On The Integrals $\int_0^\infty \frac{x^s}{(1+e^x)^n}dx$ and $\int_0^\infty \frac{x^s\ln(x)}{(1+e^x)^n}dx$

By Ng Tze Beng

In this article, we shall consider the integrals $\int_0^\infty \frac{x^s}{(1+e^x)^n}dx$ for $s \geq 0, n \geq 1$, We shall present a recursive relation and compute the integrals for $0 \leq s \leq 2$ and $1 \leq n \leq 5$. Along with the integrals we consider its derivative with respect to *s*, $\int_0^\infty \frac{x^s\ln(x)}{(1+e^x)^n}dx$ and present a recursive relation for the derivatives. We compute the integrals $\int_0^\infty \frac{x^s\ln(x)}{(1+e^x)^n}dx$, for the following cases, $1 \leq n \leq 4$ and $0 \leq s \leq 2$, $n = 5$ and $s = 1$.

We list the relation the integral $\int_0^\infty \frac{x^s}{(1+e^x)^n}dx$ has with the Eta and Gamma functions below:

For $s > -1$, $\Lambda(s) = \int_0^\infty \frac{x^s}{1+e^x}dx = \eta(s+1)\Gamma(s+1)$. -- (1)

$$\int_0^\infty \frac{x^s\ln(x)}{(1+e^x)}dx = \Lambda'(s) = \eta'(s+1)\Gamma(s+1) + \eta(s+1)\Gamma'(s+1).$$

For $s \geq 0$, $G(s) = \int_0^\infty \frac{x^s}{(1+e^x)^2}dx = \left(\eta(s+1) - \eta(s)\right)\Gamma(s+1)$, ----------------------------(2)

$$\int_0^\infty \frac{x^s\ln(x)}{(1+e^x)^2}dx = G'(s) = \left(\eta'(s+1) - \eta'(s)\right)\Gamma(s+1) + \left(\eta(s+1) - \eta(s)\right)\Gamma'(s+1).$$

For $s > -1$,

$H(s) = \int_0^\infty \frac{x^s}{(1+e^x)^3}dx = \frac{1}{2}\left(\eta(s-1) + 2\eta(s+1) - 3\eta(s)\right)\Gamma(s+1)$, -------------------- (3)

$$\int_0^\infty \frac{x^s\ln(x)}{(1+e^x)^3}dx = H'(s)$$

$= \frac{1}{2}\left(\eta'(s-1) + 2\eta'(s+1) - 3\eta'(s)\right)\Gamma(s+1) + \frac{1}{2}\left(\eta(s-1) + 2\eta(s+1) - 3\eta(s)\right)\Gamma'(s+1)$. --------- (3)*

The above relations have been presented in the article "***The Integrals*** $\int_0^\infty \frac{\ln(x)}{1+e^x}dx$, $\int_0^\infty \frac{(\ln(x))^2}{1+e^x}dx$, $\int_0^\infty \frac{x\ln(x)}{1+e^x}dx$, $\int_0^\infty \frac{x(\ln(x))^2}{1+e^x}dx$ $\int_0^\infty \frac{x\ln(x)}{e^x-1}dx$, **etc.**"

For $s \ge 0$ and $n \ge 1$,

$$\int_0^\infty \frac{x^s}{(1+e^x)^n}\,dx = \left[\frac{1}{s+1}x^{s+1}\frac{1}{(1+e^x)^n}\right]_0^\infty - \frac{1}{s+1}\int_0^\infty x^{s+1}(-n)\frac{e^x}{(1+e^x)^{n+1}}\,dx$$

$$= 0 + \frac{n}{s+1}\int_0^\infty \frac{x^{s+1}(e^x+1)-x^{s+1}}{(1+e^x)^{n+1}}\,dx = \frac{n}{s+1}\int_0^\infty \frac{x^{s+1}}{(1+e^x)^n}\,dx - \frac{n}{s+1}\int_0^\infty \frac{x^{s+1}}{(1+e^x)^{n+1}}\,dx.$$

Therefore,

$$\int_0^\infty \frac{x^{s+1}}{(1+e^x)^{n+1}}\,dx = \int_0^\infty \frac{x^{s+1}}{(1+e^x)^n}\,dx - \frac{s+1}{n}\int_0^\infty \frac{x^s}{(1+e^x)^n}\,dx,$$

or for $s \ge 1$ and $n \ge 1$, $\int_0^\infty \frac{x^s}{(1+e^x)^{n+1}}\,dx = \int_0^\infty \frac{x^s}{(1+e^x)^n}\,dx - \frac{s}{n}\int_0^\infty \frac{x^{s-1}}{(1+e^x)^n}\,dx.$

For $s \ge 1$ and $n \ge 2$, let $H_n(s) = \int_0^\infty \frac{x^s}{(1+e^x)^n}\,dx$.

