CHAPTER TWO

A Cut, An Instant

Because I know that time is always time
And place is always and only place
And what is actual is actual only for one time
And only for one place
.... T.S. Eliot

To begin the construction of the real numbers, we shall take another look at the object we shall use, namely the rational numbers in a different way. As we stand at the threshold of time dividing the past and the future, there is an urge to move and yet not moved for time's ceaseless motion like an invisible hand divides the past and future at any instant now and forever. Pondering upon the total ordering on the rational numbers, we have a situation that mirrors the ever present time. At any point P (either to be made precisely or a rational number), there is according to the total ordering a division of the rational numbers into two classes of numbers, the class whose numbers are all greater than P mimicking the future of P and the class whose numbers are all less than P mimicking the past of P, *a cut in time.* The nature of the point P or the *cut* is the subject to be discussed. This cut in time is infallibly associated with an instant, the present instant. The untethered instant has order amidst the chaos. We cannot tell when it is and yet we know. The set of rational numbers then has a new meaning and in this new meaning is born the seed of the real numbers.

A Cut in Time

We cannot quite yet describe the present but we can for practical reason describe what comes close to being the present. A measure of time, the second, the humanly imaginable unit of time is our yardstick. As we speak, the present becomes in a fraction of the second the past and the future seems in a fraction of the second like the present. Time never stands still. We can in our humanly possible way describe what comes close to an instant as a cut in time.

Definition 1. A *cut* of $\mathbf{Q}$ is an ordered pair of subsets of the rational numbers (L, R) satisfying the following three properties:

(1) $L \neq \varnothing$, $R \neq \varnothing$.
(2) $L \cap R = \varnothing$, $L \cup R = \mathbf{Q}$.
(3) If x is in L and y is in R, then $x < y$.

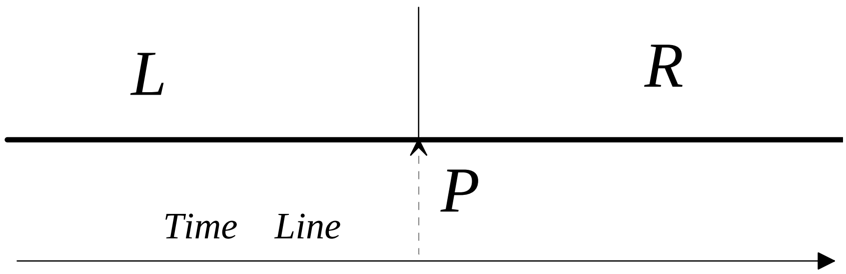
The subset L is called the left set of the cut and R the right set of the cut. By Property (2), every cut is determined by its left and right sets, each of which determines the other. Therefore, we shall identify a cut with its left set.

Thus each 'point' or each instant is given by a cut. Each 'point' as time has its past and future. If we are given a reference point, then we can begin to think of each rational number as a cut. This then gives new meaning to the rational numbers.

Definition 2. A subset ξ of **Q** is called a *real number* if it satisfies the following two properties.

(A) ξ is the left set of some cut; and

(B) ξ has no greatest element with respect to the total ordering on **Q**.



The present instant, the fleeting moment, once you think you have it in your grasp, it has already escaped you; yet we know the present instant. A point, an instant has taken on a new meaning as a cut in time. A 'real' number is not a number yet but part of a continuum of time in analogy. The abstract notions of addition and multiplication have yet to be invented. Property **B** models the fleeting nature of an instant.

Technically not all cuts are real numbers. Take for example the following cuts.

(1) $(L, R) = \{ x \in \mathbf{Q}: x^2 < 2\} \cup \{ x \in \mathbf{Q}: x^2 > 2\}$.
(2) $(L, R) = \{ x \in \mathbf{Q}: x < 2\} \cup \{ x \in \mathbf{Q}: x \geq 2\}$.
(3) $(L, R) = \{ x \in \mathbf{Q}: x \leq 2\} \cup \{ x \in \mathbf{Q}: x > 2\}$.

