A de La Vallée Poussin's Decomposition

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The de La Vallée Poussin's Theorem for a function of bounded variation *f* on arbitrary subset of $\mathbb{R}$ gives an exceptional null set *N* outside of which both the function *f* and its total variation function are differentiable finitely or infinitely and that both *f* and the total variation of *f* map the exceptional set to a null set. For an arbitrary measurable set *E*, we can give a description of the measure of the image of *E* under the total variation function. This is the classical de La Vallée Poussin's Decomposition of the measure of the image of the total variation function of a function of bounded variation in terns of the integral of the absolute value of the derivative of *f* and the image of the set of points with infinite derivatives. It is often phrased in the setting of induced measure. We state the theorem for what it is in terms of Lebesgue and Lebesgue outer measure, more assessible and readily understood.

We adopt the notation and definition from "*Absolutely Continuous Function on Arbitrary Domain and Function of Bounded variation*".

de La Vallée Poussin's Decomposition

The first result concerns the measure of a measurable subset under the total variation. The second deals with the positive and negative variation function of the function of bounded variation.

Theorem 1. Suppose *A* is a measurable subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ is a finite-valued function of bounded variation on *A*. Suppose $E \subseteq A$ is a measurable subset of *A*. Let $A_{fin} = \{x \in A : {}_A Df(x) \text{ exists and is finite}\}$ and $A_\infty = \{x \in A : |{}_A Df(x)| = +\infty\}$. Then

$$m^*\left(v_f(E)\right) = m^*\left(v_f(E \cap A_\infty)\right) + \int_{E \cap A_{fin}} |{}_A Df(x)| \, dx = m^*\left(v_f(E \cap A_\infty)\right) + \int_E |{}_A Df(x)| \, dx .$$

Proof.

Since only one-sided limit points of *A* constitute at most a countable set, we may assume that every point of *A* is a two-sided limit point of *A*.

By Theorem 18 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, there is a null set $N \subseteq A$ such that $m(N) = m(f(N)) = m\left(v_f(N)\right) = 0$ and for all $x \in A - N$, ${}_A Df(x)$ and ${}_A Dv_f(x)$ exist finitely or infinitely and that $\left|{}_A Df(x)\right| = {}_A Dv_f(x)$. Moreover, $A_\infty = \left\{x \in A : \left|{}_A Df(x)\right| = +\infty\right\}$ is a null set. Note that $A - \left(N \cup A_\infty\right)$ is measurable and the restriction of *f* to $A - \left(N \cup A_\infty\right)$ is a Lusin function. The function *f* is continuous on $A - \left(N \cup A_\infty\right)$ since ${}_A Df(x)$ exists finitely for all *x* in $A - \left(N \cup A_\infty\right)$. Since, $m(N) = m(f(N)) = m\left(v_f(N)\right) = 0$, without los of generality, we may assume that $A_{fin} = A - \left(N \cup A_\infty\right)$ and that $E \subseteq A - N$. Thus, for all $x \in A_{fin} = A - \left(N \cup A_\infty\right)$, ${}_A Df(x)$ is finite and $\left|{}_A Df(x)\right| = {}_A Dv_f(x)$. Since *E* is measurable, $E \cap A_{fin}$ is measurable. Therefore, by Theorem 10 of "*Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*", $v_f(E \cap A_{fin})$ is measurable as the restriction of the total variation function of *f*, v_f, is continuous on $A_{fin} = A - \left(N \cup A_\infty\right)$. Therefore,

$$m^*(v_f(E)) = m^*(v_f(E) \cap v_f(E \cap A_{fin})) + m^*(v_f(E) - v_f(E \cap A_{fin}))$$

$$= m^*\left(v_f(E) \cap v_f(E \cap A_{fin})\right) + m^*\left(\left(v_f(E) \cap \left(v_f(E \cap A_{fin})\right)^c\right)\right).$$

We claim that $v_f(E) \cap \left(v_f(E \cap A_{fin})\right)^c = v_f(E - E \cap A_{fin})$.

Take $a \in v_f(E) \cap \left(v_f(E \cap A_{fin})\right)^c$. Then $a \in v_f(E)$ and $a \notin v_f(E \cap A_{fin})$. Therefore, there exists $x \in E$ and $x \notin E \cap A_{fin}$ such that $a = v_f(x)$. Hence, $a \in v_f(E - E \cap A_{fin})$.

