On The Primitive Of Product Of Two Functions

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Recently I came across an interesting result of Daniel Lesnic, which one can prove using results not beyond that of the ideas of derivative and the Mean Value Theorem. I state the result as follows.

Theorem 1. Suppose $a < b$ and $[a, b]$ is a closed and bounded interval. If $f : [a, b] \to \mathbf{R}$ is a continuous function of bounded variation and g: $[a, b] \to \mathbf{R}$ is a function that has a primitive, then the product $f \cdot$ g has a primitive on $[a, b]$.

Remark. 1. If g is continuous, then Theorem 1 is trivial. This is because $f \cdot$ g is continuous and so it follows by the Fundamental Theorem of Calculus, that $f \cdot$ g has a primitive on $[a, b]$.

2. If f is continuous and g is Riemann integrable, then $f \cdot$ g is Riemann integrable and so has a primitive almost everywhere on $[a, b]$, i.e., there exists a function H such that $H' = f g$ almost everywhere on $[a, b]$.

3. If f is continuous and g is Lebesguue integrable, then $f \cdot$ g is Lebesgue integrable and $f \cdot$ g has a primitive almost everywhere on $[a, b]$.

4. Note that if f is Lebesgue integrable and g is bounded, then $f \cdot$ g has a primitive almost everywhere on $[a, b]$. This is because g has a primitive and so g is measurable by a Theorem of Banach and if g is bounded, then $f \cdot$ g is Lebesgue integrable.

We shall prove Theorem 1 for the special case when $f : [a, b] \to \mathbf{R}$ is a continuous injective function. Now a continuous injective function on $[a, b]$ is strictly monotonic. Thus we shall prove the special case of theorem 1 when f is continuous and strictly monotonic on $[a, b]$.

Lemma 2. Suppose $f : [a, b] \to \mathbf{R}$ is a continuous strictly increasing function and g: $[a, b] \to \mathbf{R}$ is a function that has a primitive. Then the product $f \cdot$ g has a primitive on $[a, b]$.

The conclusion is also valid if f is a strictly decreasing function.

Proof. Since $f : [a, b] \to \mathbf{R}$ is a strictly increasing and continuous function, the image $f([a, b])$ is a non-trivial interval $[c, d]$ with $c < d$. The inverse function $f^{-1} : [c, d] \to [a, b]$ is also continuous and strictly increasing. Let G: $[a, b] \to \mathbf{R}$ be a primitive of g. Then $G' =$ g on $[a, b]$.

G is continuous and so the composite $h = G \circ f^{-1} : [c, d] \to \mathbf{R}$ is a continuous function. Hence h has a primitive H: $[c, d] \to \mathbf{R}$. This means $H' = h = G \circ f^{-1}$.

Define a function K: $[a, b] \to \mathbf{R}$ by $K(x) = f(x)G(x) - H \circ f(x) = f(x)G(x) - H(f(x))$ for x in $[a, b]$. Then we claim that $K'(x) = f(x)\, g(x)$ for all x in $[a, b]$.

Tale x in (a, b). We shall show that the left and right derivatives of K at x is $f(x)\, g(x)$.

Let $y > x$ and y is in $[a, b]$. Then

$$K(y) - K(x) = f(y)G(y) - H(f(y)) - (f(x)G(x) - H(f(x)))$$

$$= f(x)(G(y) - G(x)) + (f(y) - f(x))G(y) - (H(f(y)) - H(f(x)))$$

-------------------------------------- (1)

Now since H is differentiable and $f(y) > f(x)$ because f is strictly increasing, by the Mean Value Theorem, there exists ζ such that $f(y) > \zeta > f(x)$ and

$$H(f(y)) - H(f(x)) = (f(y) - f(x))\, H'(\zeta) = (f(y) - f(x))\, G(f^{-1}(\zeta)) \text{ --------- (2)}$$

But since f is continuous and strictly increasing, by the Intermediate Value Theorem, there exists ζ_y such that $y > \zeta_y > x$ and $f(\zeta_y) = \zeta$. Hence it follows from (2) that

$$H(f(y)) - H(f(x)) = (f(y) - f(x))\, G(f^{-1}(f(\zeta_y))) = (f(y) - f(x))\, G(\zeta_y). \quad \text{----- (3).}$$

Therefore, it follows from (1) and (3) that

$$K(y) - K(x) = f(x)(G(y) - G(x)) + (f(y) - f(x))G(y) - (f(y) - f(x))\, G(\zeta_y)$$

$$= f(x)(G(y) - G(x)) + (f(y) - f(x))(G(y) - G(x)) - (f(y) - f(x))\, (G(\zeta_y) - G(x))$$

$$= f(y)(G(y) - G(x)) - (f(y) - f(x))\, (G(\zeta_y) - G(x)) \;.$$

Therefore,

$$\frac{K(y) - K(x)}{y - x} = f(y)\frac{G(y) - G(x)}{y - x} - (f(y) - f(x))\frac{G(\zeta_y) - G(x)}{y - x} \quad \text{------------- (4)}$$

