When is a continuous function on a closed and bounded interval be of bounded variation, absolutely continuous? The answer and application to generalized change of variable for Lebesgue integral.

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This is an intriguing question. Besides checking if the variation of a function is actually bounded above and the condition of absolute continuity is satisfied, we may use a useful but little-known criterion for deciding when a function is of bounded variation and absolutely continuous, given by Saks in his monograph "*Theory of The Integral*". We state the result as Theorem 1 below.

Theorem 1. Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function. Suppose E is a measurable subset of $[a, b]$ such that at each point x outside of E, f is differentiable, i.e., $f'(x)$ exists finitely and that the Lebesgue measure of $f(E)$, $m(f(E))$, is zero. Suppose further there exists a Lebesgue integrable function g: $[a, b] \to \mathbf{R}$ such that

$$f'(x) \leq g(x)$$

for $x \in [a, b] - E$. Then f is of bounded variation and absolutely continuous.

Remark. If f is absolutely continuous, then f is of bounded variation. The condition given in Theorem 1 is sufficient to prove both bounded variation and absolute continuity. On the other hand, if f is absolutely continuous, then the condition in Theorem 1 is fulfilled with g taken to be f' and E the complement of the set on which f is differentiable.

An immediate consequence is the following.

Corollary 2. Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function, differentiable everywhere except perhaps on a subset E of $[a, b]$, which is at most denumerable.
If f' is Lebesgue integrable or summable, then f is absolutely continuous.

Proof. Note that trivially $m(f(E)) = 0$. Let g be f'. Then by Theorem 1, f is absolutely continuous.

Remark. The condition of Corollary 2 implies that f is a N function. (See Lemma 4 below.) Thus, Corollary 2 can be deduced from Theorem 7 of " *Functions having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallee Poussin's Theorem*".

Next, we present a simple technical Lemma**,** a weaker form of a similar result in Saks monograph (Theorem 6.6 Chapter 9) restated by F. S. Cater replacing the Banach's condition (T_2) by a stronger condition which implies that the function is an N function or a function satisfying Lusin's condition. A function is a N function or Lusin function if it maps sets of measure zero to sets of measure zero. Banach has proved that any continuous N function necessarily satisfies Banach condition (T_2). f is said to have Banach's condition (T_2) if each value of the image of f, *except possibly for a set of measure zero,* is assumed by at most a denumerable number of points in the domain. In this weaker form it is much easier to prove than the stronger version.

Lemma 3. Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function. Suppose P is a subset of $[a, b]$ such that f is differentiable at each point of P and that $m(f([a, b] - P)) = 0$. Let $P_+ = \{x \in P : f'(x) \geq 0\}$ and $P_- = \{x \in P : f'(x) \leq 0\}$. Then, $\max(0, f(b) - f(a)) \leq m^*(f((a, b) \cap P_+))$ and $\max(0, f(a) - f(b)) \leq m^*(f((a, b) \cap P_-))$. Consequently,

$$-m^*(f((a, b) \cap P_-)) \leq f(b) - f(a) \leq m^*(f((a, b) \cap P_+)).$$

Proof. Suppose $f(a) < f(b)$. By hypothesis, $m(f((a, b) - P)) = 0$. Therefore,

$$m^*((f(a), f(b)) - f((a, b) - P)) = m^*((f(a), f(b))) = f(b) - f(a).$$

Next, we show that

$$(f(a), f(b)) - f((a, b) - P) \subseteq f((a, b) \cap P_+).$$

Take y in $(f(a), f(b)) - f((a, b) - P)$. Then $f(a) < y < f(b)$ and $y \notin f((a, b) - P)$. Then $f^{-1}(y)$ is a subset of P and so f is differentiable at each point of $f^{-1}(y)$. Now, since f is continuous, $f^{-1}(y)$ is compact and so is closed and bounded. Let $e = \max\{x : x \in f^{-1}(y)\}$. Then $f'(e) \geq 0$. This is because if $f'(e) < 0$, then by the definition of the derivative, there exists $x' > e$ such that

$$b > x' \text{ and } f(b) > y = f(e) > f(x').$$

Thus, by the Intermediate Value Theorem, there exists a point d such that $b > d > x'$ and $f(d) = y$. Hence, $d \in f^{-1}(y)$ and $d > e$. This contradicts that $e = \max\{x : x \in f^{-1}(y)\}$. Hence, $y = f(e) \in f((a, b) \cap P_+)$. It follows that

$$(f(a), f(b)) - f((a, b) - P) \subseteq f((a, b) \cap P_+).$$

Therefore,

$$m^*((f(a), f(b)) - f((a, b) - P)) \leq m^*(f((a, b) \cap P_+))$$

and consequently,

$$f(b) - f(a) \leq m^*(f((a, b) \cap P_+)).$$

Suppose $f(a) > f(b)$. Using a similar argument, we show that

$$f(a) - f(b) \leq m^*(f((a, b) \cap P_-)).$$

It follows that $\max(0, f(b) - f(a)) \leq m^*(f((a, b) \cap P_+))$ and

$$\max(0, f(a) - f(b)) \leq m^*(f((a, b) \cap P_-)).$$

Before we embark on the proof of Theorem 1, we show that under the hypothesis of Lemma 3, f is a N function.

Lemma 4. Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function. Suppose P is a subset of $[a, b]$ such that f is differentiable (finitely) at each point of P and that $m(f([a, b] - P)) = 0$. Then f is an N function.

Proof. Let E be a subset of $[a, b]$ of measure 0. Write $E = (E \cap P) \cup (E \cap ([a, b] - P))$. Then $m(E \cap P) = 0$ and $m(E \cap ([a, b] - P)) = 0$. By hypothesis,

$$m(f(E \cap ([a, b] - P)) = 0.$$

Since $m(E \cap P) = 0$, $E \cap P$ is measurable. Then by Theorem 2 of " *Functions having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallee Poussin's Theorem*",

$$m^*\big(f(E \cap P)\big) \leq \int_{E \cap P} |f'| = 0.$$

It follows that $m(f(E \cap P)) = 0$. Since

$$m^*(f(E)) \leq m^*(f(E \cap P)) + m^*(f(E \cap ([a, b] - P)) = 0,$$

$m(f(E)) = m^*(f(E)) = 0$. Hence, f is a N function.

Proof of Theorem 1.

The key is to show that either f has bounded positive variation or bounded negative variation. Then, since f is continuous and bounded it follows that f is of bounded variation. We can deduce this as follows.

Suppose f has bounded positive variation. Take any partition Q: $a = x_0 < x_1 \ldots < x_n = b$. Let $p(Q)$ be the positive variation with respect to Q and $n(Q)$ be the negative variation with respect to Q. Then

$$f(b) - f(a) = p(Q) - n(Q).$$

It follows that $n(Q) = p(Q) + f(a) - f(b) \le f(a) - f(b) + p(f)$, where $p(f)$ is the positive variation of f, for any partition Q. This shows that f has bounded negative variation and so f has bounded total variation. Conversely, we can show similarly, that if f has bounded negative variation, then f has bounded positive variation and so is of bounded variation.

