A Note on Partial Fraction Expansion

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The subject of this note is the following theorem.

Theorem 1. Suppose $p(x)$ and $q(x)$ are real polynomial functions, i.e., functions defined by polynomial with real coefficients. Suppose the degree of p < degree of q and $p(x)$ and $q(x)$ have no common factors other than the non zero constants. Then for the partial fraction expansion of $\frac{p(x)}{q(x)}$, (i) corresponding to a real factor $(x-a)^k$, $k \geq 1$, of $q(x)$ we have an expansion

$$\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_k}{(x-a)^k},$$

and (ii) corresponding to a real quadratic factor $(x^2+cx+d)^k$, $k \geq 1$, where $c^2 < 4d$, we have the expansion

$$\frac{C_1\,x+D_1}{(x^2+cx+d)}+\frac{C_2\,x+D_2}{(x^2+cx+d)^2}+\cdots+\frac{C_k\,x+D_k}{(x^2+cx+d)^k}.$$

This result is often used in the integration of a rational function offered in almost any elementary text on the calculus but the proof in general terms is seldom offered and tends to be glossed over. We shall prove this result using complex analysis.

Proof. Consider the polynomial functions p and q as functions on the complex numbers. Let L = degree of $q(z)$ and by hypothesis, L > degree of $p(z)$. We can write

$$f(z)=\frac{p(z)}{q(z)}=\frac{p(z)}{z^L(q(z)/z^L)}.$$

Note that $\left|\frac{q(z)}{z^L}\right| \to |b_L| \neq 0$ as $z \to \infty$, where b_L is the coefficient of z^L in $q(z)$.
Since L > degree of $p(z)$,

$$\left|\frac{p(z)}{z^L}\right| \to 0 \text{ as } z \to \infty.$$

Hence $|f(z)|=\left|\frac{p(z)}{q(z)}\right| == \frac{p(z)}{z^L}\frac{1}{(q(z)/z^L)} \to 0\cdot\frac{1}{b_L}=0$ as $z \to \infty$.

Suppose $q(z)=(x-a_1)^{n_1}\cdot(x-a_2)^{n_2}\cdots(x-a_K)^{n_K}\cdot(x^2+c_1x+d_1)^{L_1}\cdots$
$\cdots(x^2+c_Jx+d_J)^{L_J}$,

where $a_1, a_2, \ldots, a_K, c_1, c_2, \ldots, c_J, d_1, c_2, \ldots, d_J$, are real numbers and $c_n^2 < 4d_n$, $n=1,2,\ldots,J$. Let x_n, $\overline{x_n}$ be the two non real roots of $z^2+c_nz+d_n$, $n=1,\ldots,J$. Suppose $f_1, f_2, \ldots, f_K$ are the principal parts of $f(z)$ at $a_1, a_2, \ldots, a_K$ respectively. Note that a_j is a pole of $f(z)$ of order n_j and x_j, $\overline{x_j}$ are poles of $f(z)$ of order L_j, $j=1,2,\ldots,J$.
Let g_j, h_j, $j=1,2,\ldots,J$, be the principal parts of the Laurent series of f at x_j, $\overline{x_j}$ respectively. Then

$$g=f-(f_1+f_2+\ldots+f_K)-(g_1+g_2+\ldots+g_J)-(h_1+h_2+\ldots+h_J)$$

has removable singularities at $a_1, a_2, \ldots, a_K$, x_j, $\overline{x_j}$, $j=1,2,\ldots,J$.
Hence, by defining appropriate values of g at these points by taking limits, g(z) is an entire function.
Now for $j=1,2,\ldots,J$, $|g_j(z)| \to 0$, $|h_j(z)| \to 0$ as $z \to \infty$ and $|f_n(z)| \to 0$ as $z \to \infty$, for $n=1,2,\ldots,K$ and so $|g(z)| \to 0$ as $z \to \infty$. Therefore, by Liouville's Theorem, g(z) is a constant function and so since $|g(z)| \to 0$ as $z \to \infty$, g(z) is identically zero. (We can aslo

deduce this as follows. Because | g (z) | $\to$ 0 as z $\to \infty$, $M(s)$ = sup{|g(z)|: |z| = s} $\to$ 0 as $s \to \infty$. Hence by the Cauchy Inequality, g(z) is identically zero.) Thus

$$f = f_1 + f_2 + \ldots + f_K + g_1 + g_2 + \ldots + g_J + h_1 + h_2 + \ldots + h_J. \qquad \text{-- (A)}$$

Note that if a is a pole of order k of f, then the Laurent series of f at a is given by

$$f(z) = \frac{A_1}{(z-a)} + \frac{A_2}{(z-a)^2} + \cdots + \frac{A_k}{(z-a)^k} + \sum_{n=0}^{\infty} b_n (z-a)^n \quad \text{--------------- (1)}$$

where the A_i's are complex numbers. In particular, $\frac{A_1}{(z-a)} + \frac{A_2}{(z-a)^2} + \cdots + \frac{A_k}{(z-a)^k}$ is the principal part of f at a. Thus (A) is the partial fraction decomposition of f as complex rational function. We now specialize to real rational function.

