Monotone Function, Function of Bounded Variation, Fundamental Theorem of Calculus

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Have you ever wonder about just how 'nice' is monotone function? The following fact about monotone function is not usually revealed in a first course on calculus. Firstly, we say what we meant by a monotone function.

Definition 1. Let $f : [a, b] \to \mathbf{R}$ be a real valued function defined on the closed and bounded interval $[a, b]$ with $a < b$. We say f is a *monotone function* if it is either increasing or decreasing, that is, *either* for all x and y such that $a \le x < y \le b$, $f(x) \le f(y)$ (*increasing*) *or* for all x and y such that $a \le x < y \le b$, $f(x) \ge f(y)$ (*decreasing*).

Throughout we shall assume that $[a, b]$ is a non trivial interval with $a < b$. *Before we embark on describing the points of discontinuity of f, we shall see how the values of the differences of the left and right limits of f at a finite set of points in $[a, b]$ can sum up to.* Note that if f is continuous, then this sum is always zero. This will in some sense detect some discontinuity of the function f. If f is a monotone function, then the difference of the left and right limits at a point x being zero is equivalent to the function being continuous at the point x. Why? Why do the left and right limits at x exist? An explanation is in order.

Notice that if f is increasing, then for a fixed x in (a, b), for all y in $[a, b]$ such that $y < x$, $f(y) \le f(x)$. Therefore, the set $\{ f(y) : y \text{ in } [a, b] \text{ and } y < x \}$ is bounded above by $f(x)$. Hence by the completeness property of $\mathbf{R}$, $\sup\{ f(y) : y \text{ in } [a, b] \text{ and } y < x \}$ exists and is less than or equal to $f(x)$. We claim this is the left limit of f at x. Denote $\sup\{ f(y) : y \text{ in } [a, b] \text{ and } y < x \}$ by L. Then for any $\varepsilon > 0$, $L - \varepsilon < L$. Therefore, by the definition of supremum, there exists a y_0 in $\{ f(y) : y \text{ in } [a, b] \text{ and } y < x \}$ such that $L - \varepsilon < y_0 \le L$. Therefore, there exists a x_0 in $[a, b]$ such that $x_0 < x$, $f(x_0) = y_0$. Let now $\delta = x - x_0 > 0$. Then for all z in $[a, b]$ such that $x - \delta < z < x$, i.e., $x_0 < z < x$, we have $y_0 = f(x_0) \le f(z) \le f(x)$. Since $f(z) \in \{ f(y) : y \text{ in } [a, b] \text{ and } y < x \}$, $f(z) \le \sup\{ f(y) : y \text{ in } [a, b] \text{ and } y < x \} = L$. Therefore, we have $L - \varepsilon < y_0 \le f(z) \le L$. Thus $| f(z) - L | = L - f(z) < \varepsilon$. We have finally shown that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any z in $[a, b]$ with $x - \delta < z < x$, $| f(z) - L | < \varepsilon$. This means that the left limit of f at x is $L \le f(x)$. Similarly, we can show that the right limit of f at x is the infimum of $\{ f(y) : y \text{ in } [a, b] \text{ and } y > x \}$ and is greater than or equal to $f(x)$. Thus, for any x in (a, b), $\lim_{y \to x^-} f(y) \le f(x) \le \lim_{y \to x^+} f(y)$. Now the limit of f at x exists, if and only if, the left and right limits at x exist and are the same. Therefore, if the limit of f at x exists, it must be equal to $f(x)$ and so f must be continuous at x. Hence the only possible way for f to be discontinuous at x is for the left and right limits at x to be different, that is by definition a jump discontinuity. If $x = b$, the same argument as above for the left limit shows that $\sup\{ f(y) : y \text{ in } [a, b] \text{ and } y < b \} = \lim_{y \to b^-} f(y) \le f(b)$ and if $x = a$ we shall have $\inf\{ f(y) : y \text{ in } [a, b] \text{ and } y > a \} = \lim_{y \to a^+} f(y) \ge f(a)$. A jump at the point x in (a, b) is defined to be $\lim_{y \to x^+} f(y) - \lim_{y \to x^-} f(y)$, at a, it is $\lim_{y \to a^+} f(y) - f(a)$ and at b, it is $f(b) - \lim_{y \to b^-} f(y)$. Therefore, the only possible kind of discontinuity at the end points is also a jump discontinuity, that is, when the jump is not zero. In particular when the jump is zero at x, the function must be continuous at x. By definition, when the function is continuous at x, the jump must be zero. This is the case for increasing function. When f is decreasing, we shall have the same conclusion by a similar argument.