$$H_n(s) = \int_0^\infty \frac{x^s}{(1+e^x)^n}\,dx = \int_0^\infty \frac{x^s}{(1+e^x)^{n-1}}\,dx - \frac{s}{n-1}\int_0^\infty \frac{x^{s-1}}{(1+e^x)^{n-1}}\,dx = H_{n-1}(s) - \frac{s}{(n-1)}H_{n-1}(s-1).$$

---------------------------------- (4)

Differentiating (4), for $s \ge 1$ and $n \ge 2$,

$$H_n{}'(s) = \int_0^\infty \frac{x^s\ln(x)}{(1+e^x)^n}\,dx = \int_0^\infty \frac{x^s\ln(x)}{(1+e^x)^{n-1}}\,dx - \frac{1}{n-1}\int_0^\infty \frac{x^{s-1}}{(1+e^x)^{n-1}}\,dx - \frac{s}{n-1}\int_0^\infty \frac{x^{s-1}\ln(x)}{(1+e^x)^{n-1}}\,dx, \quad \text{--- (5)}$$

Or,

$$H_n{}'(s) = \int_0^\infty \frac{x^s\ln(x)}{(1+e^x)^n}\,dx = H_{n-1}{}'(s) - \frac{1}{(n-1)}H_{n-1}(s-1) - \frac{s}{(n-1)}H_{n-1}{}'(s-1) \quad \text{------------------- (6)}$$

With $n = 4$, we have

$$H_4(s) = \int_0^\infty \frac{x^s}{(1+e^x)^4}\,dx = \int_0^\infty \frac{x^s}{(1+e^x)^3}\,dx - \frac{s}{3}\int_0^\infty \frac{x^{s-1}}{(1+e^x)^3}\,dx = H_3(s) - \frac{s}{3}H_3(s-1)$$

$$= H_3(s) - \frac{s}{3}\left(\frac{1}{2}\big(\eta(s-2)+2\eta(s)-3\eta(s-1)\big)\Gamma(s)\right)$$

$$= H_3(s) - \left(\frac{1}{6}\big(\eta(s-2)+2\eta(s)-3\eta(s-1)\big)\Gamma(s+1)\right)$$

$$= \frac{1}{2}\big(\eta(s-1)+2\eta(s+1)-3\eta(s)\big)\Gamma(s+1) - \left(\frac{1}{6}\big(\eta(s-2)+2\eta(s)-3\eta(s-1)\big)\Gamma(s+1)\right)$$

$$= \left(\eta(s-1)+\eta(s+1)-\frac{11}{6}\eta(s)-\frac{1}{6}\big(\eta(s-2)\big)\right)\Gamma(s+1). \quad \text{---------------------------- (7)}$$

Note that by analyticity we may extend relation (7) to $s > -1$.

Differentiating (7), we obtain,

$$\int_0^\infty \frac{x^s \ln(x)}{(1+e^x)^4}\,dx = H_4'(s) = \left(\eta'(s-1) + \eta'(s+1) - \frac{11}{6}\eta'(s) - \frac{1}{6}\left(\eta'(s-2)\right)\right)\Gamma(s+1)$$

$$+\left(\eta(s-1) + \eta(s+1) - \frac{11}{6}\eta(s) - \frac{1}{6}\left(\eta(s-2)\right)\right)\Gamma'(s+1)\,. \text{ --------- (8)}$$

We may proceed in the same manner with $H_5(s)$.

$$\int_0^\infty \frac{x^s}{(1+e^x)^5}\,dx = H_5(s) = H_4(s) - \frac{s}{4}H_4(s-1)$$

$$= \left(\eta(s-1) + \eta(s+1) - \frac{11}{6}\eta(s) - \frac{1}{6}\left(\eta(s-2)\right)\right)\Gamma(s+1)$$

$$-\frac{s}{4}\left(\eta(s-2) + \eta(s) - \frac{11}{6}\eta(s-1) - \frac{1}{6}\left(\eta(s-3)\right)\right)\Gamma(s)$$

$$= \left(\eta(s-1) + \eta(s+1) - \frac{11}{6}\eta(s) - \frac{1}{6}\left(\eta(s-2)\right)\right)\Gamma(s+1)$$

$$-\frac{1}{4}\left(\eta(s-2) + \eta(s) - \frac{11}{6}\eta(s-1) - \frac{1}{6}\left(\eta(s-3)\right)\right)\Gamma(s+1)$$

$$= \left((1+\frac{11}{24})\eta(s-1) + \eta(s+1) - (\frac{11}{6}+\frac{1}{4})\eta(s) - (\frac{1}{6}+\frac{1}{4})\left(\eta(s-2)\right) + \frac{1}{24}\eta(s-3)\right)\Gamma(s+1)$$

$$= \left(\frac{35}{24}\eta(s-1) + \eta(s+1) - \frac{25}{12}\eta(s) - \frac{5}{12}\left(\eta(s-2)\right) + \frac{1}{24}\eta(s-3)\right)\Gamma(s+1)\,.$$

$$\int_0^\infty \frac{x^s}{(1+e^x)^5}\,dx = H_5(s)$$

$$= \left(\frac{35}{24}\eta(s-1) + \eta(s+1) - \frac{25}{12}\eta(s) - \frac{5}{12}\left(\eta(s-2)\right) + \frac{1}{24}\eta(s-3)\right)\Gamma(s+1) \text{ ------ (9)}$$

Again, we note that relation (9) is valid for $s > -1$.