Cut (1) and (2) are real numbers but cut (3) is not a real number. The cuts (2) and (3) correspond in some sense the integer 2, but we do not want both. We identify a real number with the left cut that has no greatest element. Like the essence of an instant, knowing but unattainable, the real number is represented by the left cut that has no greatest element. Out of these special partitions of rational numbers, identified as the left hand sets of the cuts and hence out of the basic building blocks in set theory, we shall realise the real numbers. We shall have to redefine what seems so natural in the rational numbers. The identification of the cuts with their left hand sets provides us with the ease to redefine operations of addition and multiplication and others in terms of the basic operations in set theory. The following is a way of deciding when a subset of the rational numbers is a real number under our identification.

Lemma 3. A subset ξ of **Q** is a real number if and only if the following four conditions are satisfied.

1. $\xi \neq \emptyset$.
2. $\xi \neq \mathbf{Q}$.
3. For any x in ξ , if y is a rational number such that $y < x$, then y is also in ξ.
4. ξ has no greatest number.

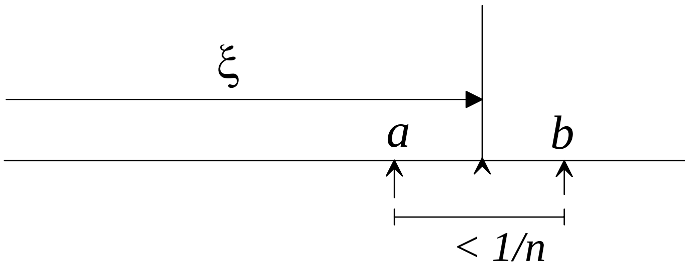
Proof. Let $L = \xi$ and $R = \mathbf{Q} - \xi$. Then conditions (1) and (2) imply that $L \neq \varnothing$ and $R \neq \varnothing$. For any x in L and any y in R, $x < y$. This is because if on the contrary that $x \geq y$, then by condition (3) above y is in ξ . But since y is in $R = \mathbf{Q} - \xi$, y is not in ξ. This contradicts that y is in ξ. Therefore, by Definition 1, (L, R) is a cut. Condition (4) then says that L has no greatest number. Therefore, ξ is a real number. Conversely if ξ is a real number, then the conditions (1) to (4) are automatically satisfied.

The essence of a cut is captured in the following lemma. It says that there are points on the left and right of a cut that are as close to one another as one wishes.

Lemma 4. **1.** If ξ is a real number and n is a counting number greater or equal to 1, then there exists (rational numbers) a in ξ and b not in ξ such that $b - a < \frac{1}{n}$.

2. For any rational number a in a real number ξ , there exists a counting number m such that $a + \frac{1}{m} \in \xi$.

Proof. Part 1. The real line will help us to visualise this lemma. Look at the following picture carefully.



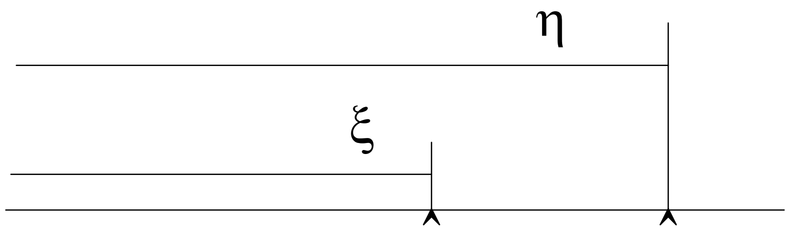
Since ξ is a real number, we can take an integer c not in ξ and an integer d in ξ. Then starting from c we shall add lengths of $\frac{1}{n^2}$ to give rational numbers that are consecutively $\frac{1}{n^2}$ apart. Define $x_i = d + \frac{i}{n^2}$. Then $x_0 = d, x_1 , x_2 , x_3 , \ldots , x_{n^2(c-d)} = c$ is a sequence of rational numbers such that $x_0 = d < x_1 < x_2 < x_3 < \ldots < x_{n^2(c-d)} = c$. Therefore, for some integer j such that $0 \leq j < n^2(c - d)$, x_j is in ξ but x_{j+1} is not in ξ. Take $b = x_{j+1}$ and $a = x_j$. Then for $n > 1$, $b - a = \frac{1}{n^2} < \frac{1}{n}$. For n =1, use n + 1 instead as above to obtain b and a such that $b - a < \frac{1}{n+1} < \frac{1}{n}$.