Thus, if f is a monotone function then this sum does detect the discontinuity of the function f at these points and to some extent can tell us something about the points of discontinuity of f.

Theorem 2. Suppose $f: [a, b] \to \mathbf{R}$ is an increasing function. Let $x_0 = a < x_1 < x_2 < \ldots < x_n = b$ be a *partition* of $[a, b]$. (See Page 121 of *Calculus, an Introduction.*) *Then the following sum*
$[f(a^+) - f(a)] + [f(x_1^+) - f(x_1^-)] + \ldots + [f(x_{n-1}^+) - f(x_{n-1}^-)] + [f(b) - f(b^-)] \le f(b) - f(a)$, where $f(x^+) = \lim_{k \to x^+} f(k)$ and $f(x^-) = \lim_{k \to x^-} f(k)$.

Proof. Note that the function f is bounded. The idea of proof is very simple. Take a point y_i in each of the open interval (x_{i-1}, x_i) for $i = 1, \ldots, n$. Then the sum of the differences of the values of f at these points would add up to $f(b) - f(a)$. Notice by the completeness property of $\mathbf{R}$, the left and right limits at the x_i 's exist. (See the above explanation.) Note that for $i = 1, \ldots, n-1$, $x_{i-1} < y_i < x_i < y_{i+1} < x_{i+1}$ and so since f is increasing $f(x_{i-1}^+) \le f(y_i) \le f(x_i^-) \le f(x_i^+) \le f(y_{i+1})$. Note also that $f(y_n) \le f(x_n^-) = f(b^-)$.
Thus, for $i = 1, \ldots, n-1$,

$$f(x_i^+) - f(x_i^-) \le f(y_{i+1}) - f(y_i).$$

Then

$$\begin{aligned}
&[f(a^+) - f(a)] + [f(x_1^+) - f(x_1^-)] + \ldots + [f(x_{n-1}^+) - f(x_{n-1}^-)] + [f(b) - f(b^-)] \\
&\le [f(x_0^+) - f(a)] + [f(y_2) - f(y_1)] + \ldots + [f(y_n) - f(y_{n-1})] + [f(b) - f(x_n^-)] \\
&\le [f(y_1) - f(a)] + [f(y_2) - f(y_1)] + \ldots + [f(y_n) - f(y_{n-1})] + [f(b) - f(y_n)] \\
&= f(b) - f(a).
\end{aligned}$$

This theorem also says that if $f: [a, b] \to \mathbf{R}$ is an increasing function, then the discontinuity of f can only be jump discontinuity not exceeding $f(b) - f(a)$. We shall use the above theorem to determine the size of the set of the points of discontinuity of f.

Theorem 3. Suppose $f: [a, b] \to \mathbf{R}$ is a monotone function. Then the set of discontinuity of f is countable.

Proof. Assume that f is increasing. As remark above any point of discontinuity of f is also a jump discontinuity. So we look at the points in (a, b), where the jump of discontinuity exceeds $1/n$ for some natural number n. This is the set

$$Dis_n = \{ x \in (a, b) : f(x^+) - f(x^-) > 1/n \}.$$

How large can this set be? Strange enough, Theorem 2 can tell us something. Take k points in this set, then for each point x the jump $f(x^+) - f(x^-) > 1/n$. Thus by theorem 2, summing over these k points would give us a sum less than or equal to $f(b) - f(a)$. That means $f(b) - f(a) \ge k/n$. Consequently $k \le n(f(b) - f(a))$. Hence the number of points in Dis_n cannot exceed $n(f(b) - f(a))$ and so is finite. Now the set of discontinuity of f is $D = \cup \{Dis_n : n = 1, \ldots, \infty\}$, that is the union of all the Dis_n. Since each Dis_n is finite and so D being a countable union of finite set is countable. (This is a result in set theory.) Hence the set of discontinuity of f is countable. If the function f is decreasing, then $-f$ is increasing. Because the sets of discontinuity of f and $-f$ are the same, the above argument applies to give that the set of discontinuity of $-f$ is countable and so the set of discontinuity of f is countable. This completes the proof of this theorem.