Let $P = [a, b] - E$.

Then f is differentiable at each point of P and $m\left(f([a,b]-P)\right) = m\left(f(E)\right) = 0$.

Let $[a_i , b_i]$ be any closed subinterval in $[a, b]$. Then by Lemma 3,

$$\max(0,\ f(b_i) - f(a_i)) \le m^*(f((a_i , b_i) \cap P_+))$$

Note that P is measurable and so $P_+ = \{x \in P: f'(x) \ge 0\}$ is also measurable. By Theorem 2 of " *Functions having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallee Poussin's Theorem*",

$$m^*\left(f((a_i,b_i)\cap P_+)\right) \le \int_{(a_i,b_i)\cap P_+} |f'| = \int_{(a_i,b_i)\cap P_+} f' .$$

But by hypothesis, $f'(x) \le |g(x)|$ for x in P_+, and so

$$\int_{(a_i,b_i)\cap P_+} f' \le \int_{(a_i,b_i)\cap P_+} |g| \le \int_{(a_i,b_i)} |g| = \int_{a_i}^{b_i} |g| .$$

Therefore, it follows by Lemma 3 that

$$\max(0,\ f(b_i) - f(a_i)) \le \int_{a_i}^{b_i} |g| . \text{ -------------------------- } (1)$$

Note that since g is Lebesgue integrable, $|g|$ is also Lebesgue integrable.

So take any partition Q: $a = x_0 < x_1 \ldots < x_n = b$. Then by (1)

$$p(Q) = \sum_{i=1}^{n} \max\left(f(x_i) - f(x_{i-1}), 0\right) \le \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |g| = \int_a^b |g| .$$

Hence, f is of bounded positive variation. Consequently, f is of bounded variation. Therefore, f is differentiable almost everywhere and f' is Lebesgue integrable. Then using the same set P as above and replacing g by f' for any closed interval $[u, v]$ in $[a, b]$, by (1) we get, if $f(u) < f(v)$,

$$f(v) - f(u) \le \int_u^v |f'| .$$

Also, if $f(u) > f(v)$, by Lemma 3,

$$f(u) - f(v) \le m^* f\left(((u,v) \cap P_-)\right) \le \int_u^v |f'| .$$

Hence,

$$|f(v) - f(u)| \le \int_u^v |f'| . \text{ --------------------------- } (2)$$

Note that the function $F(x)$ defined by $F(x) = \int_a^x |f'|$ is absolutely continuous because it is the indefinite integral of a Lebesgue integrable function. Thus, given $\varepsilon > 0$, there exists $\delta > 0$

such that for any non-overlapping sequence of closed intervals $\{[a_i , b_i] ; i =1, \ldots,n\}$ with $\sum_{i=1}^{n} |b_i - a_i| < \delta$, we have that

$$\sum_{i=1}^{n} \int_{a_i}^{b_i} |f'| < \varepsilon .$$

Therefore, for any non-overlapping sequence $\{[a_i , b_i] \}$ with $\sum_{i} |b_i - a_i| < \delta$,

$$\sum_{i=1}^{n} | f(b_i) - f(a_i)| \leq \sum_{i=1}^{n} \int_{a_i}^{b_i} |f'| < \varepsilon .$$

This shows that f is absolutely continuous. This completes the proof.

Remark. 1. The hypothesis of Theorem 1 implies that $m(f (\{x \in [a, b] : f' (x) = \pm \infty \}) = 0$. Furthermore, it also implies that f is of bounded variation. Thus, it follows by Theorem 13 of "*Functions having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallee Poussin's Theorem*", that f is absolutely continuous. Note that $m(f (\{x \in [a, b] : f' (x) = \pm \infty \}) = 0$ is a necessary condition for absolute continuity.
2. By Lemma 4, any function f satisfying the hypothesis of Theorem 1 is a N function. So, Theorem 1 resembles the Banach Zarecki Theorem since the hypothesis implies that f is of bounded variation. Recall that Banach Zarecki Theorem states that any continuous function of bounded variation on a closed and bounded interval, which is also a N function is absolutely continuous. Theorem 1 is a little more convenient since one need not verify that the function f is of bounded variation.

If it is known that the function $f : [a, b] \to \mathbf{R}$ is a continuous N function, which is differentiable almost everywhere on $[a, b]$, then f maps its set of non-differentiability (finite or infinite) into a set of measure zero. Consequently, by Theorem 1, for f to be absolutely continuous it is sufficient and necessary that f' be dominated from above by a Lebesgue integrable function. We state this result below.

Theorem 5. Suppose $f : [a, b] \to \mathbf{R}$ is a continuous N function. Furthermore, suppose that f is differentiable almost everywhere on $[a, b]$. Then f is absolutely continuous if, and only if, there exists a Lebesgue integrable function g such that $f' \leq g$ almost everywhere on $[a, b]$.

The following is a consequence of Theorem 5.

Corollary 6. Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function. Furthermore, suppose that f is differentiable everywhere on $[a, b]$ except perhaps on a subset which is at most denumerable. Then f is absolutely continuous if there exists a Lebesgue integrable function g such that $f' \leq g$ almost everywhere on $[a, b]$.

Proof. Note that if $f : [a, b] \to \mathbf{R}$ is continuous and differentiable everywhere on $[a, b]$ except perhaps on a subset which is at most denumerable, then f is a N function. The result then follows immediately from Theorem 5.

Remark. 1. Compare Theorem 5 with Theorem 7 of " *Functions having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallee*

Poussin's Theorem" which states that, given the hypothesis of Theorem 5, f is absolutely continuous if, and only if, f' is Lebesgue integrable. Theorem 5 is easier to use as we only need to look for a Lebesgue integrable function dominating the derivative of f, where the derivative of f exists (finitely).

We now apply Theorem 1 to prove Goodman's version of a change of variable formula for the Lebesgue integral. We state the Theorem as follows.

Theorem 7. Suppose g: $[a, b] \to \mathbf{R}$ is continuous and $f : [c, d] \to \mathbf{R}$ is a Lebesgue integrable function such that the range of g is contained in $[c, d]$. Let $F : [c, d] \to \mathbf{R}$ be defined by $F(x) = \int_c^x f(t)dt$. Suppose g maps its set of non-differentiability into a set of measure zero. Define the function $g^*:[a, b] \to \mathbf{R}$ by

$$g^*(x) = \begin{cases} g'(x), & \text{when } g'(x) \text{ exists (finitely),} \\ 0, & \text{when } g'(x) \text{ does not exist or is infinite} \end{cases}$$

Then $\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g^*(x)dx$ if the integral on the right exists.

Proof. Note that F and g are both continuous. It is sufficient to show that if $\int_a^b f(g(x))g^*(x)dx$ exists, then the function $F \circ g$ is absolutely continuous and that

$$(F \circ g)'(x) = f(g(x))g^*(x)$$

almost everywhere on $[a, b]$.

