On Kestelman Change of Variable Theorem for Riemann Integral

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Kestelman gave the most general form of the change of variable theorem for Riemann integral. We present here a proof of this theorem involving a result about the chain rule for composition and the properties of absolute continuity.

Theorem 1 (Kestelman). Suppose h is Riemann integrable on the closed and bounded interval [a, b] and H: [a, b] $\to$ **R** is an indefinite integral of h, i.e., $H(x) = H(a) + \int_a^x h(t)dt$ for x in [a, b]. Suppose f is a bounded function on H([a, b]). Then f is Riemann integrable on H([a, b]) if, and only if, $f \circ H(x)h(x)$ is Riemann integrable on [a, b]. Moreover, whenever f is Riemann integrable on H([a, b]) or $f \circ H(x)h(x)$ is Riemann integrable on [a, b], we have the change of variable formula for Riemann integral,

$$\int_a^b f(H(x))h(x)dx = \int_{H(a)}^{H(b)} f(x)dx \ .$$

The first assertion in Theorem 1 of the simple connection between the functions in the above integrands is stated below in the following Theorem.

Theorem 2. Suppose h is a function Riemann integrable on the closed and bounded interval [a, b] and H: [a, b] $\to$ **R** is an indefinite integral of h, i.e., $H(x) = H(a) + \int_a^x h(t)dt$ for x in [a, b]. Suppose f is a bounded real valued function on H([a, b]) = [c, d]. Then f is Riemann integrable on [c, d] if, and only if, $f(H(x))h(x)$ is Riemann integrable on [a, b].

Poof.
Since the function h is Riemann integrable on [a, b], h is continuous almost everywhere on [a, b]. Therefore, there exists a subset A of [a, b] of measure zero and h is continuous on [a, b] − A. Thus, by the Fundamental Theorem of Calculus, for x in [a, b] − A, H is differentiable at x and $H'(x) = h(x)$.

Suppose f : [c, d] $\to$ **R** is Riemann integrable. Then f is continuous almost everywhere on [c, d]. Hence, there exists a subset E in [c, d] of measure zero such that f is continuous on [c, d]− E. Note that f is bounded on [c, d] and h is bounded on [a, b], it follows that $f(H(t))h(t)$ is bounded on [a, b].

Suppose $x \in [a,b] - A$ and $h(x) = 0$. Since the function h is continuous at x and f is bounded so that $f \circ H$ is also bounded, $\lim_{y \to x} f(H(y))h(y) = \lim_{y \to x} h(y) = 0$. Hence $f(H(t))h(t)$ is continuous at x for $x \in \{t \in [a,b] - A : H'(t) = h(t) = 0\}$.

Let $L = \{t \in [a,b] - A : H'(t) = h(t) \neq 0\}$. It remains to show that $f(H(t))h(t)$ is continuous almost everywhere in L. Let $B = H^{-1}(E)$. For $x \in L - B$, $H(x) \notin E$ x so that f is continuous at $H(x)$ and since H is continuous at x, it follows that $f(H(t))$ is continuous at x. Therefore, $f(H(t))h(t)$ is continuous on $L - B$.

By Theorem 2 of *Change of Variables Theorems*, since $m\left(H\left(B\cap L\right)\right)=0$ because $m\left(H(B)\right)=0$, $H'(t)=h(t)=0$ almost everywhere on $B\cap L$. It follows that $f(H(t))h(t)$ is continuous almost everywhere on $B\cap L$. Hence, $f(H(t))h(t)$ is continuous almost everywhere on L and so on $[a,b]-A$ and as $m(A)=0$, it is continuous almost everywhere on $[a, b]$. This means that $f(H(t))h(t)$ is Riemann integrable on $[a, b]$.
Suppose f is bounded and $f(H(t))h(t)$ is Riemann integrable on $[a, b]$. We shall show that f is continuous almost everywhere on $[c, d]$ and so is Riemann integrable. To do this we use the following proposition.

Proposition 3. Suppose h is Riemann integrable on the closed and bounded interval $[a, b]$ and H: $[a, b] \to \mathbf{R}$ is an indefinite integral of h, i.e., $H(x)=H(a)+\int_a^x h(t)dt$ for x in $[a, b]$. Suppose f is bounded on $H([a, b])$.
Suppose A is a subset of $[a, b]$ of measure zero such that h is continuous on $[a, b] - A$ and $H'(x)=h(x)$. Then for x in $[a, b] - A$, $f(H(t))h(t)$ is continuous at x, if, and only if, $h(x) = 0$ or f is continuous at $H(x)$.