Differentiating (9) we get,

$$\int_0^\infty \frac{x^s \ln(x)}{(1+e^x)^5}\,dx = H_5'(s)$$

$$= \left(\frac{35}{24}\eta'(s-1) + \eta'(s+1) - \frac{25}{12}\eta'(s) - \frac{5}{12}\eta'(s-2) + \frac{1}{24}\eta'(s-3)\right)\Gamma(s+1)$$

$$+\left(\frac{35}{24}\eta(s-1)+\eta(s+1)-\frac{25}{12}\eta(s)-\frac{5}{12}\eta(s-2)+\frac{1}{24}\eta(s-3)\right)\Gamma'(s+1)\,. \quad \text{---------------} \quad (10)$$

The integrals.

$$\int_0^\infty \frac{1}{1+e^x}dx=\int_0^\infty \frac{e^{-x}}{1+e^{-x}}dx=\left[-\ln(1+e^{-x})\right]_0^\infty=\ln(2)\,.$$

$$\int_0^\infty \frac{1}{(1+e^x)^2}dx=\int_0^\infty \frac{e^{-x}e^{-x}}{(1+e^{-x})^2}dx=\left[\frac{e^{-x}}{(1+e^{-x})}\right]_0^\infty-\int_0^\infty \frac{1}{(1+e^{-x})}(-e^{-x})dx$$

$$=-\frac{1}{2}-\int_0^\infty \frac{-e^{-x}}{(1+e^{-x})}dx=-\frac{1}{2}-\left[\ln(1+e^{-x})\right]_0^\infty=-\frac{1}{2}+\ln(2)\,.$$

$$\int_0^\infty \frac{1}{(1+e^x)^3}dx=\int_0^\infty \frac{e^{-x}e^{-2x}}{(1+e^{-x})^3}dx=\left[\frac{e^{-2x}}{2(1+e^{-x})^2}\right]_0^\infty-\frac{1}{2}\int_0^\infty \frac{1}{(1+e^{-x})^2}(-2e^{-2x})dx$$

$$=-\frac{1}{8}+\int_0^\infty \frac{e^{-x}}{(1+e^{-x})^2}e^{-x}dx=-\frac{1}{8}+\left[\frac{e^{-x}}{1+e^{-x}}\right]_0^\infty+\int_0^\infty \frac{e^{-x}}{1+e^{-x}}dx=-\frac{1}{8}-\frac{1}{2}+\left[-\ln(1+e^{-x})\right]_0^\infty$$

$$=-\frac{5}{8}+\ln(2)\,.$$

$$\int_0^\infty \frac{1}{(1+e^x)^4}dx=\int_0^\infty \frac{e^{-x}e^{-3x}}{(1+e^{-x})^4}dx=\left[\frac{e^{-3x}}{3(1+e^{-x})^3}\right]_0^\infty-\frac{1}{3}\int_0^\infty \frac{1}{(1+e^{-x})^3}(-3e^{-3x})dx$$

$$=-\frac{1}{24}+\int_0^\infty \frac{e^{-3x}}{(1+e^{-x})^3}dx=-\frac{1}{24}-\frac{5}{8}+\ln(2)=-\frac{2}{3}+\ln(2)\,.$$

$$\int_0^\infty \frac{1}{(1+e^x)^5}dx=\int_0^\infty \frac{e^{-x}e^{-4x}}{(1+e^{-x})^5}dx=\left[\frac{e^{-4x}}{4(1+e^{-x})^4}\right]_0^\infty-\frac{1}{4}\int_0^\infty \frac{1}{(1+e^{-x})^4}(-4e^{-4x})dx$$

$$=-\frac{1}{64}+\int_0^\infty \frac{e^{-4x}}{(1+e^{-x})^4}dx=-\frac{1}{64}-\frac{2}{3}+\ln(2)=-\frac{131}{192}+\ln(2)\,.$$

$$\int_0^\infty \frac{x}{1+e^x}dx=\Lambda(1)=\eta(2)\Gamma(2)=\frac{\pi^2}{12}\,.$$

$$\int_0^\infty \frac{x^2}{1+e^x}dx=\Lambda(2)=\eta(3)\Gamma(3)=\frac{3}{2}\zeta(3) \text{ since } \eta(3)=(1-2^{-2})\zeta(3)=\frac{3}{4}\zeta(3)\,.$$

$$\int_0^\infty \frac{\ln(x)}{1+e^x}dx=\Lambda'(0)=-\frac{\left(\ln(2)\right)^2}{2}$$

$$\int_0^\infty \frac{x\ln(x)}{1+e^x}dx = \Lambda'(1) = \frac{1}{2}\zeta'(2) + \frac{\pi^2}{12}\ln(2) + \frac{\pi^2}{12}(1-\gamma_0),$$