Conversely suppose $b \in v_f(E - E \cap A_{fin})$. Then there exists $x \in E - E \cap A_{fin}$ such that $b = v_f(x)$. If $b \in v_f(E \cap A_{fin})$, then there exists $e \in E \cap A_{fin}$ such that $b = v_f(e)$. Hence, $v_f(x) = v_f(e)$. Note that $e \neq x$. If $e < x$, then v_f is a constant function on $[e, x]$ and ${}_A D_- v_f(x) = {}_A D^- v_f(x) = 0$, contradicting ${}_A Dv_f(x) = \infty$. Similarly, if $e > x$,we would get that ${}_A D_+ v_f(x) = {}_A D^+ v_f(x) = 0$ and obtain a similar contradiction. Thus, $b \notin v_f(E \cap A_{fin})$ and so $b \in v_f(E) \cap \left(v_f(E \cap A_{fin})\right)^c$. It follows that $v_f(E) \cap \left(v_f(E \cap A_{fin})\right)^c = v_f(E - E \cap A_{fin}) = v_f(E \cap A_\infty)$. Therefore, we have that

$$m^*\left(v_f(E)\right) = m^*\left(v_f(E \cap A_{fin})\right) + m^*\left(v_f(E \cap A_\infty)\right).$$

By Theorem 6 of "*Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*",

$$m\left(\nu_f(E\cap A_{fin})\right)=\int_{E\cap A_{fin}}\left|{}_A Df(x)\right|dx=\int_E\left|{}_A Df(x)\right|dx,$$

since $m(E-E\cap A_{fin})=0$. It follows that

$$m^*\left(\nu_f(E)\right)=m^*\left(\nu_f(E\cap A_\infty)\right)+\int_E\left|{}_A Df(x)\right|dx.$$

If we let $A_{+\infty}=\left\{x\in A:\ {}_A Df(x)=+\infty\right\}$ and $A_{-\infty}=\left\{x\in A:\ {}_A Df(x)=-\infty\right\}$, then $A_\infty=A_{+\infty}\cup A_{-\infty}$.

If we are interested in signed measure, then the next theorem is a useful decomposition.

Theorem 2. Suppose *A* is a measurable subset of $\mathbb{R}$ and $f:A\to\mathbb{R}$ is a finite-valued function of bounded variation on *A*. Suppose $E\subseteq A$ is a measurable subset of *A*. Let $A_+=\left\{x\in A:\ {}_A Df(x)\text{ exists is finite and }{}_A Df(x)\geq 0\right\}$, $A_-=\left\{x\in A:\ {}_A Df(x)\text{ exists is finite and }{}_A Df(x)<0\right\}$, $A_{+\infty}=\left\{x\in A:\ {}_A Df(x)=+\infty\right\}$ and $A_{-\infty}=\left\{x\in A:\ {}_A Df(x)=-\infty\right\}$. Let $p(x)=\frac{1}{2}\left(\nu_f(x)+f(x)\right)-\frac{f(x_0)}{2}$ and $n(x)=\frac{1}{2}\left(\nu_f(x)-f(x)\right)+\frac{f(x_0)}{2}$, where $\nu_f(x)$ is defined using a fixed anchor point x_0 in *A* as in "*Functions of Bounded Variation and de La Vallée Poussin's Theorem*". Note that $p(x)=\varphi_1(x)$ and $n(x)=\varphi_2(x)$ as given in Theorem 6 of "*Functions of Bounded Variation and de La Vallée Poussin's Theorem*". Note that $p(x)+n(x)=\nu_f(x)$ and $p(x)-n(x)=f(x)-f(x_0)$. Then

$$m^*(p(E))-m^*(n(E))=m^*(p(E\cap A_\infty)-m^*(n(E\cap A_\infty)+\int_E {}_A Df(x)dx\ .$$

Proof.

As for the proof of Theorem 1, we may assume that every point of *A* is a two-sided limit point of *A*.

By Theorem 18 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, there is a null set $N\subseteq A$ such that $m(N)=m(f(N))=m\left(\nu_f(N)\right)=0$ and for all $x \in A - N$, ${}_A Df(x)$ and ${}_A Dv_f(x)$ exist finitely or infinitely and that $|{}_A Df(x)| = {}_A Dv_f(x)$. Moreover, $A_\infty = \{x \in A : |{}_A Df(x)| = +\infty\}$ is a null set. The function *f* is a Lusin function on $A - (N \cup A_\infty)$. Therefore, by Theorem 10 of *Function of Bounded Variation on Arbitrary Subset and Johnson's Indicatrix*, $v_f(x)$ is a Lusin function on $A - (N \cup A_\infty)$. Moreover, *f* is continuous on $A - (N \cup A_\infty)$. Note that $A_\infty = A_{+\infty} \cup A_{-\infty}$. We may assume that $E \subseteq A - N$ and that $A - (N \cup A_\infty) = A_+ \cup A_-$.

