National University of Singapore

Department of Mathematics

Semester 1 (2006/2007) **MA3110 Mathematical Analysis II** Tutorial 7

1. Suppose that the function $f : (-1,1) \to \mathbf{R}$ has n derivatives and that its nth derivative $f^{(n)} : (-1,1) \to \mathbf{R}$ is bounded. Assume also that
$$f(0) = f'(0) = \ldots = f^{(n-1)}(0) = 0.$$
Prove that there is a positive constant K such that
$$|f(x)| \le K|x|^n \text{ for all } x \text{ in } (-1, 1).$$
2. Consider the partition $P = \{0, 1/4, 1/2, 1\}$ of the interval $[0, 1]$. Compute the lower and upper Darboux sums, i.e., $L(f, P)$ and $U(f, P)$ for the following three choices of function $f : [0,1] \to \mathbf{R}$:
(i) $f(x) = x$ for all x in $[0,1]$.
(ii) $f(x) = 19$ for all x in $[0, 1]$.
(iii) $f(x) = -x^2$ for all x in $[0, 1]$.

3. Suppose that the bounded function $f : [a, b] \to \mathbf{R}$ is such that $f(x) = 0$ for rational x in $[a, b]$.
Prove that $$L\int_a^b f \le 0 \le U\int_a^b f.$$

4. Suppose $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = \begin{cases} -\frac{1}{2}x^2, & x \le 0 \\ \frac{1}{2}x^2, & x \ge 0 \end{cases}$. Determine $f'(x)$ for each x in $\mathbf{R}$. What is the most general anti-derivative of the function $g(x) = |x|$?

5. Show that a necessary and sufficient condition for $f : [a, b] \to \mathbf{R}$ to be integrable is:
For any $\varepsilon > 0$, there exist integrable functions g and h on $[a, b]$ such that
$$g \le f \le h$$
and $$\int_a^b h - \int_a^b g < \varepsilon$$
[Hint: use one of the equivalent condition in Theorem 21.]
6. Use only the definition or equivalent definition of the integral to prove that
$$\int_0^x t^3 dx = \frac{1}{4}x^4$$
7. Show that $\int_{-1}^1 \mu(x)dx = 1$ if $\mu(x) = |x|$.
8. Let $[x]$ be the largest integer $\le x$. Do the following functions have anti-derivatives on the whole of $\mathbf{R}$?
(i) $f(x) = x$, $x \ne 0$, $f(0) = 1$.
(ii) $f(x) = [x]$.
(iii) $f(x) = \dfrac{|x|}{1/2 + [x]}$.
9. Prove that the function $f(x) = \begin{cases} \sin(\frac{1}{x}), & x \ne 0 \\ 0, & x = 0 \end{cases}$ is integrable on $[0, 1]$.