Proof.
We have already shown that if $x \in [a, b] - A$ and $h(x) = 0$ and f is bounded on $H([a, b])$, then $f(H(t))h(t)$ is continuous at x.
Take $x \in [a, b] - A$. Plainly, if f is continuous at $H(x)$, then $f(H(t))$ is continuous at x since H is continuous at x and so $f(H(t))h(t)$ is continuous at x.
Suppose $f(H(t))h(t)$ is continuous at $x\in[a,b]-A$ and $h(x) \neq 0$. Then plainly, $f(H(t))$ is continuous at x. We shall show that f is continuous at $H(x)$. Note that
$\lim_{t\to x}\frac{H(t)-H(x)}{t-x}=H'(x)=h(x)\neq 0$.
We may assume without loss of generality that x is in the interior of $[a, b]$.
Suppose $h(x) > 0$. Then there exists $\delta > 0$ such that $(x-\delta,x+\delta)\subseteq(a,b)$ and

$$\left|t-x\right|<\delta \Rightarrow \left|h(t)-h(x)\right|<\frac{1}{2}h(x)\Rightarrow h(t)>\frac{1}{2}h(x)>0.$$

Therefore, for all $t_1<t_2$ and $t_1,t_2\in(x-\delta,x+\delta)$,

$$H(t_2)-H(t_1)=\int_{t_1}^{t_2}h(t)dt\geq\int_{t_1}^{t_2}\frac{h(x)}{2}dt=\frac{h(x)}{2}(t_2-t_1)>0.$$

Hence, H is a continuous and strictly increasing function on $(x-\delta,x+\delta)$. This means that the restriction of H to the interval $(x-\delta,x+\delta)$ has a strictly increasing continuous inverse g. As $f\circ H(t)$ is continuous at x, $\lim_{t\to x} f\circ H(t)=f(H(x))$ and since g is continuous at $H(x)$, $\lim_{y\to H(x)}g(y)=g(H(x))=x$. Therefore, $\lim_{y\to H(x)}f(y)=\lim_{y\to H(x)}\left(f\circ H\right)\circ g(y)=f(H(x))$.

This means f is continuous at $H(x)$.
We deduce similarly that if $h(x) < 0$, f is continuous at $H(x)$.
This concludes the proof of Proposition 3.

Completion of the proof of Theorem 2.

Now suppose $f(H(t))h(t)$ is Riemann integrable on $[a, b]$ and so $f(H(t))h(t)$ is continuous almost everywhere on $[a, b] - A$.

Now for x in $[a, b] - A$, by Proposition 3, $f(H(t))h(t)$ is not continuous at x if, and only if, $h(x) \neq 0$ and f is not continuous at $H(x)$.
Let $C = \{t \in [a,b] - A : f \text{ is not ontinuous at } H(t)\}$ and

$$D = \{t \in [a,b] - A : h(t) \neq 0\} = \{t \in [a,b] - A : H'(t) = h(t) \neq 0\} = L .$$

Thus, for x in $[a, b] - A$, $f(H(t))h(t)$ is not continuous at x if, and only if $x \in C \cap D$.
Therefore, since $f(H(t))h(t)$ is continuous almost everywhere on $[a, b] - A$, $m(C \cap D) = 0$.
Since H is absolutely continuous on $[a, b]$, $m\left(H(C \cap D)\right) = 0$.
Let $\widetilde{D} = \{t \in [a,b] - A : h(t) = 0\}$. Then $[a,b] - A = D \cup \widetilde{D}$ and

$$C = C \cap \left(D \cup \widetilde{D}\right) = (C \cap D) \cup \left(C \cap \widetilde{D}\right) .$$

By Theorem 3 of *Functions Having Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*,

$$m(H(\widetilde{D})) = m\left(H\left(\{t \in [a,b] - A : H'(t) = h(t) = 0\}\right)\right) = 0 .$$

Thus, since $H(C) = H\left(C \cap D\right) \cup H\left(C \cap \widetilde{D}\right)$, $m\left(H(C)\right) = 0$. Since A is of measure zero and H is absolutely continuous, $m\left(H(A)\right) = 0$. If $y \in [c,d] - H(A)$ and f is not continuous at y, then there exists $t \in [a,b] - A$ such that $y = H(t)$ and so $t \in C$. Consequently, if $y \in [c,d] - H(A)$ and f is not continuous at y, $y \in H(C)$. As H maps $[a, b]$ onto $[c, d]$, it follows that

$$E = \{y \in [c,d] : f \text{ is not continuous at } y\} \subseteq H(A) \cup H(C) .$$

Hence, $m(E) = 0$. Thus, f is continuous almost everywhere on $[c, d]$ and so is Riemann integrable on $[c, d]$.