Proof of Part 2. Take any rational number a in the real number ξ. Suppose on the contrary that we cannot find such a number m with $a + \frac{1}{m} \in \xi$. Then for all counting number n, $a + \frac{1}{n} \notin \xi$. That is, for all counting number n, $x < a + \frac{1}{n}$ for all x in ξ . This implies that for each x in ξ, $x \leq a$. This is shown by way of contradiction.

Suppose on the contrary that $x > a$. Then by the Archimedean property of **Q**, there exists a counting number p such that $p \times (x - a) > 1$. That is $\frac{1}{p} < x - a$. Hence $x > a + \frac{1}{p}$. This contradicts $x < a + \frac{1}{p}$. Therefore, for each x in ξ, $x \leq a$. Since a is in ξ, a is a maximum element in ξ, contradicting that ξ does not have a maximum element because it is a real number. Thus for each a in ξ, there exists a counting number m such that $a + \frac{1}{m} \in \xi$.

A cut has a natural ordering arising out of a subset inclusion. Like the marching of time extending into the future, this analogy aptly describes the ordering. We make the definition below.

Definition 5. For any two real numbers ξ and η, we say $\xi \leq \eta$ if and only if $\xi \subseteq \eta$.



Let the set of real numbers be denoted by **R**. Clearly this ordering ' $\leq$ ' is a *reflexive*, *transitive* and *antisymmetric* relation on the set **R**. What does this mean? ' $\leq$ ' is *reflexive* means that for all ξ in **R**, $\xi \leq \xi$. It is obviously true since $\xi \subseteq \xi$ for any set ξ. Transitivity means if $\xi \leq \eta$ and $\eta \leq \kappa$, then $\xi \leq \kappa$. Clearly if $\xi \leq \eta$ and $\eta \leq \kappa$, then $\xi \subseteq \eta$ and $\eta \subseteq \kappa$ and so $\xi \subseteq \kappa$ which means that $\xi \leq \kappa$. ' $\leq$ ' is antisymmetric means that if $\xi \leq \eta$ and $\eta \leq \xi$, then $\xi = \eta$. This is plainly true since $\xi \subseteq \eta$ and $\eta \subseteq \xi$ imply that $\xi = \eta$.

Lemma 6. This ordering ' $\leq$ ' on **R** is a total ordering.

Proof. We have shown above that ' $\leq$ ' is a partial ordering on **R**. We need to show that any two real numbers ξ and η are comparable. That is either $\xi \leq \eta$ or $\eta \leq \xi$. Since both ξ and η are subsets of **Q**, either $\xi \subseteq \eta$ or $\xi \not\subseteq \eta$. If $\xi \subseteq \eta$, then $\xi \leq \eta$ and we have nothing to prove. It remains to show that if $\xi \not\subseteq \eta$, then $\eta \leq \xi$. Now if $\xi \not\subseteq \eta$, then there exists a rational number x in ξ such that x is not in η ($x \notin \eta$). Then for any y in η, $y < x$ by Property 3 of Definition 1 because η is a cut. Therefore, by Property 3 of Lemma 3, for any y in η, y is in ξ since x is in ξ. Thus $\eta \subseteq \xi$. Therefore, $\eta \leq \xi$.

Then **R** is complete in the sense of order. We shall show this below. First, let us examine what we have not done so far. We have not yet define the operations of addition and multiplication on **R**. The only structure we have on **R** at the moment is a total ordering.