$$\int_0^\infty \frac{x^2\ln(x)}{1+e^x}dx = \Lambda'(2) = 2\left(\frac{3}{4}\zeta'(3) + \frac{1}{4}\ln(2)\zeta(3)\right) + \frac{3}{4}\zeta(3)\left(3-2\gamma_0\right)$$

$$= \frac{3}{2}\zeta'(3) + \frac{1}{2}\ln(2)\zeta(3) + \frac{9}{4}\zeta(3) - \frac{3}{2}\zeta(3)\gamma_0 = \left(\frac{1}{2}\ln(2) + \frac{9}{4} - \frac{3}{2}\gamma_0\right)\zeta(3) + \frac{3}{2}\zeta'(3).$$

Using (4) with s =1 and n =2,

$$\int_0^\infty \frac{x}{(1+e^x)^2}dx = \int_0^\infty \frac{x}{(1+e^x)}dx - \frac{1}{1}\int_0^\infty \frac{1}{(1+e^x)}dx = \frac{\pi^2}{12} - \ln(2).$$

Using (4) with s =1 and n =2,

$$\int_0^\infty \frac{x^2}{(1+e^x)^2}dx = \int_0^\infty \frac{x^2}{(1+e^x)}dx - \frac{2}{1}\int_0^\infty \frac{x}{(1+e^x)}dx = \frac{3}{2}\zeta(3) - 2\frac{\pi^2}{12} = \frac{3}{2}\zeta(3) - \frac{\pi^2}{6}.$$

$$\int_0^\infty \frac{\ln(x)}{(1+e^x)^2}dx = G'(0) = \left(\eta'(1)-\eta'(0)\right)\Gamma(1) + \left(\eta(1)-\eta(0)\right)\Gamma'(1)$$

$$= \left(\ln(2)\gamma_0 - \frac{\left(\ln(2)\right)^2}{2} - \frac{1}{2}\ln\left(\frac{\pi}{2}\right)\right)\times 1 + \left(\ln(2) - \frac{1}{2}\right)\left(-\gamma_0\right)$$

$$= \left(-\frac{1}{2}\ln\left(\frac{\pi}{2}\right) - \frac{\left(\ln(2)\right)^2}{2}\right)\times 1 + \frac{1}{2}\gamma_0$$

$$= \frac{1}{2}\left(\gamma_0 - \ln\left(\frac{\pi}{2}\right) - \left(\ln(2)\right)^2\right).$$

Using (5), with s =1 and n =2,

$$\int_0^\infty \frac{x\ln(x)}{(1+e^x)^2}dx = \int_0^\infty \frac{x\ln(x)}{(1+e^x)}dx - \int_0^\infty \frac{1}{(1+e^x)}dx - \int_0^\infty \frac{\ln(x)}{(1+e^x)}dx$$

$$= \frac{1}{2}\zeta'(2) + \frac{\pi^2}{12}\ln(2) + \frac{\pi^2}{12}(1-\gamma_0) - \ln(2) + \frac{\left(\ln(2)\right)^2}{2}$$

$$= \frac{1}{2}\zeta'(2) + \frac{\pi^2}{12}\left(1+\ln(2)-\gamma_0\right) - \ln(2) + \frac{\left(\ln(2)\right)^2}{2}.$$

Using (5) with s = 2 and n =2,

$$\int_0^\infty \frac{x^2\ln(x)}{(1+e^x)^2}dx = \int_0^\infty \frac{x^2\ln(x)}{(1+e^x)}dx - \int_0^\infty \frac{x}{(1+e^x)}dx - 2\int_0^\infty \frac{x\ln(x)}{(1+e^x)}dx.$$

$$=\left(\frac{1}{2}\ln(2)+\frac{9}{4}-\frac{3}{2}\gamma_0\right)\zeta(3)+\frac{3}{2}\zeta'(3)-\frac{\pi^2}{12}-2\left(\frac{1}{2}\zeta'(2)+\frac{\pi^2}{12}\ln(2)+\frac{\pi^2}{12}(1-\gamma_0)\right)$$

$$=\left(\frac{1}{2}\ln(2)+\frac{9}{4}-\frac{3}{2}\gamma_0\right)\zeta(3)+\frac{3}{2}\zeta'(3)-\frac{\pi^2}{12}-\zeta'(2)-\frac{\pi^2}{6}\ln(2)-\frac{\pi^2}{6}+\frac{\pi^2}{6}\gamma_0$$

$$=\left(\frac{1}{2}\ln(2)+\frac{9}{4}-\frac{3}{2}\gamma_0\right)\zeta(3)+\frac{3}{2}\zeta'(3)-\zeta'(2)-\frac{\pi^2}{4}-\frac{\pi^2}{6}\ln(2)+\frac{\pi^2}{6}\gamma_0 .$$