Observe that if g is differentiable at x and F is differentiable at $g(x)$, then $F \circ g$ is differentiable at x. Now, F is absolutely continuous and so F is differentiable almost everywhere on $[c, d]$. Thus, there exists a subset E such that $m(E) = 0$, F is differentiable on $[c, d] - E$ and $F' = f$ on $[c, d] - E$. By hypothesis, g is differentiable except on a set A, where $m(g(A)) = 0$. By lemma 4, g is a N function. Let $B = A \cup g^{-1}(E)$. Since the Lebesgue measure on $[a, b]$ is regular, there exists a measurable subset $C \supseteq B$ such that $m(C - B) = 0$. Then for $x \notin C$, g is differentiable at x and F is differentiable at $g(x)$ and so $F \circ g$ is differentiable at *every* x in $[a, b] - C$ and

$$(F \circ g)'(x) = f(g(x))g'(x).$$

Now $g(B) \subseteq E \cup g(A)$. Since $m(E) = 0$ and $m(g(A)) = 0$, $m(g(B)) = 0$. Since F is absolutely continuous, F is a N function and so $m(F \circ g(B)) = 0$. As $m(C–B) = 0$, we have too that $m(F \circ g(C - B)) = 0$. It follows that $m(F \circ g(C)) = 0$. Moreover, for every x in $[a, b] - C$,

$$(F \circ g)'(x) = f(g(x))g'(x) \le f(g(x))g^*(x)$$

and by hypothesis $f(g(x))g^*(x)$ is Lebesgue integrable. Therefore, by Theorem 1, $F \circ g$ is absolutely continuous on $[a, b]$. Consequently, $F \circ g$ is differentiable almost everywhere on $[a, b]$ and so $F \circ g$ is differentiable almost everywhere on B. As $m(F \circ g(B)) = 0$, by Theorem 2 of "*Change of Variables Theorem*", $(F \circ g)' = 0$ almost everywhere on $B = A \cup g^{-1}(E)$. Note that g is differentiable on $g^{-1}(E) - A$ and $m\left(g(g^{-1}(E) - A)\right) = 0$. Then by Theorem 2 of "*Change of Variables Theorem*", $g' = 0$ almost everywhere on $g^{-1}(E) - A$ and so $(F \circ g)'(x) = f(g(x))g'(x) = 0$ almost everywhere on $g^{-1}(E) - A$.

Hence,

$$(F \circ g)'(x) = \begin{cases} f(g(x))g'(x), & \text{when } g'(x) \text{ exists (finitely),} \\ 0, & \text{when } g'(x) \text{ does not exist or is infinite} \end{cases}$$

almost everywhere on $[a, b]$

$= f(g(x))g^*(x)$ almost everywhere on $[a, b]$. -------------- (1)

By the absolute continuity of F,

$$\int_{g(a)}^{g(b)} f(x)dx = F(g(b)) - F(g(a)). \qquad \text{---------------------- (2)}$$

By the absolute continuity of $F \circ g$,

$$F(g(b) - F(g(a)) = \int_a^b (F \circ g)'(x)dx$$

$$= \int_a^b f(g(x))g^*(x)dx \qquad \text{------------------------ (3)}$$

by (1).
It then follows from (2) and (3) that

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g^*(x)dx.$$

Goodman stated a more generalized change of variable theorem, requiring only that g satisfies Lusin's condition, i.e., g is a continuous N function.

Theorem 8. Suppose g: $[a, b] \to \mathbf{R}$ is a continuous N function and $f : [c, d] \to \mathbf{R}$ is a Lebesgue integrable function such that the range of g is contained in $[c, d]$. Let $F : [c, d] \to \mathbf{R}$ be defined by $F(x) = \int_c^x f(t)dt$. Define the function $g^* :[a, b] \to \mathbf{R}$ by

$$g^*(x) = \begin{cases} g'(x), & \text{when } g'(x) \text{ exists (finitely),} \\ 0, & \text{when } g'(x) \text{ does not exist or is infinite} \end{cases}.$$

Then $\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g^*(x)dx$ if the integral on the right exists, or more precisely, if the integral $\int_D f(g(x))g^*(x)dx$ exists, where D is the set on which g is differentiable finitely.

As in the proof of Theorem 1 in "*Change of Variables Theorems*", the main step is to show that $F \circ g$ is absolutely continuous and that an extended generalized chain rule for the composite function $F \circ g$ holds almost everywhere on $[a, b]$.

Firstly, we state the following chain rule for $F \circ g$.

Theorem 9. Suppose g: $[a, b] \to \mathbf{R}$ is a continuous N function and $f : [c, d] \to \mathbf{R}$ is a Lebesgue integrable function such that the range of g is contained in $[c, d]$. Let $F : [c, d] \to \mathbf{R}$ be defined by $F(x) = \int_c^x f(t)dt$.

Then there is (i) a subset L such that $m(F\circ g(L))=0$ and for any x in L, if $(F\circ g)'(x)$ exists, $(F\circ g)'(x)=0$ except for such x in a set of measure zero and if $g'(x)$ exists, either $g'(x)=0$ except for such x in a subset of L of measure zero or $f(g(x))=0$, (ii) a subset K in the complement of L, where both $(F\circ g)'(x)$ and $g'(x)$ do not exist for every x in K and (iii)

$$(F\circ g)'(x)=f(g(x))g'(x)$$

for x in the complement of $L\cup K$.

Proof. Since F is an indefinite integral of an integrable function, F is absolutely continuous. Therefore, F is differentiable almost everywhere on $[c, d]$. Thus, there exists a subset E of $[c, d]$ such that $m(E) = 0$ and F is differentiable (finitely) on $[c, d] - E$ *and* $F'(x) = f(x)$ for x in $[c, d] - E$. Note that F is also a N function since it is absolutely continuous and so $m(F(E)) = 0$. Let $E_0 = \{x \in [c, d]\colon F$ is differentiable at x and $F'(x) = 0\}$. Then by Theorem 3 of " *Functions having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallee Poussin's Theorem*", $m(F(E_0)) = 0$.
Let $B = g^{-1}(E\cup E_0)$. Suppose $g'(x)$ does not exist finitely at every point of a subset A and differentiable (finitely) at every point outside of A. Let A_∞ be the subset of A where the derivative is $\pm\infty$. Then by the Theorem of Denjoy, Saks and Young (Theorem 14), $m(A_\infty) = 0$.
Let $C = A\cup B$. Then for any x in $[a, b] - C$, F is differentiable at $g(x)$ and g is differentiable at x, consequently, $F\circ g$ is differentiable at x and

$$(F\circ g)'(x)=f(g(x))g'(x).$$

Note that since both F and g are N functions, the composite, $F\circ g$ is also a N function. Let $C_1 = A_\infty \cup B$. Then $m\big(F\circ g(C_1)\big)=0$ because $m\big(F\circ g(A_\infty)\big)=0$ and because $g(B)\subseteq E\cup E_0$ so that $m\big(F\circ g(B)\big)\le m\big(F(E\cup E_0)\big)=0$.
Now consider $K = A - C_1$. Then F is differentiable at $g(x)$ for every x in K and $F'(g(x)) \neq 0$. Thus, for x in K, we can write,