Now $\lim\limits_{y \to x^+} f(y)\dfrac{G(y) - G(x)}{y - x} = f(x)g(x)$. -------------------- (5)

Note that $\dfrac{G(\zeta_y) - G(x)}{y - x} = \dfrac{G(\zeta_y) - G(x)}{\zeta_y - x} \cdot \dfrac{\zeta_y - x}{y - x}$. Since $G'(x)$ exists and is equal to $g(x)$, there exists $\delta > 0$ such that for $x < y < x + \delta$, $\left|\dfrac{G(y) - G(x)}{y - x}\right| < |g(x)| + 1$. It follows that for $0\, x < y < x + \delta$, $\left|\dfrac{G(\zeta_y) - G(x)}{\zeta_y - x} \cdot \dfrac{\zeta_y - x}{y - x}\right| < |g(x)| + 1$. That is, $\dfrac{G(\zeta_y) - G(x)}{y - x}$ is bounded on $(x, x + \delta)$. Therefore,

$$\lim_{y \to x^+} (f(y) - f(x)) \frac{G(\zeta_y) - G(x)}{y - x} = 0 \text{ ---------------------- (6)}$$

because $\lim_{y \to x^+} (f(y) - f(x)) = 0$ by continuity at x.

Thus it follows from (4), (5) and (6) that,

$$\lim_{y \to x^+} \frac{K(y) - K(x)}{y - x} = f(x)g(x).$$

This proves that the right derivative of K at x is $f(x)$ g(x).

If $x = a$, the above argument shows that $K'(a) = f(a)$ g(a). Similarly we can show that the left derivative of K at x is $f(x)$ g(x). The same argument shows that $K'(b) = f(b)$ g(b).

Hence for x in (a, b), $K'(x) = f(x)$ g(x) and so $K' = f$ g. That is, K is a primitive of f g.

If f is strictly decreasing and continuous, then $-f$ is strictly increasing and continuous. Therefore, by what we have just proved, $-fg$ has a primitive say K, Then $-K$ is a primitive for fg.

Next we show that the conclusion of Lemma 2 is valid for an increasing and continuous function.

Corollary 3. Suppose $f : [a, b] \to \mathbf{R}$ is a continuous increasing function and g: $[a, b] \to \mathbf{R}$ is a function that has a primitive. Then the product $f \cdot$ g has a primitive on $[a, b]$.

Proof, If f is increasing and continuous, then $h(x) = f(x) + x$ is strictly increasing and continuous on $[a, b]$. Therefore, by Lemma 2, $h(x)$ g(x) has a primitive, *say* K(x) on $[a, b]$. Also since x is strictly increasing and continuous, x g(x) has a primitive $H(x)$ on $[a, b]$. Then $K - H$ is a primitive for $f \cdot$ g as $(K - H)'(x) = K'(x) - H'(x) = h(x)g(x) - xg(x) = f(x)g(x)$.

Proof of Theorem 1.

If $f : [a, b] \to \mathbf{R}$ is continuous and of bounded variation, then f is the difference of two continuous increasing functions. Hence $f = f_1 - f_2$, where f_1 and f_2 are increasing continuous functions. If g has a primitive, then by Corollary 3, f_1 g and f_2 g both have primitives. It follows that f g $= f_1$ g $- f_2$ g has a primitive.

We next present a special characterization of a function of bounded variation satisfying the conclusion of Theorem 1.

First a technical lemma.

Lemma 4. Suppose $f : [a, b] \to \mathbf{R}$ is of bounded variation and has the Darboux property, i.e., f has the intermediate value property. Then f is continuous.

Proof. If f is of bounded variation, then it can have only jump discontinuities. But since f has the intermediate value property, f cannot have any jump discontinuity and so f is continuous.

Lemma 5. Suppose $f : [a, b] \to \mathbf{R}$ can be represented as the quotient of two functions having primitives. That is, $f = g / h$, g and h have primitives and $h \neq 0$. Then f is a Darboux function, i.e., f has the intermediate value property.

Proof. Suppose $f(c) < f(d)$ for some c, d in $[a, b]$, and $c \neq d$. For any k such that $f(c) < k < f(d)$. That is,

$$g(c) / h(c) < k < g(d) / h(d). \text{ ----------------------------------- } (1)$$

Define $H(x) = g(x) - k \cdot h(x)$ for x in $[a, b]$. Then since g and h have primitives, H too has a primitive. Therefore, by Darboux Theorem, H has the intermediate value property.