Using (4), taking s = 1, n = 3, we get

$$\int_0^\infty \frac{x}{(1+e^x)^3}dx=\int_0^\infty \frac{x}{(1+e^x)^2}dx-\frac{1}{2}\int_0^\infty \frac{1}{(1+e^x)^2}dx=\frac{\pi^2}{12}-\ln(2)-\frac{1}{2}\left(-\frac{1}{2}+\ln(2)\right)$$

$$=\frac{1}{4}+\frac{\pi^2}{12}-\frac{3}{2}\ln(2) .$$

Using (4), taking s = 2 and n =3,

$$\int_0^\infty \frac{x^2}{(1+e^x)^3}dx=H_3(2)=\int_0^\infty \frac{x^2}{(1+e^x)^2}dx-\frac{2}{2}\int_0^\infty \frac{x}{(1+e^x)^2}dx .$$

$$=\frac{3}{2}\zeta(3)-\frac{\pi^2}{6}-\left(\frac{\pi^2}{12}-\ln(2)\right)=\ln(2)-\frac{\pi^2}{4}+\frac{3}{2}\zeta(3) .$$

Using (5) with s =1 and n =3,

$$\int_0^\infty \frac{x\ln(x)}{(1+e^x)^3}dx=\int_0^\infty \frac{x\ln(x)}{(1+e^x)^2}dx-\frac{1}{2}\int_0^\infty \frac{1}{(1+e^x)^2}dx-\frac{1}{2}\int_0^\infty \frac{\ln(x)}{(1+e^x)^2}dx$$

$$=\frac{1}{2}\zeta'(2)+\frac{\pi^2}{12}\left(1+\ln(2)-\gamma_0\right)-\ln(2)+\frac{\left(\ln(2)\right)^2}{2}-\frac{1}{2}\left(-\frac{1}{2}+\ln(2)\right)-\frac{1}{2}\left(\frac{1}{2}\left(\gamma_0-\ln\left(\frac{\pi}{2}\right)-\left(\ln(2)\right)^2\right)\right)$$

$$=\frac{1}{4}-\frac{7}{4}\ln(2)+\frac{3}{4}\left(\ln(2)\right)^2+\frac{1}{4}\ln(\pi)+\frac{\pi^2}{12}\left(1+\ln(2)\right)-\left(\frac{\pi^2}{12}+\frac{1}{4}\right)\gamma_0+\frac{1}{2}\zeta'(2) .$$

Using (3)* with s = 2, we get,

$$\int_0^\infty \frac{x^2\ln(x)}{(1+e^x)^3}dx=H'(s)$$

$$=\frac{1}{2}\left(\eta'(1)+2\eta'(3)-3\eta'(2)\right)\Gamma(3)+\frac{1}{2}\left(\eta(1)+2\eta(3)-3\eta(2)\right)\Gamma'(3)$$

$$=\frac{1}{2}\left(\left(\ln(2)\gamma_0-\frac{\left(\ln(2)\right)^2}{2}\right)+2\left(\frac{3}{4}\zeta'(3)+\frac{1}{4}\ln(2)\zeta(3)\right)-3\left(\frac{1}{2}\zeta'(2)+\frac{\pi^2}{12}\ln(2)\right)\right)\times 2$$

$$+\frac{1}{2}\left(\ln(2)+2\left(\frac{3}{4}\zeta(3)\right)-3\left(\frac{\pi^2}{12}\right)\right)\left(3-2\gamma_0\right)$$

$$=\ln(2)\gamma_0-\frac{\left(\ln(2)\right)^2}{2}+\frac{3}{2}\zeta'(3)+\frac{1}{2}\ln(2)\zeta(3)-\frac{3}{2}\zeta'(2)-\left(\frac{\pi^2}{4}\ln(2)\right)$$

$$+\frac{3}{2}\ln(2)+\frac{9}{4}\zeta(3)-\frac{3}{8}\pi^2-\left(\ln(2)+\left(\frac{3}{2}\zeta(3)\right)-\left(\frac{\pi^2}{4}\right)\right)\gamma_0$$

$$=-\frac{\left(\ln(2)\right)^2}{2}+\frac{3}{2}\zeta'(3)+\frac{1}{2}\ln(2)\zeta(3)-\frac{3}{2}\zeta'(2)-\left(\frac{\pi^2}{4}\ln(2)\right)$$

$$+\frac{3}{2}\ln(2)+\frac{9}{4}\zeta(3)-\frac{3}{8}\pi^2-\left(\left(\frac{3}{2}\zeta(3)\right)-\left(\frac{\pi^2}{4}\right)\right)\gamma_0$$

$$=\frac{3}{2}\ln(2)-\frac{3}{8}\pi^2-\frac{1}{4}\pi^2\ln(2)-\frac{\left(\ln(2)\right)^2}{2}+\frac{1}{4}\pi^2\gamma_0+\left(\frac{1}{2}\ln(2)+\frac{9}{4}-\frac{3}{2}\gamma_0\right)\zeta(3)+\frac{3}{2}\zeta'(3)-\frac{3}{2}\zeta'(2)$$

We may express $H_n(s)=\int_0^\infty \frac{x^s}{(1+e^x)^n}\,dx$ for n = 4 in terms of the Eta, Gamma functions.