$$\frac{F\circ g(x+h)-F\circ g(x)}{h}=H\big(g(x+h)-g(x)\big)\bullet\frac{g(x+h)-g(x)}{h},$$

where, $H(k)=\begin{cases}\dfrac{F(g(x)+k)-F(g(x))}{k}, & \text{when } k\neq 0,\\ F'(g(x)), & \text{when } k=0\end{cases}$, ------------------- (1)

and H is continuous.
It follows from (1) that both $F\circ g$ and g are not differentiable finitely or infinitely at every point in K.
Let $L = C_1$. Then $L\cup K = C$.
Now, since $m\big(F\circ g(C_1)\big)=0$, on the subset of C_1, where $F\circ g$ is differentiable finitely or infinitely, $(F\circ g)'(x)=0$ almost everywhere by Theorem 2 of "*Change of Variables Theorems*". If x is in $C_1 - A$, then $g'(x)$ exists finitely. Since $m(g(g^{-1}(E)\cap (C_1 - A))) = 0$, by Theorem 2 of "*Change of Variables Theorems*", $g'(x) = 0$ almost everywhere on $g^{-1}(E)\cap (C_1 - A)$ and since $g((C_1 - A) - (g^{-1}(E)\cap (C_1 - A))) \subseteq E_0 - E$, $f(g(x)) = 0$ for every x in $(C_1 - A) - (g^{-1}(E)\cap (C_1 - A))$.
Consequently,

$$(F \circ g)'(x) = f(g(x))g^*(x) = 0,$$

whenever $\left(F \circ g\right)'(x)$ exists in L. This completes the proof.

We shall need the following theorem due to Banach (see Saks monograph chapter 9, Theorem 7.3).

Theorem 10 (Banach). Suppose $f : [a, b] \to \mathbf{R}$ is a continuous N function. Then f satisfies Banach condition (T_2) on $[a, b]$.

I thank Thierry Jeulin for pointing out that a previous version of Theorem 11 is incorrect. The current version is needed to prove a stronger version (Lemma 12) of Lemma 3.

Theorem 11. Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function. Let K be the subset of $[a, b]$ such that $f'(x)$ does not exist finitely or infinitely for any x in K. Suppose for each y in $f(K)$, there exists an isolated point x in K of $f^{-1}(y)$. Then $m\big(f(K)\big) = 0$.

Proof

By hypothesis with each y in $f(K)$, there exists a point x_y in K such that x_y is an isolated point of $f^{-1}(y)$. Thus, by examining the sign of $f(x) - f(x_y)$ for x in a small neighbourhood of x_y and not equal to x_y, $f(x) - f(x_y)$ either changes sign as x passes through x_y or the sign remains the same.

If $f(x) - f(x_y)$ changes sign as x passes through x_y, then $f(x_y)$ is either a strict local maximum or a strict local minimum. It follows that such a point x_y belongs to a set which is at most denumerable and hence of measure zero. (See Theorem 16 below.)

If the sign of $f(x) - f(x_y)$ remains the same in a small punctured neighbourhood of x_y, then the four Dini derivatives have the same sign. As $f'(x_y) \neq \pm\infty$, we have that either $0 \leq \min(D_+ f(x_y), D_- f(x_y)) < \infty$ or $0 \geq \max(D^+ f(x_y), D^- f(x_y)) > -\infty$. Since f is not differentiable at x_y, by Theorem 15 below, x_y belongs to a set E of measure zero and $m(f(E)) = 0$. Hence the collection $\{x_y : y \in f(K)\}$ is a set of measure zero and that $m(f(K)) = 0$.

To prove Theorem 13, a result about when a continuous function is of bounded variation, we shall need a stronger result than Lemma 3.

Lemma 12. Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function that satisfies Banach's condition T_2. Let $P_+ = \{x \in [a,b] : f'(x) \geq 0\}$ and $P_- = \{x \in [a,b] : f'(x) \leq 0\}$.

Then, $\max(0, f(b) - f(a)) \leq m^*(f((a, b) \cap P_+))$ and $\max(0, f(a) - f(b)) \leq m^*(f((a, b) \cap P_-))$. Consequently,

$$-m^*(f((a, b) \cap P_-)) \leq f(b) - f(a) \leq m^*(f((a, b) \cap P_+)).$$

Let P be the subset of $[a, b]$ such that f is differentiable finitely or infinitely at each point of P. The set P is uncountable. Moreover, unless f is a constant function one of P_+ or P_- must have positive measure.

Proof. Suppose $f(a) < f(b)$. Let Y be the subset of the values in the range of f, which are assumed by f at most an enumerable number of times. By removing values from Y if need be, we may assume that the values of $f(a)$ and $f(b)$ is not assumed only once. Then since f is

T2 , $m(f((a,b))-Y)=0$. For each y in $f((a,b))$, let $E_y = f^{-1}(y)$. Since f is continuous, E_y is closed in [a, b]. We shall show that for each $y \in Y$, we can choose a point $x_y \in E_y$ such that (i) the *upper derivate* of f at x, $\overline{D}f(x_y) \geq 0$ and (ii) x_y is an isolated point of E_y.
If E_y is a single point and since $f(a) < f(b)$, we choose x_y to be this point and plainly, (i) and (ii) are satisfied for x_y.
Suppose now E_y contains more than one point. Note that E_y is at most denumerable.
If E_y is finite, then each point of E_y is an isolated point of E_y. Pick any two points, α_y and β_y in E_y such that $\alpha_y < \beta_y$ and $(\alpha_y , \beta_y) \cap E_y = \varnothing$. That is, there are no other isolated point of E_y between α_y and β_y. Therefore, for all x in $\alpha_y < \beta_y$, $f(x) \neq y$. Therefore, one of $\overline{D}f(\alpha_y)$ or $\overline{D}f(\beta_y) \geq 0$. Choose x_y to be one of α_y or β_y when the *upper derivate* of f at the point is greater or equal to 0. Note that since $f^{-1}(f(a))$ and $f^{-1}(f(b))$ each contains more than one point, we may choose x_y to be in (a, b).
If E_y is an infinite set, then it is at most denumerable and closed in [a, b].
Let E_y' be the set of limit points of E_y. Since E_y is closed, $E_y' \subseteq E_y$ and so E_y' is at most denumerable. Note that on account of Bolzano Weierstrass Theorem, E_y' is non empty and is closed. Since E_y' is at most denumerable, it cannot be a perfect set and must contain an isolated point x_0. Thus, there exists an open interval (c, d) such that $(c,d) \cap E_y' = x_0$.
Therefore, there are no other limit points of E_y in (c, d) except x_0. Hence there are only isolated points of E_y in $(c,d)-\{x_0\}$. Since there are no limit points of E_y in $(c,d)-\{x_0\}$ we can find a pair of consecutive isolated points $\alpha_y < \beta_y$ in $(c,d)-\{x_0\}$. Now we choose x_y to be one of α_y or β_y when the *upper derivate* of f at the point is greater or equal to 0. Note that $x_y \in (a,b)$. Since x_0 is a limit point of E_y, at least one of $(c,x_0) \cap E_y$ or $(x_0,d) \cap E_y$ is infinite and contains no limit points of E_y.
Now, let $X = \{x_y : y \in Y\}$. Then on account of (i) f is not differentiable finitely or infinitely on $X - P_+$. By Theorem 11, $m(f(X-P_+))=0$. Hence, $m(f(X-P))=0$.
Therefore, $m(Y) = m(f(X)) = m(f(X \cap P_+) \leq m(f(P_+))$. Since f is continuous, $f(b)-f(a) \leq m(f([a,b]) = m(Y) \leq m(f(P_+))$.