Since $h \neq 0$ and h has the intermediate value property, $h > 0$ or $h < 0$. Therefore, it follows from (1) that

$$g(c) < k\ h(c) \quad \text{and} \quad k\,h(d) < g(d)\text{, if } h > 0 \text{ ------------------- } (2)$$

or

$$g(c) > k\ h(c) \quad \text{and} \quad k\,h(d). > g(d)\text{, if } h < 0. \text{ -------------------- } (3)$$

Hence, it follows from (2) and (3) that

$$H(c) < 0 < H(d) \quad \text{or} \quad H(c) > 0 > H(d).$$

Therefore, since H has the intermediate value property, there exists x between c and d such that $H(x) = 0$. That is, $g(x) = k \cdot h(x)$ and so $f(x) = g(x)/h(x) = k$. This shows that f has the intermediate value property.

Corollary 6. Suppose $f : [a, b] \to \mathbf{R}$ can be represented as the quotient of two functions having primitives. That is, $f = g / h$, g and h have primitives and $h \neq 0$. Furthermore if f is of bounded variation, then f is continuous.

Proof. By Lemma 5, f has the intermediate value property**.** Since f is also of bounded variation, by Lemma 4, f is continuous.

Now we state the characterization theorem.

Theorem 7. Suppose $f : [a, b] \to \mathbf{R}$ is of bounded variation. Suppose there exists a non-zero function $h : [a, b] \to \mathbf{R}$ possessing primitives such that the product $f \cdot h$ possesses primitives. Then $f \cdot g$ possesses primitives for any g possessing primitives.

Proof. Let $K = f \cdot h$. Then K has primitives by hypothesis. Then $f = K / h$ since $h \neq 0$. And so f is a quotient of two functions possessing primitives and is also of bounded variation and so by Corollary 6, f is continuous. This means that f is a continuous function of bounded variation. Therefore, by Theorem 1, $f \cdot g$ possesses primitives for any g possessing primitives.

Remark. By Theorem 7, a function of bounded variation having the property that there exists a non-zero function h possessing primitives such that the product $f \cdot h$ has primitives is necessarily continuous. Thus if f is a discontinuous function of bounded variation, then for any non zero function h possessing primitives, $f \cdot h$ has no primitives.

Corollary 6 is a criterion of deciding when a function of bounded variation is continuous. By isolating the use of the intermediate value property we can prove the following weaker result in exactly the same way.

Theorem 8. Suppose $f : [a, b] \to \mathbf{R}$ is a function of bounded variation that can be represented as the quotient of two Darboux functions, i.e., $f = \text{g} / h$, where h is a non-zero Darboux function possessing primitives and g is a Darboux function satisfying that $\text{g} + k$ is a Darboux function for any non-zero Darboux function k possessing primitives. Then f is continuous.

Following A. Bruckner, we can use the product formula for derivatives to deduce the following result.

Theorem 9. Suppose $\text{g} : [a, b] \to \mathbf{R}$ is a function possessing primitives and is Lebesgue integrable. Then for any differentiable function $F: [a, b] \to \mathbf{R}$, $F\cdot g$ possesses primitives, which are all absolutely continuous. In particular, if g is the derivative of a differentiable function of bounded variation or equivalently a differentiable absolutely continuous function, then for any differentiable F, $F\cdot g$ possesses primitives, which are all absolutely continuous..

Proof. Suppose $G : [a, b] \to \mathbf{R}$ is a primitive of g. Then since g is Lebesgue integrable, G is absolutely continuous and so G is continuous of bounded variation.. Since F is differentiable, by Theorem 1, $G\cdot F'$ possesses primitives. Now by the product rule for derivatives,

$$(G\cdot F)' = G'\cdot F + G\cdot F' = F\cdot g + G\cdot F' .$$

Thus, if H is a primitive of $G\cdot F'$, then $G\cdot F - H$ *is* a primitive of $F\cdot g$ since

$$(G\cdot F\ - H)\ ' = (G\cdot F)'\ -\ H' =\ F\cdot g\ + G\cdot F'\ -\ G\cdot F' = F\cdot g\ .$$

Since F is differentiable and so is continuous, F is a bounded Lebesgue integrable function. Because g is Lebesgue integrable, $F\cdot g$ is Lebesgue integrable. It follows by Theorem 6 of "Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallé Poussin;s Theorem" that $G\cdot F\ - H$ is absolutely continuous. Therefore, $F\cdot g$ possesses an absolutely continuous primitive for any differentiable F. Hence all its primitives are absolutely continuous.

In particular if G is differentiable and is of bounded variation, then $G' =$ g is Lebesgue integrable and so G is absolutely continuous. Note that a differentiable function on $[a, b]$ is absolutely continuous if and only if it is of bounded variation. It follows as above that $F\cdot g$ possesses absolutely continuous primitives for any differentiable F.