Corollary 4. Suppose $f : [a, b] \to \mathbf{R}$ is a monotone function. Then f is Riemann integrable.

Proof. We shall give a non standard proof without using the definition of the Riemann integral. The function f is obviously bounded since its range lies between $f(a)$ and $f(b)$. By Theorem 3, f is continuous except perhaps on a countable set. Since any countable set has Lebesgue measure zero, f is bounded and continuous almost every where on $[a, b]$ and so f is Riemann integrable by Lebesgue's Theorem.

Defiinition 5. Let $f : [a, b] \to \mathbf{R}$ be a real valued function. Suppose $\Delta : x_0 = a < x_1 < x_2 < \ldots < x_n = b$ is a partition of $[a, b]$. Define Δf_j for $j = 1,\ldots, n$ by $\Delta f_j = f(x_j) - f(x_{j-1})$. The function f is said to be of bounded variation if there exists a real number $K > 0$ such that $\sum_{j=1}^{n} |\Delta f_j| \leq K$ for any partition Δ of $[a, b]$.

Denote the set of functions on $[a, b]$ of bounded variation by $BV(a, b)$.

The following is an easy consequence of the definition.

Theorem 6. If f is of bounded variation on $[a, b]$, then f is bounded.

Proof. Choose any y in (a, b), let $\Delta : x_0 = a < x_1 < x_2 = b$ be a partition with $x_1 = y$. Then since f is of bounded variation, there exists $K > 0$ such that $\sum_{j=1}^{2} |\Delta f_j| \leq K$. Therefore, $|f(y)| - |f(a)| \leq |f(y) - f(a)| = |\Delta f_1| \leq \sum_{j=1}^{2} |\Delta f_j| \leq K$. Hence $|f(y)| \leq |f(a)| + K$. This is obviously true for $y = a$ and also true for $y = b$, since $\Delta : a < b$ is also a partition. Therefore, f is bounded by $|f(a)| + K$).

Theorem 7. If f is monotone on $[a, b]$, then f is of bounded variation.

Proof. Assume f is increasing. Then for any partition $\Delta : x_0 = a < x_1 < x_2 < \ldots < x_n = b$, $\sum_{j=1}^{n} |\Delta f_j| = \sum_{j=1}^{n} \Delta f_j = f(b) - f(a)$ because $\Delta f_j \geq 0$. Hence f is of bounded variation.

Total Variation

Definition 8. Let $f : [a, b] \to \mathbf{R}$ be a real valued function of bounded variation, that is, f is in BV(a, b). This means that there is a positive real number K such that for any partition $\Delta : x_0 = a < x_1 < x_2 < \ldots < x_n = b$, $\sum_{j=1}^{n} |\Delta f_j| \leq K$. Hence the set $\{ \sum_{j=1}^{n} |\Delta f_j| : \Delta$ is a partition of $[a, b] \}$ is bounded above and so by the completeness property of $\mathbf{R}$, $\sup\{ \sum_{j=1}^{n} |\Delta f_j| : \Delta$ is a partition of $[a, b] \}$ exists. This is called the *total variation* of f on $[a, b]$ and is denoted by $V(f; a, b)$. Obviously, $V(f; a, b) \geq 0$ and $V(f; a, b) = V(-f; a, b)$.

Let c be in (a, b). Then it is obvious by adding the end point b (the beginning point a) that any partition for $[a, c]$ ($[c, b]$) can be extended to a partition for $[a, b]$. And so it is trivial to conclude that if f is of bounded variation on $[a, b]$, then it is also of bounded variation on $[a, c]$ and on $[c, b]$. We have then the following theorem.

Theorem 9. Let $f : [a, b] \to \mathbf{R}$ be a real valued function of bounded variation, that is f is in BV(a, b). Then for any c in (a, b), $V(f ; a, b) = V(f ; a, c) + V(f ; c, b)$.