Suppose $f(a) > f(b)$. Let Y be the subset of the values in the range of f which are assumed by f at most an enumerable number of times. We assume that if $f(a)$ is assumed only once by a, then we remove $f(a)$ from Y and also if $f(b)$ is assumed only once by b in [a, b], then we remove $f(b)$ from Y too. Then since f is T2 , $m(f((a,b))-Y)=0$. For each y in $f((a,b))$, let $E_y = f^{-1}(y)$. Since f is continuous, E_y is closed in [a, b]. As before, we can show that for each $y \in Y$, we can choose a point $x_y \in E_y$ such that (i) the *lower derivate* of f at x, $\underline{D}f(x_y) \leq 0$ and (ii) x_y is an isolated point of E_y. With these two properties, we show similarly as above that

$$f(a)-f(b) \leq m(f([a,b]) = m(Y) \leq m(f(P_-)).$$

It follows that

$$-\,m^*(f((a,b)\cap P_-)) \le f(b)-f(a) \le m^*(f((a,b)\cap P_+)).$$

Thus, for any $c \in (a,b]$,

$$-\,m^*(f((a,c)\cap P_-)) \le f(c)-f(a) \le m^*(f((a,c)\cap P_+)). \text{ ---------- (1).}$$

If $m(f(P_-)) = m(f(P_+)) = 0$, it follows from (1) that $f(c) = f(a)$ for $c \in (a,b]$ and so f is a constant function. Therefore, if f is not a constant function, one of $f(P_-)$ or $f(P_+)$ must have positive measure and so $m(f(P_- \cup P_+)) > 0$ and the set $P = P_+ \cup P_-$ is non-denumerable.

Note that assuming that the interval $[a, b]$ is non-degenerate, (i) if $f(a) < f(b)$, then $m(f([a,b]) = m(f(P_+))$ and if $m(f(P_+)) = 0$, then $m(f([a,b]) = 0$ and so f is a constant function (ii) if $f(a) > f(b)$, then $m(f([a,b]) = m(f(P_-))$ and if $m(f(P_-)) = 0$, then $m(f([a,b]) = 0$ and so f is a constant function (iii) if $f(a) = f(b)$ and f is not constant, then for any c in (a, b) such that $f(a) < f(c)$ $m(f([a,c]) = m(f(P_+ \cap (a,c)))$ and $m(f([c,b]) = m(f(P_- \cap (c,b)))$ so that $m(f([a,b]) = 0$ if $m(f(P_+)) = m(f(P_-)) = 0$ giving a contradiction and (iv) if $f(a) = f(b)$ and f is not constant, then for any c in (a, b) such that $f(a) > f(c)$, $m(f([a,c]) = m(f(P_- \cap (a,c)))$ and $m(f([c,b]) = m(f(P_+ \cap (c,b)))$ so that $m(f([a,b]) = 0$ if $m(f(P_+)) = m(f(P_-)) = 0$ giving a contradiction. Hence, one of $f(P_-)$ or $f(P_+)$ must have positive measure unless f is a constant function.

Theorem 13. Suppose $f: [a, b] \to \mathbf{R}$ is a continuous function satisfying condition T_2. Let $g:[a,b] \to \mathbb{R}$ be a Lebesgue integrable function. Suppose $f'(x) \le g(x)$ at each point x for which the derivative exists except perhaps those of a denumerable subset or those of a measurable subset E for which $m(f(E)) = 0$. Then f is of bounded variation and for any sub-interval $[c, d]$ in $[a, b]$, $f(d) - f(c) \le \int_c^d f'(x)dx$.

Proof. The proof is almost similar to the proof of Theorem 1.
We shall show that f has bounded positive variation. Then, since f is continuous and bounded it follows that f is of bounded variation.

Let $P_+ = \{x \in [a,b] : f'(x) \ge 0\}$, that is P_+ is the subset of points in $[a, b]$, at which derivative of f exists and is non-negative. Now for each point in $P_+ - E$, by hypothesis, $f'(x) \le g(x)$. This means that f is differentiable finitely in $P_+ - E$. Therefore, by Theorem 2 of "*Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*",

$$m^*(f(P_+ - E)) \le \int_{P_+ - E} |f'| = \int_{P_+ - E} f' \le \int_{P_+ - E} g = \int_{P_+ - E} |g| .$$

By hypothesis, $m(f(E)) = 0$. Therefore, for any interval $[c, d]$, by Lemma 12,

$$\max(0, f(d) - f(c)) \le m^*\big(f((c,d) \cap P_+)\big) = m^*\big(f((c,d) \cap (P_+ - E)\big)$$
$$\le \int_{(c,d)\cap(P_+ - E)} |g| \le \int_c^d |g(x)|\,dx \text{ --------------------- (1)}$$

Note that since g is Lebesgue integrable, $|g|$ is also Lebesgue integrable.

For any partition Q: $a = x_0 < x_1 \ \ldots < x_n = b$, by (1),

$$p(Q) = \sum_{i=1}^{n} \max\left(f(x_i) - f(x_{i-1}), 0\right) \le \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |g| = \int_a^b |g| .$$

Hence, f is of bounded positive variation. Consequently, f is of bounded variation.
It follows that f is a continuous function of bounded variation. Therefore, for any two points c and d in $[a, b]$ with $c < d$,

$$f(d) - f(c) \le \left| f(d) - f(c)\right| \le v_f(d) - v_f(c) \le \int_c^d v_f{}'(x)dx = \int_c^d \left| f'\right|(x)dx .$$

Much more is true. Since f is continuous with bounded variation, by Corollary 24 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*", the Lebesgue Stieltjes measure λ_f generated by f satisfies

$$\lambda_f([c,d]) = \int_c^d f' + m\left(v_f(I_{+\infty} \cap [c,d])\right) - m\left(v_f(I_{-\infty} \cap [c,d])\right)$$

where $I_{+\infty} = \{x \in [a,b]: f$ is continuous at x and $f'(x) = +\infty\}$ and
$I_{-\infty} = \{x \in [a,b]: f$ is continuous at x and $f'(x) = -\infty\}$.