Since $f(z) = \frac{p(z)}{q(z)}$ and p and q have real coefficients,

$$\overline{f(\bar{z})} = \frac{\overline{p(\bar{z})}}{\overline{q(\bar{z})}} = \frac{p(z)}{q(z)} = f(z).$$

Thus from (1) that if a is a pole of order k of f, then

$$f(z) = \frac{\overline{A_1}}{(z-\bar{a})} + \frac{\overline{A_2}}{(z-\bar{a})^2} + \cdots + \frac{\overline{A_k}}{(z-\bar{a})^k} + \sum_{n=0}^{\infty} \overline{b_n} (z-\bar{a})^n \quad \text{----------------- (2)}$$

If a is real then $\bar{a} = a$ and so (2) becomes

$$f(z) = \frac{\overline{A_1}}{(z-a)} + \frac{\overline{A_2}}{(z-a)^2} + \cdots + \frac{\overline{A_k}}{(z-a)^k} + \sum_{n=0}^{\infty} \overline{b_n} (z-a)^n \quad \text{--------------- (3).}$$

This means that if a is real, (3) is also a Laurent series for f at a and so by the uniqueness of Laurent series, $\overline{A_j} = A_j$ for j =1, …k. It follows that the principal part of f at z = a for real a is given by

$$\frac{A_1}{(z-a)} + \frac{A_2}{(z-a)^2} + \cdots + \frac{A_k}{(z-a)^k} \quad \text{------------------------------ (B)}$$

where A_j, j =1 , 2, …k, are real numbers.
Therefore, f_1, f_2, … , f_K are of the form given by (B).
Note that if a is non real, then (2) is a Laurent series for f at $\bar{a}$. Consequently, the sum of the principal part of f at a and the principal part of f at $\bar{a}$ is

$$\left(\frac{A_1}{(z-a)} + \frac{\overline{A_1}}{(z-\bar{a})}\right) + \left(\frac{A_2}{(z-a)^2} + \frac{\overline{A_2}}{(z-\bar{a})^2}\right) + \cdots + \left(\frac{A_k}{(z-a)^k} + \frac{\overline{A_k}}{(z-\bar{a})^k}\right)$$

$$= \left(\frac{A_1(z-\bar{a}) + \overline{A_1}(z-a)}{(z^2 - 2\operatorname{Re} a + |a|^2)}\right) + \left(\frac{A_2(z-\bar{a})^2 + \overline{A_2}(z-a)^2}{(z^2 - 2\operatorname{Re} a + |a|^2)^2}\right) + \cdots + \left(\frac{A_k(z-\bar{a})^k + \overline{A_k}(z-a)^k}{(z^2 - 2\operatorname{Re} a + |a|^2)^k}\right)$$

$$\text{--------------------------------------- (4)}$$

Note that because $A_j \bar{a}^n$ and $\overline{A_j} a^n$ are conjugate pairs, the coefficients of $A_j(z-\bar{a})^j + \overline{A_j}(z-a)^j$ are real. Hence $A_j(z-\bar{a})^j + \overline{A_j}(z-a)^j$ is a real polynomial of degree j. Therefore, by successively dividing out by $z^2 - 2\operatorname{Re} a + |a|^2$, starting from $\left(\frac{A_k(z-\bar{a})^k + \overline{A_k}(z-a)^k}{(z^2 - 2\operatorname{Re} a + |a|^2)^k}\right)$, (4) can be written as

$$\left(\frac{C_1 z + D_1}{(z^2 - 2\operatorname{Re} a + |a|^2)}\right) + \left(\frac{C_2 z + D_2}{(z^2 - 2\operatorname{Re} a + |a|^2)^2}\right) + \cdots + \left(\frac{C_k z + D_k}{(z^2 - 2\operatorname{Re} a + |a|^2)^k}\right)$$

where C_j and D_j are real numbers for j =1, 2, …, k. This then implies that $g_j + h_j$, j =1 ,2, …, J , is of the form

$$\frac{C_1\ z + D_1}{(z^2 + cz + d)} + \frac{C_2\ z + D_2}{(z^2 + cz + d)^2} + \cdots + \frac{C_k\ z + D_k}{(z^2 + cz + d)^k} \text{ ---------------- } \quad \text{(C)}$$

where c, d, C_n and D_n , n =1, 2, …, $k = L_j$, are real numbers. By specializing to real variable x this proves the theorem.