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Proof. Take any partition $\Delta_1 : x_0 = a < x_1 < x_2 < \ldots < x_n = c$ for $[a, c]$ and any partition $\Delta_2 : y_0 = c < y_1 < y_2 < \ldots < y_m = b$ for $[c, b]$. Then the partition,

$$\Delta : x_0 = a < x_1 < x_2 < \ldots < x_n = c = y_0 = c < y_1 < y_2 < \ldots < y_m = b ,$$

is a partition for $[a, b]$ and so

$$\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| + \sum_{j=1}^{m} |f(y_j) - f(y_{j-1})| \leq V(f ; a, b).$$

That means for any partition, $\Delta_1 : x_0 = a < x_1 < x_2 < \ldots < x_n = c,$ for $[a, c]$,

$$\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \leq V(f ; a, b) - \sum_{j=1}^{m} |f(y_j) - f(y_{j-1})|$$

and so $V(f ; a, b) - \sum_{j=1}^{m} |f(y_j) - f(y_{j-1})|$ is an upper bound for the set

$$\{ \sum_{j=1}^{n} |\Delta f_j| : \Delta \text{ is a partition of } [a, c] \}.$$

Thus, $V(f ; a, c) = \sup\{ \sum_{j=1}^{n} |\Delta f_j| : \Delta \text{ is a partition of } [a, c] \}$

$$\leq V(f ; a, b) - \sum_{j=1}^{m} |f(y_j) - f(y_{j-1})|.$$

Hence, we have for any partition, $\Delta_2 : y_0 = c < y_1 < y_2 < \ldots < y_m = b$ for $[c, b]$,

$$\sum_{j=1}^{m} |f(y_j) - f(y_{j-1})| \leq V(f ; a, b) - V(f ; a, c).$$

That means $V(f ; a, b) - V(f ; a, c)$ is an upper bound for the set $\{\sum_{j=1}^{n} |\Delta f_j| : \Delta$ is a partition of $[c, b] \}$. It follows, by the definition of supremum, that

$V(f ; c, b) = \sup\{\sum_{j=1}^{n} |\Delta f_j| : \Delta \text{ is a partition of } [c, b] \} \leq V(f ; a, b) - V(f ; a, c)$.

Therefore, $V(f ; a, c) + V(f ; c, b) \leq V(f ; a, b)$.

Now we start with a partition $\Delta : x_0 = a < x_1 < x_2 < \ldots < x_n = b$ for $[a, b]$. Note that $c \neq x_0, x_n$ because $c \in (a, b) = (x_0 , x_n)$. If for some $k \neq 0, n$, $c = x_k$, then $\Delta_1 : x_0 = a < x_1 < x_2 < \ldots < x_k = c$ is a partition for $[a, c]$ and $\Delta_2 : x_k = c < x_{k+1} < x_2 < \ldots < x_n = b$ is a partition for $[c, b]$. Therefore,

$$\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| = \sum_{j=1}^{k} |f(x_j) - f(x_{j-1})| + \sum_{j=k+1}^{n} |f(x_j) - f(x_{j-1})|$$
$$\leq V(f ; a, c) + V(f ; c, b) .$$

If $c \neq x_j$, $j = 1, \ldots, n-1$, then c must be in the interior of one of the subintervals defined by the partition and so for some integer k, $1 \leq k \leq n$, $x_{k-1} < c < x_k$. Then $\Delta_1 : x_0 = a < x_1 < x_2 < \ldots < x_{k-1} < c$ is a partition for $[a, c]$ and $\Delta_2 : c < x_k < x_{k+1} < x_2 < \ldots < x_n = b$ is a partition for $[c, b]$. Thus,

$$\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|$$
$$= \sum_{j=1}^{k-1} |f(x_j) - f(x_{j-1})| + |f(x_k) - f(x_{k-1})| + \sum_{j=k+1}^{n} |f(x_j) - f(x_{j-1})|$$
$$\leq (\sum_{j=1}^{k-1} |f(x_j) - f(x_{j-1})| + |f(c) - f(x_{k-1})|)$$
$$+ (|f(x_k) - f(c)| + \sum_{j=k+1}^{n} |f(x_j) - f(x_{j-1})|) \text{ by the triangle inequality}$$
$$\leq V(f ; a, c) + V(f ; c, b) .$$

In the above summation, if $k = 1$, then $\sum_{j=1}^{k-1} |f(x_j) - f(x_{j-1})|$ is taken to be 0 and if $k = n$, then $\sum_{j=k+1}^{n} |f(x_j) - f(x_{j-1})|$ is taken to be 0.