Hence, $f(d) - f(c) = \lambda_f([c,d]) = \int_c^d f' + m\left(v_f(I_{+\infty} \cap [c,d])\right) - m\left(v_f(I_{-\infty} \cap [c,d])\right)$ --------- (1)

By hypothesis, $m(f(I_{+\infty})) = 0$ and so $m(v_f(I_{+\infty})) = 0$ by Theorem 1 of "*Functions of Bounded Variation and Johnson's Indicatrix*". Hence,

$$f(d) - f(c) \le \int_c^d f' + m\left(v_f(I_{+\infty} \cap [c,d])\right) = \int_c^d f' .$$

Remark.
If f is continuous of bounded variation on $[a, b]$ and $m(f(I_{+\infty} \cup I_{-\infty})) = 0$, then by identity (1) above, $f(d) - f(c) = \int_c^d f'$ for any two points c, d with $c < d$. Consequently, f is absolutely continuous on $[a, b]$. If f is absolutely continuous, then $m(f(I_{+\infty} \cup I_{-\infty})) = 0$.
Hence, $m(f(I_{+\infty} \cup I_{-\infty})) = 0$ is a necessary and sufficient condition for a continuous function of bounded variation on $[a, b]$ to be absolutely continuous.
(See Theorem 16 in of "*Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*" and its proof.)

We now state a version of the theorem of Denjoy, Saks and Young for reference. (This is Theorem 13 of "*Denjoy Saks Young Theorem for Arbitrary Function*".)

Theorem 14. (Denjoy Saks Young Theorem.)
Suppose $f : A \to \mathbb{R}$ is a finite valued function. Let
$N = \left\{x \in A: {}_AD^+ f(x) = -\infty \text{ or } {}_AD^- f(x) = -\infty \text{ or } {}_AD_+ f(x) = \infty \text{ or } {}_AD_- f(x) = \infty\right\}$,
$S = \left\{x \in A: {}_ADf(x) \text{ exists and is finite}\right\}$,
$T = \left\{x \in A: {}_AD^+ f(x) \text{ and } {}_AD_- f(x) \text{ are finite and equal }, {}_AD_+ f(x) = -\infty \text{ and } {}_AD^- f(x) = \infty\right\}$,
$U = \left\{x \in A: {}_AD^- f(x) \text{ and } {}_AD_+ f(x) \text{ are finite and equal }, {}_AD^+ f(x) = \infty \text{ and } {}_AD_- f(x) = -\infty\right\}$
and $V = \left\{x \in A: {}_AD^+ f(x) = {}_AD^- f(x) = \infty \text{ and } {}_AD_- f(x) = {}_AD_+ f(x) = -\infty\right\}$.

Then $A = N \cup S \cup T \cup U \cup V \cup E$, where E is a null set and $m\big(f(E)\big) = 0$.
Moreover, $m(N) = 0$ and f is a Lusin function on $S \cup T \cup U$.

The next result is used in the proof of Theorem 11.

Theorem 15. Suppose $f : A \to \mathbb{R}$ is a finite valued function. Suppose B is a subset of A such that at each point x of B, f has either both finite Dini derivates on the same side or finite bilateral derivates ${}_A\overline{D}f(x)$ or ${}_A\underline{D}f(x)$. Then, f is differentiable almost everywhere on B, i.e., for almost all x in B, ${}_ADf(x)$ exists and is finite. Moreover, for the subset E of B, where ${}_ADf(x)$ does not exists, $m^*(E) = m^*\big(f(E)\big) = 0$. f is a Lusin function on B.

(See Theorem 11 of "*Denjoy Saks Young Theorem for Arbitrary Function*".)

Note that $\min\big(D_+ f(x), D_- f(x)\big) = \underline{D}f(x)$ and $\max\Big(D^+ f(x), D^- f(x)\Big) = \overline{D}f(x)$ the lower and upper derivate of f at x respectively

Theorem 16. Suppose $f : [a, b] \to \mathbf{R}$ is a real valued function defined on the closed and bounded interval $[a, b]$. Then the set of points at which f assumes a strict maximum or minimum is at most denumerable.

Proof. Let E be the set of points, where f assumes a strict maximum**.** For any such local maximizer x *in* E, there exists an integer n such that for all $y \neq x$ in $(x - 1/n, x + 1/n)$, $f(y) < f(x)$. Moreover, it is obvious that $(x - 1/n, x + 1/n)$ cannot contain more than one maximizer. Consequently, the collection $A_n = \{ x \in E : f(y) < f(x)$ for all y in $(x - 1/n, x + 1/n)\}$ is a set of isolated points. Therefore, A_n is at most denumerable. This can be seen as follows. The collection $\mathcal{C} = \{(x - 1/(2n), x + 1/(2n)) : x \in A_n\}$ is a collection of disjoint open intervals covering A_n , such that each interval $(x - 1/(2n), x + 1/(2n))$ contains exactly one point in A_n . Since the set of real numbers is of the second countable, the collection $\mathcal{C}$ is at most denumerable. Now $E = \bigcup_{n=1}^{\infty} A_n$ and so it follows that E is at most denumerable. In a similar way we can show that the set of strict local minimizers is at most denumerable. Therefore, the union of these two sets is at most denumerable. This means that the set of points at which f assumes a strict maximum or minimum is at most denumerable.

Variation of Theorem 1.

Theorem 17. Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function. Suppose E is a measurable subset of $[a, b]$ such that at each point x outside of E, f is differentiable, i.e., $f'(x)$ exists finitely and that the Lebesgue measure of $f(E)$, $m(f(E))$, is zero.
Let $P_+ = \{ x \in [a, b] - E : f'(x) \geq 0\}$ and suppose further there exists a function $g : [a, b] \to \mathbf{R}$ such that

$$f'(x) \leq g(x)$$

for $x \in P_+$ and g is integrable or summable on P_+. Then f is absolutely continuous.

Proof. Exactly as in the proof of Theorem 1, we apply Lemma 3 to conclude that for any partition Q: $a = x_0 < x_1 \ldots < x_n = b$, of $[a, b]$, the positive variation $p(Q)$ satisfies,

$$\begin{aligned} p(Q) &= \sum_{i=1}^{n} \max\left(f(x_i) - f(x_{i-1}), 0\right) \le \sum_{i=1}^{n} m^*\left(f\left((x_{i-1}, x_i) \cap P_+\right)\right) \\ &\le \sum_{i=1}^{n} \int_{(x_{i-1},x_i)\cap P_+} |f'| = \sum_{i=1}^{n} \int_{(x_{i-1},x_i)\cap P_+} f' \\ &\le \sum_{i=1}^{n} \int_{(x_{i-1},x_i)\cap P_+} |g| = \int_{(a,b)\cap P_+} |g| . \end{aligned}$$

Hence, f is of bounded positive variation. Since f is continuous, f is of bounded variation. By Lemma 4, f is a Lusin function or N function. Hence, f is a continuous function of bounded variation which is also a Lusin function. Therefore, by the Banach Zarecki Theorem, (see Theorem 8 of "*Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*" for arbitrary function see Theorem 4 of "*Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*"), f is absolutely continuous.

