Monotonicity and Continuity of Inverse Function

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The following is a subtle theorem. Essentially it means that the monotonicity of a function implies the continuity of its inverse function.

Theorem 1. If I is an interval and $f: I \to \mathbf{R}$ is (strictly) monotonic, then f^{-1} is continuous.

Proof. Assume that f is (strictly) increasing. [f is (strictly) monotonic implies that f is either (strictly) increasing or is (strictly) decreasing. If f is is (strictly) decreasing, then $-f$ is (strictly) increasing. We also have that $f^{-1} = (-f)^{-1} \circ (\mathbf{-1})$, where $\mathbf{-1}$ is multiplication by -1. Therefore, since $\mathbf{-1}$ is a continuous function, if we can show that $(-f)^{-1}$ is continuous, then f^{-1}, being the composite of two continuous functions is continuous. So it suffices to prove the theorem when f is (strictly) increasing.]

Let $J = f(I)$, the range of f. Then J is the domain of the inverse function $f^{-1}: J \to I \subseteq \mathbf{R}$. Take an element b in J. Then since f is strictly monotonic, f is also injective. Hence there exists an unique element a in I such that $f(a) = b$. We shall show that given any $\varepsilon > 0$, we can find a $\delta > 0$ such that for all y in J, $|y - b| < \delta$ implies that $|f^{-1}(y) - f^{-1}(b)| < \varepsilon$. *Suppose a is in the interior of I*, i.e., there exist ξ and η, with $\xi < a < \eta$ and $[\xi, \eta] \subseteq I$. Then $(\xi, \eta)\cap(a - \varepsilon, a + \varepsilon)$ is an open interval containing a and contained in I. If need be, we can replace ξ by $\max(\xi, a - \varepsilon)$ and η by $\min(\eta, a + \varepsilon)$ and still call them ξ and η. That means we may assume that $(\xi, \eta) \subseteq (a - \varepsilon, a + \varepsilon)\cap I$. Then since f is (strictly) increasing,

$$f((\xi, \eta)) = (f(\xi), f(\eta))\cap J. \qquad \text{---------------------} \quad (1)$$

This can be deduced in the following manner. For any x such that $\xi < x < \eta$, we have that $f(\xi) < f(x) < f(\eta)$ since f is (strictly) increasing and so $f(x) \in (f(\xi), f(\eta))\cap J$. Hence $f((\xi, \eta)) \subseteq (f(\xi), f(\eta))\cap J$. Conversely, take any y in $(f(\xi), f(\eta))\cap J$. That means there exists x in I such that $y = f(x)$ and $f(\xi) < f(x) < f(\eta)$. We shall show that x actually lies in (ξ, η). It is a simple matter to check that $f^{-1}: J \to I$ is also (strictly) increasing if f is. (*Obviously* $y_1 < y_2 \Rightarrow f^{-1}(y_1) < f^{-1}(y_2)$, otherwise if $f^{-1}(y_1) \geq f^{-1}(y_2)$, then $y_1 = f(f^{-1}(y_1)) \geq f(f^{-1}(y_2)) = y_2$, contradicting $y_1 < y_2$.) Hence we can conclude that $f^{-1}(f(\xi)) < f^{-1}(f(x)) < f^{-1}(f(\eta))$, that is, $\xi < x < \eta$, deduced from the fact that $f^{-1} \circ f$ is the identity function on I. Hence $y = f(x) \in f((\xi, \eta))$. That means $(f(\xi), f(\eta))\cap J \subseteq f((\xi, \eta))$. This proves (1).

Now since $b = f(a) \in (f(\xi), f(\eta))$, and because $(f(\xi), f(\eta))$ is an open interval, we can find a $\delta > 0$ such that $(b - \delta, b + \delta) \subseteq (f(\xi), f(\eta))$. Then for all y in J and y in $(b - \delta, b + \delta)$, we have that y is in $(f(\xi), f(\eta))\cap J = f((\xi, \eta))$ and so since f is injective, $f^{-1}(y) \in (\xi, \eta) \subseteq (a - \varepsilon, a + \varepsilon)\cap I$. This means whenever y is in J and $|y - b| < \delta$, we have $|f^{-1}(y) - f^{-1}(b)| < \varepsilon$. Hence, f^{-1} is continuous at b.

We now consider the case when $a = f^{-1}(b)$ is an end point of I. Suppose a is the right hand end point of the interval I. Then as above we can find a $\xi < a$ with $[\xi, a] \subseteq I \cap (a - \varepsilon, a + \varepsilon)$. We now choose $\delta > 0$ such that $\delta < f(a) - f(\xi) = b - f(\xi) > 0$. Then for all y in J with $f(\xi) < b - \delta < y < b + \delta$ we have that $f(\xi) < y \leq b$ because f^{-1} is increasing. This is easily deduced for if $y > b$, then $f^{-1}(y) > f^{-1}(b) = a$ contradicting that $f^{-1}(y) \leq a$, the right hand end point of the interval I. Hence, applying f^{-1} we obtained $\xi = f^{-1}(f(\xi)) < f^{-1}(y) \leq f^{-1}(b) = a$. That is to say $f^{-1}(y) \in (\xi, a) \subseteq I \cap (a - \varepsilon, a + \varepsilon)$. In other words, for any y in J such that $|y - b| < \delta$ we have $|f^{-1}(y) - f^{-1}(b)| = |f^{-1}(y) - a| < \varepsilon$. This means f^{-1} is continuous at b. It can be similarly shown that f^{-1} is continuous at b *if* $a = f^{-1}(b)$ is the left hand end point of the interval I. This completes the proof.

The proof is a little longer than one using the characterization of continuity by sequences. It is chosen because we use only the definition of continuity in terms of 'ε and δ' i.e., in terms of open sets and also because the definition of continuity applying to the image J which may not be an interval is emphasized.

This theorem then raises the question whether f is itself continuous. An important condition of the theorem is that the domain I be an interval. Now since $f: I \to \mathbf{R}$ is (strictly) monotonic, its inverse $f^{-1}: J \to I$ is also (strictly) monotonic. Does this mean that the inverse of f^{-1}, whcich is f is also continuous? Obviously, if J is an interval, then Theorem 1 applies to tell us that indeed f is also continuous. Hence we can phrase our result thus:

Theorem 2. Suppose I is an interval and $f: I \to \mathbf{R}$ is (strictly) monotonic. Then f is continuous if and only if the range of f, $J = f(I)$ is an interval.

Proof. I have already shown above that if $J = f(I)$ is an interval, then f is continuous. On the other hand, if f is continuous, then $J = f(I)$ is an interval. This is because if y_1 and y_2 are in J with $y_1 < y_2$, then there exist c and d in I such that $f(c) = y_1$ and $f(d) = y_2$. Therefore, by the Intermediate Value Theorem, for any y between y_1 and y_2, *there can be found an element k between c and d such that* $f(k) = y$, *i.e.* y is also in J. This means J is an interval. This completes the proof of this theorem.