Hence we have shown that for any partition $\Delta : x_0 = a < x_1 < x_2 < \ldots < x_n = b$ for $[a, b]$, $\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \leq V(f ; a, c) + V(f ; c, b)$.

Therefore,

$V(f ; a, b) = \sup\{ \sum_{j=1}^{n} |\Delta f_j| : \Delta \text{ is a partition of } [a, b] \} \leq V(f ; a, c) + V(f ; c, b)$.

It follows that $V(f ; a, b) = V(f ; a, c) + V(f ; c, b)$. This completes the proof.

The total variation is a very useful information for a function with bounded variation. We can even use it to define a function and with this function we can show that any function of bounded variation is the difference of two monotone increasing functions.

Definition 10. Let $f: [a, b] \to \mathbf{R}$ be a real valued function of bounded variation, that is f is in BV(a, b). The *variation* of f is a function $V_f: [a, b] \to \mathbf{R}$ defined by $V_f(a) = 0$, and for x in $(a, b]$,

$$V_f(x) = \mathrm{V}(f; a, x) = \sup\{ \sum_{j=1}^{n} |\Delta f_j| : \Delta \text{ is a partition of } [a, x] \}.$$

This is well defined since for any x in $(a, b]$, f is also a function of bounded variation on $[a, x]$.

Theorem 11. If f is in BV(a, b), then $V_f: [a, b] \to \mathbf{R}$ is an increasing function.

Proof. Let $a \le x < y \le b$. Then

$V_f(y) = \mathrm{V}(f; a, y) = \mathrm{V}(f; a, x) + \mathrm{V}(f; x, y)$ by Theorem 10

$= V_f(x) + \mathrm{V}(f; x, y) \ge V_f(x)$ since $\mathrm{V}(f; x, y) \ge 0$.

Thus V_f is increasing.

Theorem 12. If f is in BV(a, b), then $V_f - f$ is an increasing function on $[a, b]$.

Proof. Let $a \le x < y \le b$. Then

$(V_f - f)(y) - (V_f - f)(x) = V_f(y) - V_f(x) - (f(y) - f(x))$

$= \mathrm{V}(f; a, y) - \mathrm{V}(f; a, x) - (f(y) - f(x))$

$= \mathrm{V}(f; x, y) - (f(y) - f(x))$, by Theorem 10,

$\ge |f(y) - f(x)| - (f(y) - f(x))$, because $x < y$ is a partition for $[x, y]$ and

$\mathrm{V}(f; x, y) = \sup\{ \sum_{j=1}^{n} |\Delta f_j| : \Delta \text{ is a partition of } [x, y] \} \ge |f(y) - f(x)|$,

≥ 0.

Therefore, $(V_f - f)(y) \ge (V_f - f)(x)$ and so $V_f - f$ is increasing.

Theorem 13. (A characterization of BV(a, b)) . BV(a, b) consists entirely of functions defined on $[a, b]$, expressible as the difference of two monotone increasing functions.

Proof. If f and g are monotone increasing functions, then by Theorem 7, f and g are in BV(a, b) and as a consequence of the triangle inequality $f -$ g is also in BV(a, b).

Suppose now f is in BV(a, b). Then both V_f and $V_f - f$ are increasing functions. Thus $f = V_f - (V_f - f)$ is the difference of two monotone increasing functions.

Theorem 14. If f is in BV(a, b), then f is Riemann integrable.

Proof. By Theorem 13, $f = g - h$ where g and h are monotone increasing functions. Since monotone increasing functions on $[a, b]$ are integrable and f being the difference of two Riemann integrable functions, f is Riemann integrable.

Below we state a deeper form of the Fundamental Theorem of Calculus involving only Riemann integrable function.