Theorem 18 (Banach).
Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function. Then f is absolutely continuous if, and only if, f is a N function and that f' is Lebesgue integrable on $P_+ = \{x \in [a, b] : f'(x)$ exists finitely and $f'(x) \ge 0\}$, i.e.,

$$\int_{P_+} f' < \infty .$$

Proof. If f is absolutely continuous, then of course, f is of bounded variation and a N function (see Lemma 2 and Lemma 3 of "*Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*"). Since f is of bounded variation, f is differentiable almost everywhere and f' is Lebesgue integrable on $[a, b]$ and so on P_+.

Conversely, suppose f is a N function and that f' is Lebesgue integrable on P_+. Then by Theorem 10, f is T_2. Let $g : [a,b] \to \mathbb{R}$ be the function equal to f on P_+ and equal to 0 outside of P_+. Let $E_\infty = \{x \in [a, b]: f'(x) = +\infty\}$. By the Denjoy-Saks-Young Theorem (Theorem 14), $m(E_\infty) = 0$. Since f is a N function, $m(f(E_\infty)) = 0$. For $x \notin E_\infty$, if $f'(x)$ exists, then $f'(x) \le g(x)$. By Theorem 13, f is of bounded variation. Hence f is a continuous Lusin function of bounded variation and so it is absolutely continuous by the Banach Zarecki Theorem.

Theorem 19.
Suppose $f : [a, b] \to \mathbf{R}$ is a continuous N function. Then f must be differentiable at every point of a set of positive measure.
Proof. By Theorem 10, f satisfies Banach condition T_2 on $[a, b]$. Suppose no subset of differentiability has positive measure. Let $P_+ = \{x : f'(x) \text{ is finite and } f'(x) \ge 0\}$ and $P_- = \{x : f'(x) \text{ is finite and } f'(x) \le 0\}$. Then $m(P_+) = 0$ and $m(P_-) = 0$. Since $m(\{x : f'(x) = +\infty \text{ or } f'(x) = -\infty\}) = 0$, $m\left(f((P_+ \cup I_{+\infty}) \cap (c,d))\right) = 0$ and $m\left(f((P_- \cup I_{-\infty}) \cap (c,d))\right) = 0$ for $c \in (a,b]$. It follows by Lemma 12, that for any $c \in (a,b]$, $f(c) = f(a)$. Hence, f must be a constant function, and so f is differentiable on (a, b) with

positive measure, contradicting that no subset of differentiability has positive measure. Therefore, f must be differentiable (finitely) at every point of a set of positive measure.

Proof of Theorem 8.

By Theorem 9, there is a subset C such that both g and $F \circ g$ are differentiable at every point outside C, and

$$(F \circ g)'(x) = f(g(x))g'(x)$$

for x not in C.

Moreover, if $f(g(x))g'(x)$ is integrable on $D = \{x \in [a, b]$: $g'(x)$ exists finitely$\}$, then $(F \circ g)'$ is integrable on $[a, b] - C$.

Now, $C = L \cup K$. $F \circ g$ and g are not differentiable finitely or infinitely at every point in K. Note that L and K are disjoint. Since $m\big(F \circ g(L)\big) = 0$, on the subset M of L, where $F \circ g$ is differentiable, $(F \circ g)'(x) = 0$ almost everywhere on M. In $L - M$, $F \circ g$ is not differentiable. We may replace L by M. We may remove the subset of measure zero, where $(F \circ g)'(x) \neq 0$ from M. Thus, we may assume that $(F \circ g)'(x) = 0$ on M. Let A be the subset of $[a, b]$ where g is not differentiable or $g'(x) = \pm\infty$. If x is in $M - A$, then $g'(x)$ exists finitely and either $F'(g(x))$ does not exist or $F'(g(x)) = 0$. Then, on the subset of $M - A$, where $F'(g(x))$ does not exist, $g'(x) = 0$ almost everywhere. Removing the appropriate subset from M, we may assume that when $F'(g(x))$ does not exist, $g'(x) = 0$. If x is in $M \cap A$, then $g'(x)$ does not exist or $g'(x) = \pm\infty$.

Hence, $(F \circ g)'(x) = f(g(x))g^*(x)$ almost everywhere outside K. Note that $F \circ g$ and g are not differentiable finitely or infinitely at every point in K. By hypothesis, the integral $\int_D f(g(x))g^*(x)dx$ exists. This implies that $\int_{P_+} (F \circ g)'(x)dx < \infty$, where $P_+ = \{x \in [a,b] : (F \circ g)'(x) \geq 0\}$.

Then by Theorem 18, $F \circ g$ is absolutely continuous. Therefore,

$$\begin{aligned} F(g(b)) - F(g(a)) &= \int_a^b (F \circ g)'(x)dx = \int_D (F \circ g)'(x)dx \\ &= \int_D f(g(x))g^*(x)dx = \int_a^b f(g(x))g^*(x)dx. \end{aligned}$$

Consequently, $\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g^*(x)dx$.

Remark. Observe that for x outside of C, $g'(x) \neq 0 \Leftrightarrow (F \circ g)'(x) \neq 0$. Hence, $(F \circ g)'(x) = f(g(x))g^*(x)$ almost everywhere outside C. $\int_D f(g(x))g^*(x)dx$ exists implies that $\int_{D_0} f(g(x))g^*(x)dx$ is finite where $D_0 = \{x : g'(x)$ exists finitely and $g'(x) \neq 0\}$. Hence, $\int_{P_0} (F \circ g)'(x)dx < \infty$, where $P_0 = \{x : (F \circ g)'(x) > 0\}$ and so $\int_{P_+} (F \circ g)'(x)dx < \infty$.

In "*Change of Variables Theorems*", I made a remark after Theorem 1, querying if there are integrable function f and finite function g not having finite derivatives almost everywhere on

[a, b] such that $F \circ$ g is absolutely continuous on [a, b] but $(F \circ g)'(x) \neq f(g(x))\, g'(x)$ almost everywhere on [a, b]. Goodman in "*N-Functions and Integration By Substitution, Milan Journal of Mathematics vol 47 (1977) 123-134,*" gave such an example due to Ruziewicz, where g is a function not differentiable on a set of positive measure but its square g^2 is absolutely continuous on [0,1]. However, we can make some interesting observation as follows.