Proof of the Second part of Theorem 1.

By Theorem 2, we may assume that f is Riemann integrable on $[c, d]$. Let $F : [c,d] \to \mathbb{R}$ be an indefinite integral of the function f. Then F is an absolutely continuous function satisfying a Lipschitz condition. Since $H : [a,b] \to [c,d]$ is absolutely continuous, $F \circ H$ is absolutely continuous on $[a, b]$. Therefore, $F \circ H$ has finite derivative almost everywhere on $[a, b]$. Since F is absolutely continuous, F is an N-function, therefore by the following Chain Rule (see Theorem 4 below),

$$\left(F \circ H\right)'(x) = \left(f \circ H\right)(x)H'(x) = \left(f \circ H\right)(x)h(x)$$

almost everywhere on $[a, b]$. By Theorem 2, $f(H(x))h(x)$ is Riemann integrable on $[a, b]$.
Therefore, $\left(F \circ H\right)'$ is Riemann integrable on $[a, b]$. Hence, by the Fundamental Theorem of Calculus, the Riemann integral,

$$\int_a^b \left(F \circ H\right)'(t)dt = F \circ H(b) - F \circ H(a) = F(H(b)) - F(H(a)) .$$

We may also deduce this as follows.

Since $F \circ H$ is absolutely continuous on [*a*, *b*], $\left(F \circ H\right)'$ is Lebesgue integrable on [*a*, *b*] and the Lebesgue integral of the derivative,

$$\text{Lebesgue} \int_a^b \left(F \circ H\right)'(t)dt = F \circ H(b) - F \circ H(a) = F(H(b)) - F(H(a)).$$

But $\left(F \circ H\right)'$ is Riemann integrable on [*a*, *b*] so that the Lebesgue integral is equal to the Riemann integral and so

$$\int_a^b \left(F \circ H\right)'(t)dt = F(H(b)) - F(H(a)).$$

Since *F* is an indefinite Riemann integral of *f*, $\int_{H(a)}^{H(b)} f(t)dt = F(H(b)) - F(H(A))$ and so

$\int_a^b \left(F \circ H\right)'(t)dt = \int_{H(a)}^{H(b)} f(t)dt$.

Hence, $\int_{H(a)}^{H(b)} f(x)dx = \int_a^b \left(F \circ H\right)'(x) = \int_a^b \left(f \circ H\right)(x)h(x)dx$.

Theorem 4. Suppose *F* has finite derivatives almost everywhere on [*c*, *d*] and *g* and $F \circ g$ have finite derivatives almost everywhere on [*a*, *b*]. It is assumed that the range of g is contained in [*c*, *d*]. Suppose *F* is an *N*-function, i.e., *F* maps sets of measure zero to sets of measure zero. Then $(F \circ g)' = (f \circ g)\, g'$ almost everywhere on [*a*, *b*], where $F' = f$ almost everywhere on [*c*, *d*], that is to say, the chain rule holds almost everywhere on [*a*, *b*].

Theorem 4 is Theorem 3 of *Change of Variables Theorems* and the proof can be found there.

A necessary condition for the Riemann integrability of the function *f* in Theorem 1 is that the function *f* be bounded on $[H(a), H(b)]$. If *f* is bounded on $[H(a), H(b)]$ and $f \circ H(t)h(t)$ is Riemann integrable on [*a*, *b*], then *f* is Riemann integrable on $[H(a), H(b)]$ and the Change of variable formula holds even though *f* may not be bounded on *H*([*a*, *b*]).

Theorem 5.

Suppose *h* is Riemann integrable on the closed and bounded interval [*a*, *b*] and *H*: [*a*, *b*] → **R** is an indefinite integral of *h*, i.e., $H(x) = H(a) + \int_a^x h(t)dt$ for *x* in [*a*, *b*].

Suppose *A* is a subset of [*a*, *b*] of measure zero such that *h* is continuous on [*a*, *b*] − *A* and $H'(x) = h(x)$ for $x \in [a,b] - A$.

Suppose f is a function defined on *H*([*a*, *b*]).

Assume that *H*(*a*) < *H*(*b*).

Suppose $f(H(t))h(t)$ is Riemann integrable on [*a*, *b*] and *f* is bounded on the interval $[H(a), H(b)]$. Then *f* is continuous almost everywhere on $[H(a), H(b)]$ and so is Riemann integrable on $[H(a), H(b)]$. Moreover, $\int_a^b f(H(x))h(x)dx = \int_{H(a)}^{H(b)} f(x)dx$.

Proof.