Theorem 15 (Fundamental Theorem of Calcullus). Let f be a Riemann integrable function on $[a, b]$. Define F: $[a, b] \to \mathbf{R}$ by $F(x) = \int_a^x f(t)\, dt$ for x in $[a, b]$. *Then*

1. F is in $BV(a, b)$,
2. F is continuous on $[a, b]$.
3. If f is continuous at x in $[a, b]$, then F is differentiable at x and $F'(x) = f(x)$.

Proof. Since f is Riemann integrable function on $[a, b]$, f is bounded on $[a, b]$. Thus by the completeness property of $\mathbf{R}$, $K = \sup\{|f(x)| : x \in [a, b]\}$ exists. Then $-K \le f(x) \le K$ for all x in $[a, b]$. Take any partition $\Delta : x_0 = a < x_1 < x_2 < \ldots < x_n = b$ of $[a, b]$. Then for all x in $[x_{i-1}, x_i]$, $-K \le f(x) \le K$. Since f is Riemann integrable on $[x_{i-1}, x_i]$ for $i = 1, \ldots, n$, by Theorem 2 or Corollary 3 of *Riemann Integral and Bounded Function*, for $i = 1, \ldots, n$, we have

$$-\int_{x_{i-1}}^{x_i} K dt \le \int_{x_{i-1}}^{x_i} f(t) dt \le \int_{x_{i-1}}^{x_i} K dt, \text{ that is,}$$

$$-K(x_i - x_{i-1}) \le \int_{x_{i-1}}^{x_i} f(t) dt \le K(x_i - x_{i-1}). \text{ ---------------------- (1)}$$

Thus, from (1) we have for $i = 1, \ldots, n$,

$$\left|\int_{x_{i-1}}^{x_i} f(t) dt\right| \le K(x_i - x_{i-1}). \text{ ----------------------------------- (2)}$$

Now, for $i = 1, \ldots, n$,

$$|F(x_i) - F(x_{i-1})| = \left|\int_a^{x_i} f(t) dt - \int_a^{x_{i-1}} f(t) dt\right| = \left|\int_{x_{i-1}}^{x_i} f(t) dt\right| \le K(x_i - x_{i-1}).$$

Therefore,

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| = \sum_{i=1}^{n} \left|\int_{x_{i-1}}^{x_i} f(t) dt\right| \le K \sum_{i=1}^{n} (x_i - x_{i-1}) = K(b-a).$$

This is true for any partition Δ of $[a, b]$ and so F is of bounded variation and the total variation by the completeness property of $\mathbf{R}$ exists and is finite. That is $F \in \mathrm{BV}(a, b)$ and $\mathrm{V}(F; a, b) = \sup\{\sum_{j=1}^{n} |\Delta F_j| : \Delta \text{ is a partition of } [a, b]\}$ is finite. This proves part (1). We have actually proved that $\mathrm{V}(F; a, b) \le \sup\{|f(x)| : x \in [a, b]\}(b-a)$.

For part (2). We shall show that F is uniformly continuous and hence continuous. For any $x < y$ such that $a \le x \le y \le b$ we have that $-K \le f(t) \le K$ for all t in $[x, y]$. *Hence* $-K(y-x) \le \int_x^y f(t) dt \le K(y-x)$ and so $\left|\int_x^y f(t) dt\right| \le K(y-x) = K|y-x|$. This is also true for $a \le y < x \le b$ because then $\left|\int_x^y f(t) dt\right| = \left|\int_y^x f(t) dt\right| \le K(x-y) = K|x-y|$. Therefore, for any x, y in $[a, b]$,

$$|F(x) - F(y)| = \left|\int_a^x f(t) dt - \int_a^y f(t) dt\right| = \left|\int_y^x f(t) dt\right| \le K|x-y|.$$

Thus for any $\varepsilon > 0$, take δ to be any real number greater than zero if $K = 0$, otherwise take any $0 < \delta < \varepsilon/K$. We have then for all x, y in $[a, b]$,

$$|x-y| < \delta \text{ implies that } |F(x) - F(y)| \le K|x-y| < K\,\varepsilon/K = \varepsilon.$$

Therefore, F is continuous on $[a, b]$,

Part (3) is proved in *Calculus, An Introduction*, page 137.

This completes the proof.