> **Lemma 7.** Let S be a subset of **R** which is bounded above. Then S has a least upper bound or supremum in **R**.

Set theoretically the supremum is easily found. But we do need to show that we can actually obtain a real number this way. Much of the proof of Lemma 7 goes in showing this. Remember each element in S is a subset of the rational numbers.

> **Proof of Lemma 7.** Define $\eta = \cup\,\{\xi : \xi \in S\}$.Then $\xi \subseteq \eta$ for all ξ in S. Remember that each ξ in S is a cut and so is a subset of the rational numbers. Therefore, η is a subset of the rational numbers **Q**. We need to show that η is a real number and that it is the supremum of S. We are given that S is bounded above. Therefore, there exists a real number κ such that $\xi \leq \kappa$ for all ξ in S. That means $\xi \subseteq \kappa$ for all ξ in S. Therefore, $\eta = \cup\,\{\xi : \xi \in S\} \subseteq \kappa$. If we can show that η is a real number, then $\eta \leq \kappa$. We shall now use Lemma 3 to show that η is a real number. Since $\xi \neq \emptyset$ for each ξ in S, we have then that $\eta \neq \emptyset$. Also $\eta \neq$ **Q** because $\eta \subseteq \kappa$ and $\kappa \neq$ **Q**. Take now any x in η. Then $x \in \xi$ for some ξ in S. Thus for any y in **Q** with $y < x$, y is in ξ by Property 3 of Lemma 3 since ξ is a real number. It now remains to show property 4 of Lemma 3 for η; that is η has no maximum element. We shall show this by contradiction. Suppose on the contrary that η has a maximum element l. Then l is in some ξ in S. Also for all $y \in \eta$, $y \leq l$. This is also true of all y in this particular ξ, since $\xi \subseteq \eta$. Thus l would be the greatest element in ξ. This contradicts that ξ has no greatest element since it is a real number. This completes the proof that η is a real number. Since for any ξ in S, $\xi \subseteq \cup\,\{\xi : \xi \in S\} = \eta$, $\xi \leq \eta$ for any ξ in S. Therefore, η is an upper bound for S. Now let Ψ be any other upper bound of S. Then $\xi \leq \Psi$ for all ξ in S. That means $\xi \subseteq \Psi$ for all ξ in S. Therefore, $\eta = \cup\,\{\xi : \xi \in S\} \subseteq \Psi$. Hence $\eta \leq \Psi$. This shows that η is the least upper bound or the supremum of S.

We have thus proved the following.

> **Theorem 10.** **R** has a complete total ordering.

The construction of the real numbers makes it stand apart from the rational numbers. Firstly, it is constructed out of subsets of the rational numbers. For second, the set of rational numbers is definitely not a subset of it, at least not in a natural sense. To make sense of the rational numbers in this new as yet to be defined system of real numbers, we shall embed our rational numbers into **R** in such a way that it is compatible with the ordering on the rational numbers.

The Embedding of the Rational Numbers in The Real Numbers

Define an embedding φ: $\mathbf{Q} \to \mathbf{R}$ by $\varphi(a) = \{ x \in \mathbf{Q}: x < a\}$ for any rational number a. This is the only natural way of embedding **Q** and gives the new meaning of **Q** that was mentioned earlier. This is well defined for $\varphi(a)$ is a real number. Why? Obviously $\varphi(a)$ is a cut that does not have a maximum element and so by definition $\varphi(a)$ is a real number. The image of **Q** under φ is truly a copy of **Q**. Until addition and multiplication are defined on the set of real numbers we cannot show that the image behaves just like the rational number system. This will be done in the later chapter. We now show that φ is injective and is compatible with the ordering on the rational numbers. Mathematically this is summarised as follows.

Lemma 11. φ: $\mathbf{Q} \to \mathbf{R}$ is injective. Furthermore, for any rational numbers a and b $a \le b \iff \varphi(a) \le \varphi(b)$.

