding is unique, respects the addition, multiplication and the orderings on $\mathbb{Z}$ and $\mathbb{Q}$.

Finally, the last stage is the construction of the real numbers as described in Chapter 2, involving Dedekind cuts. Thus, the set of real numbers sits on top of a heap of other constructs. Its nature is as yet unexplained as the empty set itself. We might say the real number system has no nature but that the logical dynamic process of its creation comes close to being described as its nature. The end products say much less than the process in arriving at these products.

In Chapter 5, we have seen that the real number system contains the n-th root of any positive real number for any counting number n. This was a constructive demonstration. We can divide the real numbers into two classes of numbers, the algebraic numbers and the transcendental numbers. The algebraic numbers are those real numbers that satisfy a polynomial equation with coefficients in the rational numbers, whereas a transcendental number does not. Now, a Cantorian argument shows that the algebraic numbers can be at most countably infinite, a consequence of the result that the union of countably infinite family of countable infinite sets is countable. The union of two countable sets is countable again irrespective of whether the sets are infinite or not. An easy argument below will then prove that the set of transcendental numbers is uncountably infinite, if we can show that the set of real numbers is uncountably infinite. If the set of transcendental numbers is countable, then the set of real numbers being the union of the algebraic numbers and the transcendental numbers, would then be the union of two countable sets and is therefore countable, contradicting that the set of real numbers is uncountable. Hence, the set of transcendental numbers is uncountably infinite. We shall next show that the set of real numbers is uncountably infinite by employing a very simple argument of Cantor. Firstly, we shall need a common representation of the non-zero real numbers by non-terminating decimals.

6.4. Decimal Expressions.

We shall redefine each real number > 0 as the supremum of a sequence of numbers. Take any $\xi > 0$ in $\mathbb{R}$. Choose the greatest integer $a_0 < \xi$. We can verify its existence as in Corollary 9 of Chapter 1. Then we have $a_0 + 1 \geq \xi$. Consider next the numbers $a_0 + \frac{0}{10}$, $a_0 + \frac{1}{10}$, $a_0 + \frac{2}{10}$, $a_0 + \frac{3}{10}$, $a_0 + \frac{4}{10}$, $a_0 + \frac{5}{10}$, $a_0 + \frac{6}{10}$, $a_0 + \frac{7}{10}$, $a_0 + \frac{8}{10}$, $a_0 + \frac{9}{10}$. Then take the largest of these which is $< \xi$. That is, we choose a_1 in $\{0,1,2,\ldots,9\}$ such that $a_0 + \frac{a_1}{10} < \xi \leq a_0 + \frac{a_1}{10} + \frac{1}{10}$. Let $d_1 = a_0 + \frac{a_1}{10}$. Then $d_1 < \xi \leq d_1 + \frac{1}{10}$. Now, choose integer a_2, $0 \leq a_2 \leq 9$, such that $d_1 + \frac{a_2}{10^2} < \xi \leq d_1 + \frac{a_2}{10^2} + \frac{1}{10^2}$. Let now $d_2 = d_1 + \frac{a_2}{10^2} = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2}$. We have, $d_2 < \xi \leq d_2 + \frac{1}{10^2}$. Continuing like this, we shall obtain a sequence $a_0, a_1, a_2, \ldots$, of integers, $0 \leq a_i \leq 9$, such that if $d_n = d_{n-1} + \frac{a_n}{10^n} = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n}$, then $d_n < \xi \leq d_n + \frac{1}{10^n}$. The symbol $a_0 \cdot a_1\ a_2\ a_3 \ldots$ is called the *non-terminating decimal expansion* of ξ. Note that the set $\{d_1, d_2, d_3, d_4, \ldots\}$ is bounded above by ξ. Then the least upper bound or supremum of this set is ξ. We deduce this as follows. If $\xi \neq \sup\{d_1, d_2, d_3, d_4, \ldots\} = M$, then $\xi > M$. Then by Corollary 6 of Chapter 1, there exists a counting number m such that $\frac{1}{m} < \xi - M$ and so $M + \frac{1}{m} < \xi$. Now, take a non-terminating decimal expansion $0 \cdot c_1\ c_2\ c_3 \ldots$ for $\frac{1}{m}$. Let L be the first integer such that $c_L \neq 0$. Then $\frac{1}{10^L} < \frac{1}{m}$. Therefore, $M + \frac{1}{10^L} < M + \frac{1}{m} < \xi$. Hence, since $M = \sup\{d_1, d_2, d_3, d_4 \ldots\}$, $d_L + \frac{1}{10^L} \leq M + \frac{1}{10^L} < \xi$ and this contradicts $\xi \leq d_L + \frac{1}{10^L}$. Therefore, $\xi = \sup\{d_1, d_2, d_3, d_4 \ldots\}$.