For n = 4, recall the relation (7),

$$\int_0^\infty \frac{x^s}{(1+e^x)^4}\,dx=H_4(s)=\int_0^\infty \frac{x^s}{(1+e^x)^3}\,dx-\frac{s}{3}\int_0^\infty \frac{x^{s-1}}{(1+e^x)^3}\,dx=H_3(s)-\frac{s}{3}H_3(s-1)$$

$$=\left(\eta(s+1)-\frac{11}{6}\eta(s)+\eta(s-1)-\frac{1}{6}\eta(s-2)\right)\Gamma(s+1)$$

Hence,

$$\int_0^\infty \frac{x}{(1+e^x)^4}\,dx=\left(\eta(2)-\frac{11}{6}\eta(1)+\eta(0)-\frac{1}{6}\eta(-1)\right)\Gamma(2)$$

$$=\left(\frac{\pi^2}{12}-\frac{11}{6}\ln(2)+\frac{1}{2}-\frac{1}{6}\frac{1}{4}\right)=\frac{\pi^2}{12}-\frac{11}{6}\ln(2)+\frac{11}{24}.$$

$$\int_0^\infty \frac{x^2}{(1+e^x)^4}\,dx=\left(\eta(3)-\frac{11}{6}\eta(2)+\eta(1)-\frac{1}{6}\eta(0)\right)\Gamma(3)$$

$$=\left(\frac{3}{4}\zeta(3)-\frac{11}{6}\frac{\pi^2}{12}+\ln(2)-\frac{1}{6}\left(\frac{1}{2}\right)\right)\times 2=2\ln(2)+\frac{3}{2}\zeta(3)-\frac{11}{36}\pi^2-\frac{1}{6}.$$

Recall (8),

$$\int_0^\infty \frac{x^s \ln(x)}{(1+e^x)^4}dx = \left(\eta'(s+1)-\frac{11}{6}\eta'(s)+\eta'(s-1)-\frac{1}{6}\eta'(s-2)\right)\Gamma(s+1)$$

$$+\left(\eta(s+1)-\frac{11}{6}\eta(s)+\eta(s-1)-\frac{1}{6}\eta(s-2)\right)\Gamma'(s+1).$$

Using (8), with s =2,

$$\int_0^\infty \frac{x^2 \ln(x)}{(1+e^x)^4}dx = \left(\eta'(3)-\frac{11}{6}\eta'(2)+\eta'(1)-\frac{1}{6}\eta'(0)\right)\Gamma(3)$$

$$+\left(\eta(3)-\frac{11}{6}\eta(2)+\eta(1)-\frac{1}{6}\eta(0)\right)\Gamma'(3)$$

$$=\left(\frac{3}{4}\zeta'(3)+\frac{1}{4}\ln(2)\zeta(3)-\frac{11}{6}\left(\frac{1}{2}\zeta'(2)+\frac{\pi^2}{12}\ln(2)\right)+\left(\ln(2)\gamma_0-\frac{(\ln(2))^2}{2}\right)-\frac{1}{6}\frac{1}{2}\ln\left(\frac{\pi}{2}\right)\right)\times 2$$

$$+\left(\frac{3}{4}\zeta(3)-\frac{11}{6}\frac{\pi^2}{12}+\ln(2)-\frac{1}{12}\right)(3-2\gamma_0)$$

$$=-\frac{1}{4}+\frac{19}{6}\ln(2)-(\ln(2))^2-\frac{11}{24}\pi^2-\frac{11}{36}\pi^2\ln(2)+\frac{11}{36}\pi^2\gamma_0-\frac{1}{6}\ln(\pi)+\frac{1}{6}\gamma_0$$

$$+\left(\frac{1}{2}\ln(2)+\frac{9}{4}-\frac{3}{2}\gamma_0\right)\zeta(3)+\frac{3}{2}\zeta'(3)-\frac{11}{6}\zeta'(2).$$

Using (8) with s =1,

$$\int_0^\infty \frac{x \ln(x)}{(1+e^x)^4}dx = \left(\eta'(2)-\frac{11}{6}\eta'(1)+\eta'(0)-\frac{1}{6}\eta'(-1)\right)\Gamma(2)$$

$$+\left(\eta(2)-\frac{11}{6}\eta(1)+\eta(0)-\frac{1}{6}\eta(-1)\right)\Gamma'(2)$$

$$=\left(\frac{1}{2}\zeta'(2)+\frac{\pi^2}{12}\ln(2)-\frac{11}{6}\left(\ln(2)\gamma_0-\frac{(\ln(2))^2}{2}\right)+\frac{1}{2}\ln\left(\frac{\pi}{2}\right)-\frac{1}{6}\left(3\ln(A)-\frac{1}{4}-\frac{\ln(2)}{3}\right)\right)\times 1$$