Theorem 20. Suppose g: [a, b] $\to$ **R** is a finite function f : [c, d] $\to$ **R** is a Lebesgue integrable function such that the range of g is contained in [c, d]. Let F: [c, d] $\to$ **R** be defined by $F(x) = \int_c^x f(t)dt$. Suppose $F \circ g$ is absolutely continuous on [a, b].
Then $(F \circ g)'(x) = f(g(x))g^*(x)$ almost everywhere on [a, b], where

$$g^*(x) = \begin{cases} g'(x), & \text{when } g'(x) \text{ exists (finitely),} \\ 0\,, & \text{when } g'(x) \text{ does not exist or is infinite} \end{cases},$$

$f(g(x))g^*(x)$ is Lebesgue integrable on [a, b] and

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g^*(x)dx = \int_D f(g(x))g'(x)dx,$$

where D = {$x \in$[a, b]: $g'(x)$ exists finitely.}.
Proof.
Since F is an indefinite integral of an integrable function, F is absolutely continuous. Therefore, F is a N function, differentiable almost everywhere on [c, d] and $F'(x) = f(x)$ almost everywhere on [c, d].
Let $E = \{x \in [c, d]$: $F'(x)$ does not exist finitely or $F'(x) = \pm\infty$ or $F'(x) \neq f(x)\}$. Then $m(E) = 0$ since $m\{x : F'(x) = \pm\infty\} = 0$ by the Denjoy-Saks-Young Theorem (Theorem14). It follows that for x in [c, d] – E, $F'(x)$ exists finitely and $F'(x) = f(x)$. Let $E_0 = \{x \in [c, d]$: $F'(x) = 0\}$. By Theorem 3 of "*Functions having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallee Poussin's Theorem*", $m(F(E_0)) = 0$. Since F is a N function, $m(F(E)) = 0$. Consequently, $m(F(E_0 \cup E)) = 0$. Let $B = g^{-1}(E_0 \cup E)$. Suppose $A = \{x \in [a, b]$: $g'(x)$ does not exist finitely or infinitely$\}$. If $x \in [a, b] - A$, then either $g'(x)$ is finite or $g'(x) = \pm\infty$.
Observe that $(F \circ g)'(x) = f(g(x))g'(x)$ for x in [a, b] – ($A \cup B$).
We now examine the derivative of $F \circ g$ on $A \cup B$. By hypothesis $F \circ g$ is absolutely continuous and so $F \circ g$ is differentiable almost everywhere on [a, b].
Now for x in $A - B$, $g(x) \notin E_0 \cup E$ and so we have that F is differentiable at $g(x)$ and $F'(g(x)) = f(g(x)) \neq 0$. But for x in $A - B$, $g'(x)$ does not exist finitely or infinitely. It follows that for x in $A - B$, $(F \circ g)'(x)$ does not exist finitely or infinitely. Since $F \circ g$ is differentiable almost everywhere on [a, b], $A - B$ must be of measure zero. Thus, we may assume without loss of generality that $A \subseteq B$. (We may simply remove the set $A - B$ of measure zero from A.)
Next, we examine the set B. Since $g(B) \subseteq E_0 \cup E$ and $m(F(E_0 \cup E)) = 0$, it follows that $m(F \circ g(B)) = 0$. Since $F \circ g$ is differentiable almost everywhere on B, it follows then by Theorem 2 of "*Change of Variables Theorems*", that $(F \circ g)'(x) = 0$ almost everywhere on B. Hence $(F \circ g)'(x) = 0$ almost everywhere on A.

Consider $g^{-1}(E) - A$. Since $m(g(g^{-1}(E) - A)) = 0$ and g is differentiable finitely on $g^{-1}(E) - A$, by Theorem 2 of "*Change of Variables Theorems*", $g'(x) = 0$ almost everywhere on $g^{-1}(E) - A$.
Note that for x in $g^{-1}(E_0) - (g^{-1}(E) \cup A)$, $g'(x)$ exists finitely and $f(g(x)) = 0$. Consequently, $(F \circ g)'(x) = f(g(x))g'(x) = 0$ *almost* everywhere on $B - A$. It follows that $(F \circ g)'(x) = f(g(x))g^*(x)$ almost everywhere on B. Hence, $(F \circ g)'(x) = f(g(x))g^*(x)$ almost everywhere on $[a, b]$. Therefore, $f(g(x))g^*(x)$ is Lebesgue integrable on $[a, b]$ and

$$\int_{g(a)}^{g(b)} f(x)dx = F(g(b)) - F(g(a)) = \int_a^b (F \circ g)'(x)dx$$

$$= \int_D (F \circ g)'(x)dx = \int_D f(g(x))g^*(x)dx,$$

where $D = \{x \in [a, b]: g'(x) \text{ exists finitely.}\}$ since $[a, b] - D = A$ and $(F \circ g)'(x) = 0$ almost everywhere on A.
This completes the proof.

There is a partial converse to Theorem 20. We impose the requirement that g be a continuous N function.

Theorem 21. Suppose g: $[a, b] \to \mathbf{R}$ is a continuous N function and $f : [c, d] \to \mathbf{R}$ is a Lebesgue integrable function such that the range of g is contained in $[c, d]$. Let $F : [c, d] \to \mathbf{R}$ be defined by $F(x) = \int_c^x f(t)dt$. Suppose $f(g(x))g^*(x)$ is Lebesgue integrable on $D = \{x \in [a, b]: g'(x) \text{ exists finitely.}\}$, where

$$g^*(x) = \begin{cases} g'(x), & \text{when } g'(x) \text{ exists (finitely),} \\ 0, & \text{when } g'(x) \text{ does not exist or is infinite} \end{cases}.$$

Then $F \circ g$ is absolutely continuous on $[a, b]$ and

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g^*(x)dx = \int_D f(g(x))g'(x)dx.$$

Proof. This is just Theorem 8. We deduce as in the proof of Theorem 8 that under the hypothesis of Theorem 21, $F \circ g$ is absolutely continuous on $[a, b]$. The remaining conclusion then follows from Theorem 20.

Remark. Note that the integrability of $f(g(x))g^*(x)$ is not sufficient to ensure that $F \circ g$ is absolutely continuous on $[a, b]$. Take for example, $f(x) = 2x$ and g to be the Cantor ternary function. Then g is increasing and continuous, $g' = 0$ almost everywhere on $[0, 1]$ but g is not absolutely continuous on $[0, 1]$ and therefore not a N function. $F \circ g = g^2$ and g^2 is not absolutely continuous on $[0, 1]$. With the terminology of Theorem 21, we can observe this by noting that $f(g(x))g^*(x) = 0$ almost everywhere on $[0,1]$ but g^2 is not constant.

Corollary 22. Suppose g: $[a, b] \to \mathbf{R}$ is an absolutely continuous function and $f : [c, d] \to \mathbf{R}$ is a Lebesgue integrable function such that the range of g is contained in $[c, d]$. Let $F : [c, d] \to \mathbf{R}$ be defined by $F(x) = \int_c^x f(t)dt$. Suppose $f(g(x))g^*(x)$ is Lebesgue integrable on $D = \{x \in [a, b]: g'(x) \text{ exists finitely.}\}$, where

$$g^*(x) = \begin{cases} g'(x), & \text{when } g'(x) \text{ exists (finitely),} \\ 0, & \text{when } g'(x) \text{ does not exist or is infinite} \end{cases} .$$

Then $F \circ g$ is absolutely continuous on $[a, b]$ and

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g^*(x)dx = \int_D f(g(x))g'(x)dx .$$

Proof. If the function g is absolutely continuous, then it is a continuous N function. The corollary then follows from Theorem 21.

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