Since *h* is Riemann integrable on the closed and bounded interval [*a*, *b*], there is a subset *A* of measure zero in [*a*, *b*] such that *h* is continuous on [*a*, *b*] − *A* and $H'(x) = h(x)$.

$f(H(t))h(t)$ is Riemann integrable implies that there is a subset *E* of measure zero in [*a*, *b*] such that $f(H(t))h(t)$ is continuous on [*a*, *b*] − *E*. Let $C = A \cup E$. Then the measure of *C* is zero. Both $f(H(t))h(t)$ and *h*(*t*) are continuous on [*a*, *b*] − *C*. Moreover, $H'(x) = h(x)$ for all $x \in [a,b] - C$.

Now for $x \in [a,b] - C$, either $H'(x) = h(x) \neq 0$ or $H'(x) = h(x) = 0$.

Let $F = \{x \in [a,b] - C : H'(x) = h(x) = 0\}$. Then by Theorem 3 of *Functions Having Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*, $m(H(F)) = 0$. Note that $m(H(C)) = 0$ as *H* is absolutely continuous. We have shown in the proof of Proposition 3 that if $f(H(t))h(t)$ is continuous at $x \in [a,b] - C$ and $H'(x) = h(x) \neq 0$, *f* is continuous at *H*(*x*).

If *H* is not continuous at $y \in H([a,b])$ and $y \notin H(C)$, then $y \in H([a,b] - C)$ and so there exists $t \in [a,b] - C$ such that $y = H(t)$. Since $f(H(t))h(t)$ is continuous on [*a*, *b*] − *C*, by Proposition 3, $H'(t) = h(t) = 0$ and so $y \in H(F)$.

It follows that *f* is continuous almost everywhere on *H*([*a*, *b*]) and so it is continuous almost everywhere in $[H(a), H(b)]$. As *f* is bounded on $[H(a), H(b)]$, *f* is Riemann integrable on $[H(a), H(b)]$.

Let $M = \sup\{|\,|f(x)| : x \in [H(a), H(b)]\}$. For each positive integer *n* let

$$f_n(y) = \begin{cases} f(y), \text{ if } |f(y)| \leq M + n, \\ M + n, \text{ if } f(y) > M + n, \\ -M - n, \text{ if} f(y) < -M - n \end{cases}.$$

Then f_n is a bounded function on *H*([*a*, *b*]) bounded by *M* + *n*.

Then the restriction of f_n to $[H(a), H(b)]$ is equal to the restriction of *f* on $[H(a), H(b)]$.

Moreover f_n converges to f pointwise on *H*([*a*, *b*]). It follows that $f_n(H(t))h(t)$ converges pointwise to $f(H(t))h(t)$ on [*a*, *b*]. Note that $|f_n(y)| \leq |f(y)|$ for all *y* in *H*([*a*, *b*]) and so $|f_n(H(t))h(t)| \leq |f(H(t))h(t)|$ for all *t* in [*a*, *b*].

Next, we shall show that $f_n(H(t))h(t)$ is Riemann integrable on [*a*, *b*]. We shall show that $f_n(H(t))h(t)$ is continuous almost everywhere on [*a*, *b*]. Let $M_n = M + n$. Let $x \in [a,b] - C$.

If $h(x) = 0$, then $\lim_{t \to x} f_n(H(t))h(t) = \lim_{t \to x} h(t) = 0$ since $f_n \circ H$ is bounded on [*a*, *b*] and *h* is continuous at *x*. Hence $f_n(H(t))h(t)$ is continuous at *x* if $h(x) = 0$.

Suppose now $h(x) \neq 0$. We may assume without loss of generality that *x* is in the interior of [*a*, *b*].

Suppose *h*(*x*) > 0. We have either $|f(H(x))| > C_n$, $|f(H(x))| < C_n$ or $|f(H(x))| = C_n$.

Suppose $|f(H(x))| > C_n$.

Then $|f_n(H(x))| = C_n$. Since $f(H(t))h(t)$ is continuous at $x \in [a,b] - C$ and $h(x) \neq 0$, $f(H(t))$ is continuous at *x*.

Suppose $f(H(x)) > C_n$. Then by the continuity of $f(H(t))$ at *x*, there exists $\delta_1 > 0$ such that $(x - \delta_1, x + \delta_1) \subseteq (a,b)$ and $f(H(t)) > C_n$ for $t \in (x - \delta_1, x + \delta_1)$. Hence, $f_n(H(t)) = C_n$ for $t \in (x - \delta_1, x + \delta_1)$. It follows that $f_n(H(t))h(t)$ is continuous at *x*. Similarly, we can show that if $f(H(x)) < -C_n$, $f_n(H(t))h(t)$ is continuous at *x*.

