National University of Singapore

Department of Mathematics

Semester 1 (2006/2007) **MA3110 Mathematical Analysis II** Tutorial 9

1. Suppose that the function $f:\mathbf{R}\to\mathbf{R}$ is differentiable. Define the function H: $\mathbf{R}\to\mathbf{R}$ by

$$H(x)=\int_{-x}^{x}(f(t)+f(-t))dt \quad \text{for all } x \text{ in } \mathbf{R}.$$

Find $H''(x)$.

2. Suppose that the function $f:\mathbf{R}\to\mathbf{R}$ has a continuous second derivative. Prove that

$$f(x)=f(0)+f'(0)x+\int_{0}^{x}(x-t)f''(t)dt \quad \text{for all } x \text{ in } \mathbf{R}.$$

3. Suppose that the function $f:\mathbf{R}\to\mathbf{R}$ is continuous. Define

$$G(x)=\int_{0}^{x}(x-t)f(t)dt \quad \text{for all } x \text{ in } \mathbf{R}.$$

Prove that $G''(x)=f(x)$ for all x in $\mathbf{R}$.

4. Show that the conclusion of the Mean Value Theorem for Integrals (Theorem 40) can be strengthened so that we can choose the point χ to be in (a, b), not just in $[a, b]$.

5. Suppose that the function g: $\mathbf{R}\to\mathbf{R}$ is continuous and that $g(x) > 0$ for all x. define

$$h(x)=\int_{0}^{x}\frac{1}{g(t)}dt \quad \text{for all } x \text{ in } \mathbf{R}$$

and let $J = h(\mathbf{R})$. Prove that if $f: J\to\mathbf{R}$ is the inverse of h: $\mathbf{R}\to\mathbf{R}$, then $f: J\to\mathbf{R}$ is a solution of the non-linear differential equation

$$\begin{cases} f'(x)=g(f(x)) \text{ for all } x \text{ in } J \\ f(0)=0 \end{cases}$$

6. Suppose that the function $f:[a, b]\to\mathbf{R}$ is continuous and let P be any partition of its domain $[a, b]$. Show that there is a Riemann sum $R(f, P, C)$ that equals $\int_a^b f$.

[Hint: Use the Mean Value Theorem for Integrals.]

7. (i) Let p and n be counting numbers in $\mathbf{P}$ with $n \geq 2$. Prove by induction that

$$\sum_{k=1}^{n-1} k^p \leq \frac{n^{p+1}}{p+1} \leq \sum_{k=1}^{n} k^p$$

(ii) Use (i) to prove that for a counting number p,

$$\int_0^1 x^p dx = \frac{1}{p+1}$$

8. Suppose $f:[0,\infty)\to\mathbf{R}$ is continuous and that $\lim_{x\to\infty} f(x)=a$, where a is a real number. Prove that

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x f(t)dt=a.$$

9. Suppose that f is continuous on $[a, b]$ and that $\int_a^b f(x)g(x)dx=0$ for any continuous function g on $[a, b]$ such that $g(a) = g(b) = 0$. Prove that $f = 0$ the zero constant function.