$$+\left(\frac{\pi^2}{12}-\frac{11}{6}\ln(2)+\frac{1}{2}-\frac{1}{24}\right)(1-\gamma_0)$$

$$=\frac{1}{2}\zeta'(2)+\frac{\pi^2}{12}\ln(2)+\frac{11}{12}(\ln(2))^2-\frac{11}{6}(\ln(2)\gamma_0)+\frac{1}{2}\ln\left(\frac{\pi}{2}\right)-\frac{1}{2}\ln(A)+\frac{1}{24}+\frac{\ln(2)}{18}$$

$$-\left(\frac{\pi^2}{12}+\frac{11}{24}\right)\gamma_0+\frac{11}{6}\ln(2)\gamma_0-\frac{11}{6}\ln(2)+\frac{\pi^2}{12}+\frac{11}{24}$$

$$=\frac{1}{2}\zeta'(2)+\frac{\pi^2}{12}\ln(2)+\frac{11}{12}\left(\ln(2)\right)^2+\frac{1}{2}\ln\left(\pi\right)-\frac{41}{18}\ln\left(2\right)-\frac{1}{2}\ln(A)+\frac{1}{2}-\left(\frac{\pi^2}{12}+\frac{11}{24}\right)\gamma_0+\frac{\pi^2}{12}.$$

Using (8) with s = 0,

$$\int_0^\infty \frac{\ln(x)}{(1+e^x)^4}\,dx=\left(\eta'(1)-\frac{11}{6}\eta'(0)+\eta'(-1)-\frac{1}{6}\eta'(-2)\right)\Gamma(1)$$

$$+\left(\eta(1)-\frac{11}{6}\eta(0)+\eta(-1)-\frac{1}{6}\eta(-2)\right)\Gamma'(1)$$

$$=\left(\ln(2)\gamma_0-\frac{\left(\ln(2)\right)^2}{2}-\frac{11}{6}\left(\frac{1}{2}\ln\left(\frac{\pi}{2}\right)\right)+\left(3\ln(A)-\frac{1}{4}-\frac{\ln(2)}{3}\right)-\frac{1}{6}\left(7\frac{\zeta(3)}{4\pi^2}\right)\right)$$

$$+\left(\ln(2)-\frac{11}{12}+\frac{1}{4}\right)\left(-\gamma_0\right)$$

$$=-\frac{\left(\ln(2)\right)^2}{2}-\frac{11}{12}\ln\left(\frac{\pi}{2}\right)+3\ln(A)-\frac{1}{4}-\frac{\ln(2)}{3}-7\frac{\zeta(3)}{24\pi^2}+\frac{2}{3}\gamma_0//$$

$$=-\frac{\left(\ln(2)\right)^2}{2}-\frac{11}{12}\ln\left(\pi\right)+\frac{11}{12}\ln\left(2\right)+3\ln(A)-\frac{1}{4}-\frac{\ln(2)}{3}-7\frac{\zeta(3)}{24\pi^2}+\frac{2}{3}\gamma_0$$

$$=-\frac{\left(\ln(2)\right)^2}{2}-\frac{11}{12}\ln\left(\pi\right)+\frac{7}{12}\ln\left(2\right)+3\ln(A)-\frac{1}{4}-7\frac{\zeta(3)}{24\pi^2}+\frac{2}{3}\gamma_0.$$

Recall relation (9),

$$\int_0^\infty \frac{x^s}{(1+e^x)^5}\,dx=H_5(s)$$

$$=\left(\frac{35}{24}\eta(s-1)+\eta(s+1)-\frac{25}{12}\eta(s)-\frac{5}{12}\left(\eta(s-2)\right)+\frac{1}{24}\eta(s-3)\right)\Gamma(s+1)\ .$$

With s =1, we have,

$$\int_0^\infty \frac{x}{(1+e^x)^5}\,dx=\left(\frac{35}{24}\eta(0)+\eta(2)-\frac{25}{12}\eta(1)-\frac{5}{12}\left(\eta(-1)\right)+\frac{1}{24}\eta(-2)\right)\Gamma(2)$$

$$=\left(\frac{35}{24}\frac{1}{2}+\frac{\pi^2}{12}-\frac{25}{12}\ln(2)-\frac{5}{12}\frac{1}{4}+0\right)\times1=\frac{5}{8}+\frac{\pi^2}{12}-\frac{25}{12}\ln(2).$$