Note that since *f* is continuous on $A - (N \cup A_\infty)$, by Theorem 13 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, v_f is also continuous on $A - (N \cup A_\infty)$.

Let $p(x) = \varphi_1(x)$ and $n(x) = \varphi_2(x)$ be respectively the positive and negative variation functions of *f* as defined in Theorem 6 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*. Since $p(x) = \frac{1}{2}(v_f(x) + f(x)) - \frac{f(x_0)}{2}$ and $n(x) = \frac{1}{2}(v_f(x) - f(x)) + \frac{f(x_0)}{2}$, *p*(*x*) and *n*(*x*) are also continuous on $A - (N \cup A_\infty)$. Both *p*(*x*) and *n*(*x*) are increasing bounded functions on *A*. Note that *p*(*x*) and *n*(*x*) are Lusin functions on $A - (N \cup A_\infty)$. We deduce this as follows.
Similar to the proof that for a function of bounded variation *g*, if $m(v_g(H)) = 0$, then $m(g(H)) = 0$, we can show that, if $m(v_g(H)) = 0$, then for the positive and negative variation of *g*, p_g and n_g, $m(n_g(H)) = m(p_g(H)) = 0$. We deliberate a proof of this assertion below.

Suppose *H* is a subset of *A* such that $m^*(v_f(H)) = 0$. Then given any $\varepsilon > 0$, there exists an open set *U* such that $v_{f,\ell}(H) \subseteq U$ and $m(U) < \varepsilon$. Since *U* is open, *U* is a disjoint union of at most countable number of open intervals, i.e., $U = \bigcup_n I_n$ and $m(U) = \sum_i m(I_i) < \varepsilon$. Moreover, $v_{f,\ell}^{-1}(U) \supseteq H$. Let $A_i = p_f\left(v_{f,\ell}^{-1}(I_i)\right)$. For any *x*, *y* in A_i, there exist $a, b \in v_{f,\ell}^{-1}(I_i)$ such that $x = p_f(a)$ and $y = p_f(b)$. Then

$$|x - y| = |p_f(a) - p_f(b)| \le |v_{f,\ell}(a) - v_{f,\ell}(b)| \le m^*(I_i).$$

It follows that the diameter of A_i is less than or equal to $m^*(I_i)$. Hence, $m^*(A_i) \le m^*(I_i)$.

Now, $p_f(H) \subseteq p_f\left(\nu_{f,\ell}{}^{-1}(U)\right) = p_f\left(\nu_{f,\ell}{}^{-1}\left(\bigcup_i I_i\right)\right) = p_f\left(\bigcup_i \nu_{f,\ell}{}^{-1}(I_i)\right) = \bigcup_i p_f\left(\nu_{f,\ell}{}^{-1}(I_i)\right)$.

Therefore,

$$m^*\left(p_f(H)\right) \le \sum_i m^*\left(p_f\left(\nu_{f,\ell}{}^{-1}(I_i)\right)\right) = \sum_i m^*\left(A_i\right) \le \sum_i m^*(I_i) < \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $m^*(p_f(H)) = 0$. In exactly the same manner, we deduce that $m^*(n_f(H)) = 0$.

Thus, $p(N)$ and $n(N)$ are null sets. Hence, *p* and *n* are Lusin functions on $A - A_\infty$.

By Theorem 10 of "*Absolutely Continuous Function on Arbitrary Domain and Function of Bounded*, $p(E \cap \left(A - \left(N \cup A_\infty\right)\right))$ and $n(E \cap A - \left(N \cup A_\infty\right))$ are measurable and so $p\left(E \cap \left(A - A_\infty\right)\right)$ and $n(E \cap A - A_\infty)$ are measurable.

Now for a measurable set *F* and for any set *B* we have that

$$m^*(F \cup B) + m^*(F \cap B) = m^*(F) + m^*(B) .$$

We give a quick proof of this statement.

Since *F* is measurable,

$$m^*(F \cup B) = m^*\left((F \cup B) \cap F\right) + m^*\left((F \cup B) - F\right) = m^*\left(F\right) + m^*\left(B - F\right)$$

Therefore,

$$m^*(F \cup B) + m^*(F \cap B) = m^*\left(F\right) + m^*\left(B - F\right) + m^*(F \cap B) = m^*\left(F\right) + m^*\left(B\right) .$$

Taking $F = p\left(E \cap \left(A - A_\infty\right)\right)$ and $B = p\left(E \cap A_\infty\right)$ we get

$$m^*\left(p\left(E \cap \left(A - A_\infty\right)\right) \cup p\left(E \cap A_\infty\right)\right) + m^*\left(p\left(E \cap \left(A - A_\infty\right)\right) \cap p\left(E \cap A_\infty\right)\right)$$

$$= m^*\left(p\left(E \cap \left(A - A_\infty\right)\right)\right) + m^*\left(p\left(E \cap A_\infty\right)\right).$$