Proof. We shall show that φ is *injective*. More precisely, we shall show that whenever $\varphi(a) = \varphi(b)$, then $a = b$. $\varphi(a) = \varphi(b)$ implies that $\{x \in \mathbf{Q}: x < a\} = \{x \in \mathbf{Q}: x < b\}$. If $a < b$, then $a \in \{x \in \mathbf{Q}: x < b\}$. Take $c = \frac{a+b}{2}$. We have then $a < c < b$, $c \in \{x \in \mathbf{Q}: x < b\}$ and $c \notin \{x \in \mathbf{Q}: x < a\}$. Therefore, $\{x \in \mathbf{Q}: x < b\} \neq \{x \in \mathbf{Q}: x < a\}$ contradicting $\{x \in \mathbf{Q}: x < b\} = \{x \in \mathbf{Q}: x < a\}$. Thus a must be greater or equal to b. We can show similarly that a cannot be greater than b. Thus $a = b$. Hence φ is injective.

If $a \le b$, then $\varphi(a) = \{ x \in \mathbf{Q}: x < a\} \subseteq \{ x \in \mathbf{Q}: x < b\} = \varphi(b)$. Therefore, $\varphi(a) \le \varphi(b)$. Conversely, if $\varphi(a) \le \varphi(b)$, then $\{ x \in \mathbf{Q}: x < a\} \subseteq \{ x \in \mathbf{Q}: x < b\}$. It follows that $a \le b$, for otherwise $a > b$ would imply that $\{ x \in \mathbf{Q}: x < a\} \not\subseteq \{ x \in \mathbf{Q}: x < b\}$ since we would have $c = \frac{a+b}{2} \in \{ x \in \mathbf{Q}: x < a\}$ and $c \notin \{ x \in \mathbf{Q}: x < b\}$. This completes the proof of Lemma 11.

Thus we now have a new model of the rational numbers embedded in the real numbers, We have yet to define multiplication and addition on **R** and yet to show that the embedding respects addition and multiplication. We shall come back to this in the later chapter. Having known that the embedding φ respects ordering, Lemma 4 will then has new interpretation as given below.

Corollary 12. If ξ is a real number, then for any counting number n, there exist rational numbers a and b such that

$$\varphi(a) < \xi \le \varphi(b) \quad \text{and} \quad \varphi(b - a) < \varphi(\tfrac{1}{n}),$$

where the strict ordering ' < ' is defined by $\xi < \eta$ if and only if $\xi \le \eta$ and $\xi \neq \eta$.

Proof. By Lemma 4, for any counting number n, there exist rational numbers a in ξ and $b \notin \xi$ such that $b - a < \frac{1}{n}$. Since $a \in \xi$, by Property 3 of Lemma 3, any rational numbers $x < a$ belongs to ξ. Therefore, $\{ x \in \mathbf{Q}: x < a\} \subseteq \xi$. That means $\varphi(a) \subseteq \xi$, therefore $\varphi(a) \leq \xi$. Because $a \notin \varphi(a)$ and $a \in \xi$, $\varphi(a) \neq \xi$. Thus $\varphi(a) < \xi$. Since $b \notin \xi$, for any x in ξ, $x < b$ because b is in the right set of the cut ξ. Hence $\xi \subseteq \{ x \in \mathbf{Q}: x < b\} = \varphi(b)$. Therefore, $\xi \leq \varphi(b)$. By Lemma 11, $\varphi(b - a) \leq \varphi(\frac{1}{n})$. Because $b - a < \frac{1}{n}$, $b - a \in \varphi(\frac{1}{n})$. Now that we also have $b - a \notin \varphi(b - a)$, $\varphi(b - a) \neq \varphi(\frac{1}{n})$. Therefore, $\varphi(b - a) < \varphi(\frac{1}{n})$. This completes the proof.

Corollary 12 says that for any real number ξ, there are rational numbers (the embedded kind) before and after ξ that are arbitrary close to one another. In the next chapter we shall define addition on **R**.