With s = 2, we get,

$$\int_0^\infty \frac{x^2}{(1+e^x)^5}dx = \left(\frac{35}{24}\eta(1)+\eta(3)-\frac{25}{12}\eta(2)-\frac{5}{12}\big(\eta(0)\big)+\frac{1}{24}\eta(-1)\right)\Gamma(3)$$

$$=\left(\frac{35}{24}\ln(2)+\frac{3}{4}\zeta(3)-\frac{25}{12}\frac{\pi^2}{12}-\frac{5}{12}\left(\frac{1}{2}\right)+\frac{1}{24}\frac{1}{4}\right)\times 2$$

$$=\frac{35}{12}\ln(2)+\frac{3}{2}\zeta(3)-\frac{25}{72}\pi^2-\frac{19}{48}.$$

Using relation (10) with s =1, we have,

$$\int_0^\infty \frac{x\ln(x)}{(1+e^x)^5}dx$$

$$=\left(\frac{35}{24}\eta'(0)+\eta'(2)-\frac{25}{12}\eta'(1)-\frac{5}{12}\eta'(-1)+\frac{1}{24}\eta'(-2)\right)\Gamma(2)$$

$$+\left(\frac{35}{24}\eta(0)+\eta(2)-\frac{25}{12}\eta(1)-\frac{5}{12}\eta(-1)+\frac{1}{24}\eta(-2)\right)\Gamma'(2)$$

$$=\frac{35}{24}\left(\frac{1}{2}\ln\left(\frac{\pi}{2}\right)\right)+\left(\frac{1}{2}\zeta'(2)+\frac{\pi^2}{12}\ln(2)\right)-\frac{25}{12}\left(\ln(2)\gamma_0-\frac{\big(\ln(2)\big)^2}{2}\right)$$

$$+\left(-\frac{5}{12}\left(3\ln(A)-\frac{1}{4}-\frac{\ln(2)}{3}\right)+\frac{1}{24}\left(7\frac{\zeta(3)}{4\pi^2}\right)\right)$$

$$+\left(\frac{5}{8}+\frac{\pi^2}{12}-\frac{25}{12}\ln(2)\right)(1-\gamma_0)$$

$$=\frac{35}{48}\ln\left(\frac{\pi}{2}\right)+\frac{1}{2}\zeta'(2)+\frac{\pi^2}{12}\ln(2)+\frac{25}{24}\big(\ln(2)\big)^2$$

$$-\frac{5}{4}\ln(A)+\frac{5}{48}+\frac{5}{36}\ln(2)+7\frac{\zeta(3)}{96\pi^2}+\frac{5}{8}+\frac{\pi^2}{12}-\frac{25}{12}\ln(2)-\left(\frac{5}{8}+\frac{\pi^2}{12}\right)\gamma_0$$

$$=\frac{35}{48}\ln(\pi)-\frac{385}{144}\ln(2)+\frac{1}{2}\zeta'(2)+\frac{\pi^2}{12}\ln(2)+\frac{25}{24}\big(\ln(2)\big)^2$$

$$-\frac{5}{4}\ln(A)+\frac{35}{48}+7\frac{\zeta(3)}{96\pi^2}+\frac{\pi^2}{12}-\left(\frac{5}{8}+\frac{\pi^2}{12}\right)\gamma_0$$

$$=\frac{35}{48}\big(1+\ln(\pi)\big)-\frac{385}{144}\ln(2)+\frac{1}{2}\zeta'(2)+\frac{\pi^2}{12}\big(1+\ln(2)\big)+\frac{25}{24}\big(\ln(2)\big)^2$$

$$-\frac{5}{4}\ln(A)+7\frac{\zeta(3)}{96\pi^2}-\left(\frac{5}{8}+\frac{\pi^2}{12}\right)\gamma_0$$

We list some useful identities below.

$\Gamma(1)=1$, $\Gamma(2)=1$, $\Gamma(3)=2$, $\Gamma'(1)=-\gamma_0$, $\Gamma'(2)=1-\gamma_0$, $\Gamma'(3)=3-2\gamma_0$.

$\eta(s)=(1-2^{1-s})\zeta(s)$ for $s>1$.

$\eta(0)=\frac{1}{2}$, $\eta(1)=\ln(2)$, $\eta(2)=\frac{\pi^2}{12}$, $\eta(3)=(1-2^{-2})\zeta(3)=\frac{3}{4}\zeta(3)$, $\eta(-1)=\frac{1}{4}$, $\eta(-2)=0$,

$\eta'(0)=\frac{1}{2}\ln\left(\frac{\pi}{2}\right)$, $\eta'(1)=\ln(2)\gamma_0-\frac{\left(\ln(2)\right)^2}{2}$, $\eta'(2)=\frac{1}{2}\zeta'(2)+\frac{\pi^2}{12}\ln(2)$,

$\eta'(3)=(1-2^{-2})\zeta'(3)+2^{-2}\ln(2)\zeta(3)=\frac{3}{4}\zeta'(3)+\frac{1}{4}\ln(2)\zeta(3)$, $\eta'(-1)=3\ln(A)-\frac{1}{4}-\frac{\ln(2)}{3}$,

$\eta'(-2)=7\frac{\zeta(3)}{4\pi^2}$,

$\zeta'(2)=\frac{\pi^2}{6}(\gamma_0+\ln(2\pi)-12\ln(A))$, $\zeta'(-2)=-\frac{\zeta(3)}{4\pi^2}$,

$\zeta(2)=\frac{\pi^2}{6}$, $\zeta(3)\approx 1.2020569031595942853...$

$\zeta'(2)\approx -0.93754825431584375370...$

$\zeta'(3)=-0.19812624288563685333...$

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