Remark. Note that any differentiable function $G : [a, b] \to \mathbf{R}$ is necessarily a continuous N function. (See lemma 4 of " When is a function on a closed and bounded interval be of bounded variation, absolutely continuous?".) Therefore, by Theorem 6 of "Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallé Poussin;s Theorem", such a differentiable function G is absolutely continuous if and only if G' is Lebesgue integrable.

In view of the remark above we may state Theorem 9 as follows.

Theorem 10. Suppose g : $[a, b] \to \mathbf{R}$ is the derivative of a differentiable absolutely continuous function. Then for any differentiable function F: $[a, b] \to \mathbf{R}$, $F\cdot g$ possesses primitives, which are all absolutely continuous.

If g is positive and g possesses primitives, then any primitive of g is strictly increasing. For such function we have the following result.

Lemma 11. Suppose g : $[a, b] \to \mathbf{R}$ is a function possessing primitives and g > 0 or g (x) >. 0 for $x \neq a, b$. Then for any continuous function f : $[a, b] \to \mathbf{R}$, $f\cdot g$ possesses primitives, which are all absolutely continuous.

Proof. Suppose $G : [a, b] \to \mathbf{R}$ is a primitive of g. Then $G'(x) =$ g(x) >. 0 for $x \neq a, b$. It follows that G is strictly increasing and continuous. Hence $G([a, b])$ is compact and so a closed and bounded interval $[c, d]$ and G^{-1} : $[c, d] \to [a, b]$ is also strictly increasing and continuous. Thus, for any continuous function f: $[a, b] \to \mathbf{R}$, $f \circ G^{-1}$: $[c, d] \to \mathbf{R}$ is continuous and so has a primitive H: $[c, d] \to \mathbf{R}$ such that $H\ ' = f \circ G^{-1}$. Then $H \circ G$: $[a, b] \to \mathbf{R}$ is differentiable and by the Chain Rule,

$$(H \circ G)'(x) = H'(G(x))\cdot G'(x) = f \circ G^{-1}(G(x)) \cdot \mathrm{g}(x) = f\ (x)\cdot \mathrm{g}(x).$$

It follows that $H \circ G$ is a primitive of $f\cdot g$. Note that g being the derivative of a monotone function is Lebesgue integrable. Since f is continuous and so is integrable and bounded, $f\cdot g$

is Lebesgue integrable. It follows that $H \circ G$ is absolutely continuous. Hence all primitives of $f \cdot g$ are absolutely continuous. We can also deduce that $H \circ G$ is absolutely continuous by observing that H satisfies a Lipschitz condition and G is absolutely continuous so that their composite $H \circ G$ is absolutely continuous.

Theorem 12. Suppose $g : [a, b] \to \mathbf{R}$ is a function possessing primitives and $g \geq 0$ Then for any continuous function $f: [a, b] \to \mathbf{R}$, $f \cdot g$ possesses primitives, which are all absolutely continuous.

Proof. Let $h = g + 1$. Then $h > 0$. Therefore, by Lemma 11, $f \cdot h$ possesses an absolutely continuous primitive, say H. Since f is continuous, f has an absolutely continuous primitive, say F. Then $H - F$ is absolutely continuous, differentiable and

$$(H - F)' = H' - F' = f \cdot h - f = f \cdot g + f - f = f \cdot g \ .$$

Thus, $f \cdot g$ possesses an absolutely continuous primitive and so all its primitives are absolutely continuous.

The proof of Theorem 12 suggests the following slight generalization of Theorem 12.

Theorem 13. Suppose $g : [a, b] \to \mathbf{R}$ is a function possessing primitives. Suppose there exists a function $h: [a, b] \to \mathbf{R}$ possessing primitives such that $h \leq 0$ and $g \geq h$. Then for any continuous function $f: [a, b] \to \mathbf{R}$, $f \cdot g$ possesses primitives, which are all absolutely continuous.

Proof. Let $k = g - h \geq 0$. Then for any continuous function $f: [a, b] \to \mathbf{R}$, by Theorem 12, $f \cdot k$ has an absolutely continuous primitive K *and* $-f \cdot h$ has an absolutely continuous primitive H. Hence $K - H$ is absolutely continuous and $(K - H)' = f \cdot g$ and so $f \cdot g$ has an absolutely continuous primitive. It follows that all its primitives are absolutely continuous.

Note that in the proof of Theorem 13, both K and H are differentiable increasing (therefore absolutely continuous) functions. We can prove the following in exactly the same way as Theorem 13.

Theorem 14. Suppose $g : [a, b] \to \mathbf{R}$ is a function possessing a primitive expressible as the difference of two differentiable increasing functions. Then for any continuous function $f: [a, b] \to \mathbf{R}$, $f \cdot g$ possesses primitives, which are all absolutely continuous.