Suppose now $|f(H(x))| < C_n$ and $h(x) \neq 0$. Then by continuity of $f(H(t))$ at x, there exists $\delta_2 > 0$ such that $(x-\delta_2, x+\delta_2) \subseteq (a,b)$ and $|f(H(t))| < C_n$ for $t \in (x-\delta_2, x+\delta_2)$. Hence, $f_n(H(t)) = f(H(t))$ for $t \in (x-\delta_2, x+\delta_2)$. Therefore, $f_n(H(t))h(t)$ is continuous at x.

Suppose now $h(x) > 0$ and $f_n(H(x)) = f(H(x)) = C_n$. Then since $f(H(t))$ is continuous at x, given $\varepsilon > 0$, there exists $\delta_3 > 0$ such that $(x-\delta_3, x+\delta_3) \subseteq (a,b)$ and $|f(H(t)) - f(H(x))| < \varepsilon$ for $t \in (x-\delta_3, x+\delta_3)$. Then

$$|f_n(H(t)) - f(H(x))| = \begin{cases} |f(H(t)) - f(H(x))| \text{ if } |f(H(t))| < C_n \\ 0 \text{ if } |f(H(t))| \geq C_n \end{cases} < \varepsilon .$$

Thus, $f_n(H(t))$ is continuous at x and so $f_n(H(t))h(t)$ is continuous at x.
Similarly, we can show that if $h(x) > 0$ and $f_n(H(x)) = f(H(x)) = -C_n$, then $f_n(H(t))h(t)$ is continuous at x.
Therefore, if $x \in [a,b] - C$ and $h(x) > 0$, $f_n(H(t))h(t)$ is continuous at x.
In the same manner we can show that if $x \in [a,b] - C$ and $h(x) < 0$ then $f_n(H(t))h(t)$ is continuous at x. Thus, $f_n(H(t))h(t)$ is continuous at $x \in [a,b] - C$ if $h(x) \neq 0$. We have already shown that $f_n(H(t))h(t)$ is continuous at $x \in [a,b] - C$ if $h(x) = 0$. Therefore, $f_n(H(t))h(t)$ is continuous at $x \in [a,b] - C$. Since $f_n(H(t))h(t)$ is a bounded function on [a, b], it follows that $f_n(H(t))h(t)$ is Riemann integrable on [a, b].
Note that $|f_n(H(t))h(t)| \leq |f(H(t))h(t)|$ for all t in [a, b]. Since $f(H(t))h(t)$ is bounded on [a, b], $|f(H(t))h(t)| \leq D$ for some $D > 0$ and for all t in [a, b]. Hence the sequence of Riemann integrable functions $\{f_n(H(t))h(t)\}$ is uniformly bounded. Note that $f_n(H(t))h(t) \to f(H(t))h(t)$ pointwise in [a, b]. Therefore, by Arzelà's Dominated Convergence Theorem, $\int_a^b f_n(H(t))h(t)dt \to \int_a^b f(H(t))h(t)dt$. By Theorem 1, since $f_n(H(t))h(t)$ is Riemann integrable on [a, b], $\int_a^b f_n(H(t))h(t)dt = \int_{H(a)}^{H(b)} f_n(y)dy$. But $\int_{H(a)}^{H(b)} f_n(y)dy = \int_{H(a)}^{H(b)} f(y)dy$ and so $\int_a^b f_n(H(t))h(t)dt = \int_{H(a)}^{H(b)} f(y)dy$. It follows that $\int_a^b f(H(t))h(t)dt = \int_{H(a)}^{H(b)} f(t)dt$.

Remark. The converse of Theorem 5 is false. It is not necessary that if f is Riemann integrable on $[H(a), H(b)]$, h is Riemann integrable on the closed and bounded interval [a, b] and H: [a, b] $\to$ **R** is an indefinite integral of h, then $f(H(t))h(t)$ is Riemann integrable on [a, b].

We can easily find an unbounded function f and a function H for a counterexample.

Take $H(t) = \dfrac{t(t-4)^2}{3}$ so that $H'(t) = h(t) = \dfrac{(t-4)(3t-4)}{3}$ and $f(t) = \begin{cases} \dfrac{1}{|t-3|}, \text{ if } t \neq 3, \\ 1 \text{ , if } t = 3 \end{cases}$.

$H(0)=0,\ H(3)=1$. The function f is Riemann integrable on $[H(0), H(3)] = [0,1]$ but $f(H(t))h(t)$ is unbounded on [0, 3] as $\lim_{t \to 1} f(H(t))h(t) = \infty$ and so is not Riemann integrable on [0, 3].