That is,

$$m^*\left(p\left(E \cap A\right)\right) + m^*\left(p\left(E \cap \left(A - A_\infty\right)\right) \cap p\left(E \cap A_\infty\right)\right)$$

$$= m\left(p\left(E \cap \left(A_+ \cup A_-\right)\right)\right) + m^*\left(p\left(E \cap A_\infty\right)\right).$$

We shall show that $p\left(E\cap\left(A-A_\infty\right)\right)\cap p\left(E\cap A_\infty\right)$ is a null set.

Take $a\in p\left(E\cap\left(A-A_\infty\right)\right)\cap p\left(E\cap A_\infty\right)$. Then there exists $x\in E\cap\left(A-A_\infty\right)$ and $y\in E\cap A_\infty$ such that $p(x)=p(y)=a$. Note that $x\neq y$.

Suppose $x<y$. Then *p* is a constant function on [*x*, *y*]. If $y\in E\cap A_{+\infty}$, then ${}_A Dp(y)=+\infty$. But this is impossible since ${}_A Dp^-(y)=0$. Thus, $y\notin E\cap A_{+\infty}$ and so $y\in E\cap A_{-\infty}$. In this case, either ${}_A Dp(y)=+\infty$, or ${}_A Dp(y)$ is finite or ${}_A Dp(y)$ is not defined. We have already seen that ${}_A Dp(y)\neq+\infty$. If ${}_A Dp(y)$ is finite, then it must be zero. Then by Theorem 11 of "*Arbitrary Functions, Limit Superior, Dini Derivative and Lebesgue Density Theorem*", *y* must belong to a subset *D* such that $m^*(p(D))=0$. If ${}_A Dp(y)$ is not defined, that is, *p* is not differentiable finitely or infinitely at *y*, then *y* belongs to a subset *N* of *A* such that $m(N)=m(p(N))=0$. If $x>y$, we get the same conclusion about *y*. It follows that $p\left(E\cap\left(A-A_\infty\right)\right)\cap p\left(E\cap A_\infty\right)$ is a null set. Hence,

$$m^*\left(p\left(E\cap A\right)\right)=m\left(p\left(E\cap\left(A_+\cup A_-\right)\right)\right)+m^*\left(p\left(E\cap A_\infty\right)\right).$$

For $x\in E\cap\left(A_-\right)$, ${}_A Dp(x)=\frac{1}{2}\left(\left|{}_A Df(x)\right|+{}_A Df(x)\right)=0$ and so by Theorem 11 of *Arbitrary Function, Limit Superior, Dini Derivative and Lebesgue Density Theorem*, $m\left(p\left(E\cap A_-\right)\right)=0$. Therefore,

$$m^*(p(E))=m\left(p\left(E\cap A_+\right)\right)+m^*\left(p\left(E\cap A_\infty\right)\right).$$

Similarly, we deduce that $m(n(E\cap A_+))=0$ and that

$$m^*(n(E))=m\left(n\left(E\cap A_-\right)\right)+m^*\left(n\left(E\cap A_\infty\right)\right).$$

Now by Theorem 6 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*,

$$m(p(E\cap A_+))=\int_{E\cap A_+}{}_A Dp(x)dx=\int_{E\cap A_+}{}_A Df(x)dx.$$

Similarly, we obtain

$$m(n(E\cap A_-))=\int_{E\cap A_-}{}_A Dn(x)dx=-\int_{E\cap A_-}{}_A Df(x)dx.$$

Hence, we get

$$m^*(p(E)) - m^*(n(E))$$
$$= m^*\big(p(E\cap(A_\infty))\big) - m^*\big(n(E\cap(A_\infty))\big) + \int_{E\cap A_+} {}_A Df(x)dx + \int_{E\cap A_-} {}_A Df(x)dx$$
$$= m^*\big(p(E\cap(A_\infty))\big) - m^*\big(n(E\cap(A_\infty))\big) + \int_{E\cap(A_+\cup A_-)} {}_A Df(x)dx$$
$$= m^*\big(p(E\cap(A_\infty))\big) - m^*\big(n(E\cap(A_\infty))\big) + \int_{E} {}_A Df(x)dx .$$

This completes the proof of Theorem 2.