This representation of the real number $\xi > 0$ is unique. Suppose two non-terminating decimal expressions $a_0 \cdot a_1\ a_2\ a_3 \ldots$ and $b_0 \cdot b_1\ b_2\ b_3 \ldots$ are such that for some integer $j \geq 0$, $a_j \neq$

b_j and $a_i = b_i$ for $i \le j - 1$. If $a_0 \cdot a_1\, a_2\, a_3 \ldots$ represents ξ and $b_0 \cdot b_1\, b_2\, b_3 \ldots$ represents η, we shall show that $\xi \ne \eta$. Suppose that $a_j < b_j$. By the hypothesis, we have

$$a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_{j-1}}{10^{j-1}} + \frac{a_j}{10^j} < \xi \le a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_{j-1}}{10^{j-1}} + \frac{a_j + 1}{10^j}.$$ But

$$a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_{j-1}}{10^{j-1}} + \frac{a_j + 1}{10^j} = b_0 + \frac{b_1}{10} + \frac{b_2}{10^2} + \ldots + \frac{b_{j-1}}{10^{j-1}} + \frac{a_j + 1}{10^j}$$

$$\le b_0 + \frac{b_1}{10} + \frac{b_2}{10^2} + \ldots + \frac{b_{j-1}}{10^{j-1}} + \frac{b_j}{10^j} < \eta.$$

Therefore, $\xi < \eta$ and so $\xi \ne \eta$. Similarly, if $a_j > b_j$, we can show that $\xi \ne \eta$. Thus, this way of representing any real number > 0 is unique. Notice that for $\xi = \frac{1}{2}$, the corresponding non-terminating expansion is the expression 0.499999999999……, which is a non-terminating expression with recurring '9'. To complete the picture, for any $\xi < 0$, we take the non-terminating decimal expansion for $-\xi$ and represent ξ by attaching the negative sign to the non-terminating decimal expansion for $-\xi$. This is consistent with the following observation. If $-\xi = \sup\{d_1, d_2, d_3, d_4 \ldots\}$, where the d_i's are as defined above in terms of the a_i's and $a_0 \cdot a_1\ a_2\ a_3 \ldots$ is the non-terminating decimal expansion of $-\xi$, then $\xi = -\sup\{d_1, d_2, d_3, d_4 \ldots\} = \inf\{-d_1, -d_2, -d_3, -d_4 \ldots\}$. One exception is 0 itself. It is simply represented by 0.0 a terminating decimal, that is, a decimal expression with recurring '0'.

Cantor's Theorem. The set of real numbers strictly between 0 and 1 is not countable.

Proof. The proof is a simple one, known as the Cantor's diagonal process.

Suppose on the contrary that the interval $\{x \in \mathbb{R} : 0 < x < 1\}$ is countable. Hence, there is a matching function (that is a bijective function) g: $\mathbb{N}_+ \to \{x \in \mathbb{R} : 0 < x < 1\}$ from the counting numbers $\mathbb{N}_+$ onto the interval $\{x \in \mathbb{R} : 0 < x < 1\}$. For each counting number n, let the decimal expansion of $g(n)$ be given by $0\cdot g_{n1}\, g_{n2}\, g_{n3}\, g_{n4}\, g_{n5} \ldots$. (Note here that for any real number ξ in the interval $\{x \in \mathbb{R} : 0 < x < 1\}$, the first term of the non-terminating decimal expansion of ξ is 0.) We then construct the following decimal expansion

$$0\cdot b_1\, b_2\, b_3 \ldots,$$

where b_j is chosen to be different from $g_{j\,j}$, 0 *and* 9. Then $0\cdot\ b_1\ b_2\ b_3 \ldots$ represents a real number $\zeta > 0$ and < 1 because $b_j \ne 0, 9$ for all j. In particular, $\zeta \ne g(j)$ for any counting number j, since $b_j \ne g_{j\,j}$. We have thus produced a real number, ζ, in the interval {x

$\in \mathbb{R} : 0 < x < 1\}$, which is not in the range of g, contradicting that g is bijective. This proves that the interval $\{x \in \mathbb{R} : 0 < x < 1\}$ cannot be countable and so the set of real numbers $\mathbb{R}$ is not countable.

We close this chapter with the poetic remark of Eric Temple Bell:

The algebraic numbers are spotted over the plane like stars against a black sky;

the dense blackness is the firmament of the transcendentals.

This image sums up just how little we know about the transcendentals and hence the real numbers. Apart from the well-known transcendentals, like the numbers π and e, practically none is readily available; any attempt to produce one will need quite an effort both in its construction as well as its authentication.

Epilogue

The water in a vessel is sparkling; the water in the sea is dark.

The small truth has words that are clear; the great truth has great silence.