Since $E \subseteq A - N$ and $A - N = A_\infty \cup A_+ \cup A_-$ is a disjoint union $E\cap(A-N) = (E\cap A_\infty)\cup(E\cap A_+)\cup(E\cap A_-)$ is a disjoint union and $m(E\cap A_\infty) = 0$. For x in $E\cap(A-N)$, $|{}_A Df(x)| = {}_A D\nu_f(x)$.

By Theorem 13 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*,

$$m\big(p(E\cap A_+)\big) = \int_{E\cap A_+} {}_A Dp(x)dx = \int_{E\cap A_+} {}_A Df(x)dx = \int_{E\cap A_+} \left|{}_A Df(x)\right| dx = m\big(\nu_f(E\cap A_+)\big) \text{ and}$$
$$m\big(n(E\cap A_-)\big) = \int_{E\cap A_-} {}_A Dn(x)dx = -\int_{E\cap A_-} {}_A Df(x)dx = \int_{E\cap A_-} \left|{}_A Df(x)\right| dx = m\big(\nu_f(E\cap A_-)\big).$$

Also,

$$m\big(\nu_f(E\cap(A_+\cup A_-))\big) = m\big(\nu_f(E\cap A_+)\big) + m\big(\nu_f(E\cap A_-)\big) ,$$

since $\nu_f(E\cap A_+)$ and $\nu_f(E\cap A_-)$ are measurable and disjoint,

$$= m\big(p(E\cap A_+)\big) + m\big(n(E\cap A_-)\big).$$

$$m^*(p(E)) + m^*(n(E))$$
$$= m^*\big(p(E\cap(A_\infty))\big) + m^*\big(n(E\cap(A_\infty))\big) + \int_{E\cap A_+} {}_A Df(x)dx - \int_{E\cap A_-} {}_A Df(x)dx$$
$$= m^*\big(p(E\cap(A_\infty))\big) + m^*\big(n(E\cap(A_\infty))\big) + \int_{E\cap(A_+\cup A_-)} \left|{}_A Df(x)\right| dx$$
$$= m^*\big(p(E\cap(A_\infty))\big) + m^*\big(n(E\cap(A_\infty))\big) + \int_{E} \left|{}_A Df(x)\right| dx .$$

Comparing with

$$m^*\big(\nu_f(E)\big) = m^*\big(\nu_f(E\cap A_\infty)\big) + \int_{E\cap A_{fin}} \left|{}_A Df(x)\right| dx = m^*\big(\nu_f(E\cap A_\infty)\big) + \int_{E} \left|{}_A Df(x)\right| dx \text{, we}$$

get

$m^*\left(\nu_f(E)\right)$

$= m^*(p(E)) + m^*(n(E)) + m^*\left(\nu_f(E \cap A_\infty)\right) - m^*\left(p(E \cap (A_\infty))\right) - m^*\left(n(E \cap (A_\infty))\right)$.

Thus, if *f* is absolutely continuous,

$$m\left(\nu_f(E)\right) = m(p(E)) + m(n(E)) .$$

Hence, we have proved:

Corollary 3. Suppose *A* is a measurable subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ is a finite-valued function of bounded variation on *A*. Suppose $E \subseteq A$ is a measurable subset of *A*. Then

$m^*(p(E)) + m^*(n(E)) = m^*\left(p(E \cap (A_\infty))\right) + m^*\left(n(E \cap (A_\infty))\right) + \int_E \left| {}_A Df(x) \right| dx$ and

$m^*\left(\nu_f(E)\right)$

$= m^*(p(E)) + m^*(n(E)) + m^*\left(\nu_f(E \cap A_\infty)\right) - m^*\left(p(E \cap (A_\infty))\right) - m^*\left(n(E \cap (A_\infty))\right)$.

If *f* is absolutely continuous or if $E \subseteq A - A_\infty$, then
$m^*\left(\nu_f(E)\right) = m^*(p(E)) + m^*(n(E))$.

Remark.

Of theoretical interest to the proponent that all sets are measurable, we have the following refinement.

Note that $\nu_f(E \cap A_{+\infty})$ and $\nu_f(E \cap A_{-\infty})$ are disjoint. If we take the view point that all sets are measurable, then we may write

$$m(\nu_f(E \cap A_\infty)) = m\left(\nu_f(E \cap A_{+\infty})\right) + m\left(\nu_f(E \cap A_{-\infty})\right).$$

Similarly, under the assumption that all sets are measurable,

$$m(p(E \cap A_\infty)) = m\left(p(E \cap A_{+\infty})\right) + m\left(p(E \cap A_{-\infty})\right)$$

and

$$m\left(n(E \cap A_\infty)\right) = m\left(n(E \cap A_{+\infty})\right) + m\left(n(E \cap A_{-\infty})\right).$$