Rabindranath Tagore

We have constructed the real numbers and have recovered the natural numbers, integers and rational numbers in this model of the real numbers. Henceforth, these numbers will take on new meanings. We have also shown that the properties of the real numbers are the essence of a complete totally ordered field. Are there other models? Yes, there are, and they are constructs of quite different nature. Each model has its own merit. Cantor based his model on fundamental sequences (Cauchy sequences), where every Cauchy sequence converges, a notion that is called *metric complete*. Weierstrass' model, based on nested intervals, is another that allows us to zoom in the position of a real number on the infinite real line. The more recent model of John Conway, in terms of Conway games, has its origin in a different conceptual consideration, that of calling number a game and is a subset of a bigger field that includes infinitely small and infinitely large 'numbers'. It allows one to look forward to an extension of the real numbers to non-Archimedean, non-standard numbers. Whatever models that we choose, we can give an axiomatic definition of the real numbers, based on the axioms for a complete totally ordered field. Undoubtedly, there is some gain in this approach, but there is a loss of familiarity with the object itself.

Let us think about this creation of the real numbers in an abstract and foundational way, free from what we may think numbers or indeed what natural numbers really are. Let us pause and think about the consequence of this creation. Starting from the empty set we have obtained the real numbers, using entirely set theoretic construction from von Neumann's natural number system. From the real numbers, we can give definitions of complex numbers, vector spaces, real and complex algebras and so on. These are the building blocks of more complicated mathematical objects, which in turn, are also building blocks of even more complicated mathematical objects and so on. If we accept the consistency of (Zermelo-Frankel) set theory, we can view the whole of mathematics as being built up from set theory, that is, giving it a unified axiomatic basis. This view, though controversial, gives us the assured meaning of rigour, at least, in a restricted portion of *analysis*. The question, whether the whole of mathematics can be based on an axiomatised logical footing, will continue to be debated well into the next century. On a note of pragmatism, the path that we have traversed in reaching the real number system is truly one of logical clarity and that is, perhaps how one should view the makings of the other branches of mathematics in this way. Gödel's incompleteness theorem does restrict us from saying that everything in mathematics is provable or not provable, but it does not make the slightest blemish on the beauty of theorems such as the *Fundamental Theorem of Calculus* and the *Euler-Poincaré*

Index Theorem. For the question on the foundation of mathematics, a good account is given in chapter 14 of *Numbers* by Ebbinghaus et al and a very rich extensive historical account in Chapter 51 of *Mathematical Thoughts from Ancient to Modern Times* (Volume 3) by Morris Kline.

The Archimedean property allows us to think of the real numbers as an infinite line. Thus, we can think of the real numbers as a continuum. How useful is this fact? When we say that the real number system is a line, what do we mean? Does the set of real numbers, being a continuum, gives us a justification to equate real numbers with the infinite line? Yes, to some extent it does. The question of what a real number is, fundamentally, is a philosophical one. To reconcile with the technicality of its construction, we think of the real numbers, in very different ways. Natural numbers and the integers and to some extent the rational numbers are something we have learnt and used so effectively since our school days. We do not view them in a way towards a foundational understanding; we do not think of them as objects or, in the case of the rational numbers, as embedded subfield, in the real number system we have constructed. When we talk about the real numbers, we think of them more in terms of the properties they possessed, that is the properties of a complete totally ordered field, but the embedded rational number field is not thought of in the same way. This is rightly so and we have two ways of thinking that suit our convenience. It is easy for a mathematician to switch back and forth from just plain rational numbers to embedded rational numbers in the real numbers (whether being thought of as a complete totally ordered field or a continuum). The layman would find this task difficult, to say the least and the real numbers remain mysterious to fathom.

Suggested Readings

The books listed here meant for further readings are good source for someone who wants to know what mathematics is about. Some of them are extremely technical and some are controversial. They all have something to say about the foundation of mathematics and in particular, the real numbers. They are chosen for the richness and depth of thoughts and the abundance of historical content, both inviting and challenging.

William Dunham, *Journey through Genius*, Penguin Books 1990.

(This is a very fascinating and captivating book to begin with.)

Ebbinghaus et al., *Numbers*, Springer Verlag 1995.

(This is a collection of the works by some prominent mathematicians on different aspects and areas pertaining to number systems and their ramification. It is a very technical book, mathematically very demanding and is recommended for further reading.)

Morris Kline, *Mathematical Thought From Ancient To Modern Times*, Oxford University Press 1972.

(A monumental work of a kind that treats the history of mathematics in a readable and thought-provoking manner.)

Morris Kline, *Mathematics The Loss of Certainty*, Oxford University Press 1980.

Reinhard Laubenbacher and David Pengelly, *Mathematical Expeditions*, Springer Verlag 1999.

(An amazing book that invites you to participate in the foot-steps of some famous mathematicians.)